Electronic Journal of Qualitative Theory of Differential Equations
2020, No. **83**, 1–12; https://doi.org/10.14232/ejqtde.2020.1.83 www.math.u-szeged.hu/ejqtde/

Periodic solutions with long period for the Mackey–Glass equation

Dedicated to Professor Jeffrey R. L. Webb on the occasion of his 75th birthday

Tibor Krisztin ✉

Bolyai Institute, University of Szeged, Aradi vértanúk tere 1, Szeged, H–6720, Hungary

Received 8 November 2020, appeared 21 December 2020
Communicated by Gennaro Infante

Abstract. The limiting version of the Mackey–Glass delay differential equation $x'(t) = -ax(t) + bf(x(t-1))$ is considered where a, b are positive reals, and $f(\xi) = \xi$ for $\xi \in [0, 1)$, $f(1) = 1/2$, and $f(\xi) = 0$ for $\xi > 1$. For every $a > 0$ we prove the existence of an $\varepsilon_0 = \varepsilon_0(a) > 0$ so that for all $b \in (a, a + \varepsilon_0)$ there exists a periodic solution $p = p(a, b) : \mathbb{R} \to (0, \infty)$ with minimal period $\omega(a, b)$ such that $\omega(a, b) \to \infty$ as $b \to a+$. A consequence is that, for each $a > 0$, $b \in (a, a + \varepsilon_0(a))$ and sufficiently large n, the classical Mackey–Glass equation $y'(t) = -ay(t) + by(t-1)/[1 + y^n(t-1)]$ has an orbitally asymptotically stable periodic orbit, as well, close to the periodic orbit of the limiting equation.

Keywords: Mackey–Glass equation, periodic solution, limiting nonlinearity, discontinuous right-hand side, long period.

2020 Mathematics Subject Classification: 34K13, 34K39, 34K06.

1 Introduction

The Mackey–Glass equation

$$y'(t) = -ay(t) + b\frac{y(t-\tau)}{1 + y^n(t-\tau)}$$

with positive parameters a, b, τ, n was proposed to model blood production and destruction in the study of oscillation and chaos in physiological control systems by Mackey and Glass [13]. This simple-looking differential equation with a single delay attracted the attention of many mathematicians since its hump-shaped nonlinearity causes entirely different dynamics compared to the case where the nonlinearity is monotone. See [16] for a similar equation. There exist several rigorous mathematical results, numerical and experimental studies on the Mackey–Glass equation showing convergence of the solutions, oscillations with different

✉ Email: krisztin@math.u-szeged.hu

characteristics, and the complexity of the dynamics, see e.g. [1,3,6,7,9,15,17–19,22,23]. Despite the intense research, the dynamics is not fully understood yet.

The recent paper [2] studies the classical Mackey–Glass delay differential equation

$$y'(t) = -ay(t) + bf_n(y(t-1)) \tag{E_n}$$

where a, b, n are positive reals, $f_n(\xi) = \xi/[1+\xi^n]$ for $\xi \geq 0$, $\tau = 1$ can be assumed by rescaling the time. [2] constructs stable periodic solutions of (E_n) for some $b > a > 0$ and large n. The periodic solutions can have complicated shapes, see [2]. A limiting version of (E_n) plays a key role in the construction. The function $f(\xi) = \lim_{n\to\infty} f_n(\xi)$ is given by $f(\xi) = \xi$ for $\xi \in [0,1)$, $f(1) = 1/2$, and $f(\xi) = 0$ for $\xi > 1$. The equation

$$x'(t) = -ax(t) + bf(x(t-1)) \tag{E_∞}$$

is called the limiting Mackey–Glass equation.

Let $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{N}$ denote the set of real numbers, complex numbers and positive integers, respectively. Let C be the Banach space $C([-1,0],\mathbb{R})$ equipped with the norm $\|\varphi\| = \max_{s\in[-1,0]} |\varphi(s)|$. For a continuous function $u : I \to \mathbb{R}$ defined on an interval I, and for $t, t-1 \in I$, $u_t \in C$ is given by $u_t(s) = u(t+s)$, $s \in [-1,0]$. Introduce the subsets

$$C^+ = \{\psi \in C : \psi(s) > 0 \text{ for all } s \in [-1,0]\},$$
$$C_r^+ = \left\{\psi \in C^+ : \psi^{-1}(c) \text{ is finite for all } c \in (0,1]\right\}$$

of C where $\psi^{-1}(c) = \{s \in [-1,0] : \psi(s) = c\}$. C^+ and C_r^+ are metric spaces with the metric $d(\varphi,\psi) = \|\varphi - \psi\|$.

A solution of equation (E_n) on $[-1,\infty)$ with initial function $\psi \in C^+$ is a continuous function $y : [-1,\infty) \to \mathbb{R}$ so that $y_0 = \psi$, the restriction $y|_{(0,\infty)}$ is differentiable, and equation (E_n) holds for all $t > 0$. The solutions are easily obtained from the variation-of-constants formula for ordinary differential equations on successive intervals of length one,

$$y(t) = e^{-a(t-k)}y(k) + b\int_k^t e^{-a(t-s)} f_n(y(s-1))\,ds \tag{1.1}$$

where $k \in \mathbb{N} \cup \{0\}$, $k \leq t \leq k+1$. Hence it is well known that each $\psi \in C^+$ uniquely determines a solution $y = y^{n,\psi} : [-1,\infty) \to \mathbb{R}$ with $y_0^{n,\psi} = \psi$, and $y^{n,\psi}(t) > 0$ for all $t \geq 0$.

For equation (E_∞) with the discontinuous f, we use formula (1.1) with f instead of f_n to define solutions. A solution of equation (E_∞) with initial function $\varphi \in C^+$ is a continuous function $x = x^\varphi : [-1, t_\varphi) \to \mathbb{R}$ with some $0 < t_\varphi \leq \infty$ such that $x_0 = \varphi$, the map $[0, t_\varphi) \ni s \mapsto f(x(s-1)) \in \mathbb{R}$ is locally integrable, and

$$x(t) = e^{-a(t-k)}x(k) + b\int_k^t e^{-a(t-s)} f(x(s-1))\,ds \tag{1.2}$$

holds for all $k \in \mathbb{N} \cup \{0\}$ and $t \in [0, t_\varphi)$ with $k \leq t \leq k+1$.

It is easy to show that, for any $\varphi \in C^+$, there is a unique solution x^φ of equation (E_∞) on $[-1,\infty)$. However, comparing solutions with initial functions $\varphi > 1$, $\varphi \equiv 1$, one sees that there is no continuous dependence on initial data in C^+. Therefore we restrict our attention to the subset C_r^+ of C^+. The choice of C_r^+ as a phase space guarantees not only continuous dependence on initial data, but also allows to compare certain solutions of equations (E_∞) and (E_n) for large n. This is not used here, but it is important in [2]. [2] proves that for each $\varphi \in C_r^+$

there is a unique maximal solution $x^{\varphi} : [-1, \infty) \to \mathbb{R}$ of equation (E_∞). The maximal solution x^{φ} satisfies $x_t^{\varphi} \in C_r^+$ for all $t \geq 0$; and if $t > 0$ and $x^{\varphi}(t-1) \neq 1$, then x^{φ} is differentiable at t, and equation (E_∞) holds at t.

One of the main results of [2] is as follows.

Theorem 1.1. *If the parameters $b > a > 0$ are given so that*

(H) equation (E_∞) has an ω-periodic solution $p : \mathbb{R} \to \mathbb{R}$ with the following properties:

(i) $p(0) = 1$, $p(t) > 1$ for all $t \in [-1, 0)$,

(ii) $(p(t), p(t-1)) \neq (1, a/b)$ for all $t \in [0, \omega]$

holds then there exists an $n_* \geq 4$ such that, for all $n \geq n_*$, equation (E_n) has a periodic solution $p^n : \mathbb{R} \to \mathbb{R}$ with period $\omega^n > 0$ so that the periodic orbits

$$\mathcal{O}^n = \{p_t^n : t \in [0, \omega^n]\}$$

are hyperbolic, orbitally stable, exponentially attractive with asymptotic phase, moreover, $\omega^n \to \omega$, $\operatorname{dist}\{\mathcal{O}^n, \mathcal{O}\} \to 0$ *as $n \to \infty$, where $\mathcal{O} = \{p_t : t \in [0, \omega]\}$.*

[2] shows that in case b is large comparing to a, namely $b > \max\{ae^a, e^a - e^{-a}\}$, then (H) is satisfied. In addition, by using a rigorous computer-assisted technique, [2] gives parameter values a, b such that (H) is valid, and the obtained stable periodic orbits for the Mackey–Glass equation may have complicated structures.

[2] remarks that (H) holds if $b > a > 0$ and b is sufficiently close to a, and refers to this work for the proof. The aim of this paper is to prove this fact, namely the following result.

Theorem 1.2. *For every $a > 0$ there exists an $\varepsilon_0 = \varepsilon_0(a) > 0$ such that for the parameters a, b with $b \in (a, a + \varepsilon_0)$ condition (H) holds.*

In particular, for the periodic solution $p = p(a, b)$ of equation (E_∞) the minimal period $\omega = \omega(a, b)$ satisfies $\omega > 5$, and there exists a $\sigma = \sigma(a, b) \in (4, \omega - 1)$ so that

$$0 < p(t) < 1 \text{ for all } t \in (0, \sigma); \quad p(t) > 1 \text{ for all } t \in (\sigma, \omega).$$

Moreover, if $a > 0$ is fixed and $(b_k)_{k=1}^{\infty}$ is a sequence in $(a, a + \varepsilon_0(a))$, $\lim_{k\to\infty} b_k = a$ then $\sigma(a, b_k) \to \infty$, $\omega(a, b_k) \to \infty$ as $k \to \infty$.

Theorems 1.1 and 1.2 immediately imply the following result for equation (E_n).

Theorem 1.3. *For each $a > 0$ there exists an $\varepsilon_0 = \varepsilon_0(a) > 0$ such that for every $b \in (a, a + \varepsilon_0)$ there exists an $n^* = n^*(a, b) \geq 4$ so that, for all $n \geq n^*$, equation (E_n) has a periodic solution $p^n : \mathbb{R} \to \mathbb{R}$ with minimal period $\omega^n(a, b)$ so that the periodic orbits*

$$\mathcal{O}^n = \{p_t^n : t \in [0, \omega^n]\}$$

are hyperbolic, orbitally stable, exponentially attractive with asymptotic phase. Moreover, if $(b_k)_{k=1}^{\infty}$ is a sequence in $(a, a + \varepsilon_0(a))$ with $\lim_{k\to\infty} b_k = a$, $n_k > n^*(a, b_k)$ then $\omega^n(a, b_k) \to \infty$ as $k \to \infty$.

Note that the papers [8] by Karakostas et al. and [5] by Gopalsamy et al. give conditions for the global attractivity of the unique positive equilibrium of (E_n) for $b > a > 0$, and n is below a certain constant given in terms of a, b. Theorem 1.3 requires n to be large.

Section 2 contains the proof of Theorem 1.2. The proof requires the study of a special solution of a linear autonomous delay differential equation. If $\varphi \in C_r^+$ is any function such that $\varphi(s) > 1$ for $s \in [-1,0)$ and $\varphi(0) = 1$ then the unique solution $x = x^\varphi$ of equation (E_∞) satisfies $x(t) = e^{-at}$ for $t \in [0,1]$. In order to find a periodic solution of (E_∞) as stated in Theorem 1.2 we consider the linear autonomous equation

$$u'(t) = -au(t) + bu(t-1)$$

for $t > 1$ with $u(t) = e^{-at}$, $t \in [0,1]$. If we find a $T > 0$ such that $u(t) < 1$ for $t \in (0,T)$, $u(T) = 1$, $u(t) > 1$ for $t \in (T, T+1]$, then it is straightforward to see that $x(t) = u(t)$ for all $t \in [0, T+1]$. Then, equation (E_∞) gives $x'(t) = -ax(t)$ for all $t > T+1$ as long as $x(t-1) > 1$. Hence there exists an $\omega > T+1$ with $x(\omega) = 1$ and $x(t) > 1$ for all $t \in (T,\omega)$. By the fact $f(\xi) = 0$ for $\xi > 1$, the solution x does not change on $[0,\infty)$ if φ is replaced by x_ω, and consequently $x(t) = x(t+\omega)$ follows for all $t \geq -1$. Therefore the proof of Theorem 1.2 is reduced to the existence of a $T > 0$ with $u(t) < 1$ for $t \in (0,T)$, $u(T) = 1$, $u(t) > 1$ for $t \in (T, T+1]$. Property (H)(ii) is guaranteed by $u'(T) > 0$.

We remark that the use of a limiting equation in order to study nonlinear delay differential equations when the nonlinearity is close to its limiting function is not new. We refer to the papers [10–12, 21, 24–26] where the limiting step function reduces the search of periodic solutions to a finite dimensional problem. The limiting Mackey–Glass nonlinearity f is not a step function. The introduction of the limiting Mackey–Glass equation does not reduce the search for periodic solutions to a finite dimensional problem, nevertheless it can simplify it. The paper [14] considered the limiting Mackey–Glass nonlinearity to construct periodic solutions for an equation different from (E_n). The result of [14] is analogous to the case when b is large comparing to a, mentioned above for the Mackey–Glass equation.

2 The proof of Theorem 1.2

The proof is divided into eight steps. The desired periodic solution of equation (E_∞) will be an ω-periodic extension of a function $w : [0,\omega] \to \mathbb{R}$. We construct w in the remaining part of this section.

Step 1. Let $a > 0$ be fixed, and consider the characteristic function

$$h : \mathbb{C} \times \mathbb{R} \ni (z,\varepsilon) \mapsto z + a - (a+\varepsilon)e^{-z} \in \mathbb{C}$$

of the linear delay differential equation $v'(t) = -av(t) + (a+\varepsilon)v(t-1)$. By $h(0,0) = 0$, $D_1h(0,0) = 1 + a$, and $D_2h(0,0) = -1$, the Implicit Function Theorem can be applied to get that there are $\varepsilon_1 \in (0, \min\{a, 1/4\})$, $r_1 \in (0,1)$ and a C^1-smooth map $\lambda_0 : (-\varepsilon_1, \varepsilon_1) \to \mathbb{C}$ such that $\lambda_0(0) = 0$, $h(\lambda_0(\varepsilon),\varepsilon) = 0$, and $(\lambda_0(\varepsilon),\varepsilon)$ is the unique solution of $h(z,\varepsilon) = 0$ in the set $\{z \in \mathbb{C} : |z| < r_1\} \times (-\varepsilon_1,\varepsilon_1)$. Since a and ε are real in the equation $h(z,\varepsilon) = 0$, (z,ε) is a solution together with $(\bar{z},\varepsilon)$. Then, by uniqueness, it follows that $\lambda_0(\varepsilon) \in \mathbb{R}$, $\varepsilon \in (-\varepsilon_1,\varepsilon_1)$.

Chapter XI of [4] applies to get that the zeros of the characteristic function $h(z,\varepsilon)$ for $\varepsilon \in (-\varepsilon_1,\varepsilon_1)$ are $\lambda_0(\varepsilon) \in \mathbb{R}$ and a sequence of pairs $\big(\lambda_j(\varepsilon), \overline{\lambda_j(\varepsilon)}\big)_{j=1}^\infty$ with

$$\lambda_0(\varepsilon) > \operatorname{Re}\lambda_1(\varepsilon) > \operatorname{Re}\lambda_2(\varepsilon) > \cdots > \operatorname{Re}\lambda_j(\varepsilon) \to -\infty \quad \text{as} \quad j \to \infty$$

and

$$\operatorname{Im}\lambda_j \in \big((2j-1)\pi, 2j\pi\big) \quad (j \in \mathbb{N}).$$

If $\varepsilon = 0$ then $\lambda_0(0) = 0$, and consequently $\operatorname{Re}\lambda_1(0) < 0$. Fix $c \in (0,a)$ so that

$$\operatorname{Re}\lambda_1(0) < -2c.$$

Notice that the choice of c depends only on a.

Differentiating the equation $h(\lambda_0(\varepsilon), \varepsilon) = 0$ with respect to ε we obtain $\lambda_0'(0) = 1/(1+a)$, and thus

$$\lambda_0(\varepsilon) = \frac{\varepsilon}{1+a} + \eta(\varepsilon)$$

with a function $\eta : (-\varepsilon_1, \varepsilon_1) \to \mathbb{R}$ satisfying $\lim_{\varepsilon \to 0} \eta(\varepsilon)/\varepsilon = 0$. Applying the above representation for $\lambda_0(\varepsilon)$, we assume (in addition to the above properties of ε_1) that ε_1 is so small that

$$\lambda_0(\varepsilon) < \frac{2\varepsilon}{1+2a} \quad \text{for all } \varepsilon \in (0, \varepsilon_1), \tag{2.1}$$

where the equality $2\varepsilon/(1+2a) = \varepsilon/(1+a) + \varepsilon/[(1+a)(1+2a)]$ shows that this is possible.

By Rouché's theorem [20] there exists an $\varepsilon_2 \in (0, \varepsilon_1)$ such that

$$\operatorname{Re} \lambda_1(\varepsilon) < -2c \quad \text{for all } \varepsilon \in [0, \varepsilon_2].$$

In particular, $h(z, \varepsilon) \neq 0$ on the line $\{-c + is : s \in \mathbb{R}\}$ for all $\varepsilon \in [0, \varepsilon_2]$.

Step 2. For $\varepsilon \in (0, \varepsilon_2)$ consider the unique solution $v : [-1, \infty) \to \mathbb{R}$ of the linear equation

$$v'(t) = -av(t) + (a+\varepsilon)v(t-1) \qquad (t > 0) \tag{2.2}$$

with initial function $v_0(s) = e^{-a(s+1)}$, $-1 \leq s \leq 0$. Remark that v and λ_0 depend on ε as well. Taking the Laplace transform of both sides of (2.2) and expressing the Laplace transform $\mathcal{L}(v)(z)$ of v,

$$\mathcal{L}(v)(z) = \frac{1}{h(z,\varepsilon)} \left[e^{-a} + (a+\varepsilon) \frac{1 - e^{-(z+a)}}{z+a} \right]$$

is obtained where the right hand side can be written as $F(z, \varepsilon) = F_1(z) + F_2(z, \varepsilon)$ with

$$F_1(z) = \frac{e^{-a}}{z+a}, \qquad F_2(z,\varepsilon) = \frac{a+\varepsilon}{(z+a)h(z,\varepsilon)}.$$

According to Chapter I of [4], by taking the inverse Laplace transform, function v can be written as

$$v(t) = e^{\lambda_0 t} \operatorname{Res}_{\lambda_0} F(z,\varepsilon) + \frac{1}{2\pi} e^{-ct} \lim_{T\to\infty} \int_{-T}^{T} e^{ist} F(-c+is, \varepsilon)\, ds \quad (t > 0).$$

As $F_1(z)$ is holomorphic in a neighborhood of λ_0, one finds $\operatorname{Res}_{\lambda_0} F(z,\varepsilon) = \operatorname{Res}_{\lambda_0} F_2(z,\varepsilon)$. By using that $h(z,\varepsilon)$ has a simple zero at λ_0, and $\lambda_0 + a = (a+\varepsilon)e^{-\lambda_0}$, we get

$$\operatorname{Res}_{\lambda_0} F(z,\varepsilon) = \frac{a+\varepsilon}{(\lambda_0 + a)\, D_1 h(\lambda_0, \varepsilon)} = \frac{a+\varepsilon}{(\lambda_0+a)(1+(a+\varepsilon)e^{-\lambda_0})} = \frac{e^{\lambda_0}}{1+a+\lambda_0}.$$

For $t \geq 1$, integration by parts leads to

$$\int_{-T}^{T} e^{ist} F_1(-c+is)\, ds = \left[\frac{e^{ist}}{it} \frac{e^{-a}}{a-c+is} \right]_{s=-T}^{s=T} + \int_{-T}^{T} \frac{e^{ist}}{it} \frac{ie^{-a}}{(a-c+is)^2}\, ds.$$

Thus

$$\left| \lim_{T\to\infty} \int_{-T}^{T} e^{ist} F_1(-c+is)\, ds \right| \leq \int_{-\infty}^{\infty} \left| \frac{e^{ist}}{it} \frac{ie^{-a}}{(a-c+is)^2} \right| ds \leq K_1$$

with

$$K_1 = 2\int_0^\infty \frac{e^{-a}}{(a-c)^2+s^2}\,ds.$$

Let $s_0 = 2(a+1)e^c$. The continuous function $(s,\varepsilon) \mapsto h(-c+is,\varepsilon) \in \mathbb{C}$ is nonzero on the set $[-s_0,s_0]\times[0,\varepsilon_2]$. So there exists $k>0$ such that $|F_2(-c+is,\varepsilon)| \le k$ on the compact set $[-s_0,s_0]\times[0,\varepsilon_2]$. If $|s|\ge s_0$, $\varepsilon\in[0,\varepsilon_2]$ then, by the choice of s_0,

$$\begin{aligned}|h(-c+is,\varepsilon)| &\ge |a-c+is| - |(a+\varepsilon)e^{c-is}| \ge \left[(a-c)^2+s^2\right]^{1/2} - (a+1)e^c\\ &\ge \frac{1}{2}\left[(a-c)^2+s^2\right]^{1/2}.\end{aligned}$$

Consequently

$$\begin{aligned}\left|\lim_{T\to\infty}\int_{-T}^{T} e^{ist}F_2(-c+is,\varepsilon)\,ds\right| &\le \int_{-\infty}^{\infty}|F_2(-c+is,\varepsilon)|\,ds\\ &\le 2\int_0^{s_0} k\,ds + 2\int_{s_0}^\infty \frac{a+1}{(1/2)[(a-c)^2+s^2]}\,ds\\ &= K_2\end{aligned}$$

with

$$K_2 = 2ks_0 + 4\int_{s_0}^\infty \frac{(a+1)}{(a-c)^2+s^2}\,ds.$$

Notice that both K_1 and K_2 are independent of $\varepsilon\in(0,\varepsilon_2)$.

Summarizing the above estimations we obtain that

$$v(t) = \frac{e^{\lambda_0(t+1)}}{1+a+\lambda_0} + \hat{r}(t) \qquad (t\ge 1)$$

for some continuous function $\hat{r}:[1,\infty)\to\mathbb{R}$ satisfying

$$|\hat{r}(t)| \le \hat{K}e^{-ct} \qquad (t\ge 1)$$

with $\hat{K} = (K_1+K_2)/(2\pi)$. Note that $\hat{r}$ depends on ε, however $\hat{K}$ and c are independent of ε.

Step 3. For $\varepsilon\in(0,\varepsilon_2)$ define the function $u:[0,\infty)\to\mathbb{R}$ by $u(t) = v(t-1)$, $t\ge 0$. Then $u(t) = e^{-at}$ for $t\in[0,1]$, u is differentiable on $(1,\infty)$ and satisfies

$$u'(t) = -au(t) + (a+\varepsilon)u(t-1) \qquad (t>1). \tag{2.3}$$

Moreover, defining $r(t) = \hat{r}(t-1)$ for $t\ge 2$, $K = \hat{K}e^c$, u has the representation

$$u(t) = \frac{e^{\lambda_0 t}}{1+a+\lambda_0} + r(t) \qquad (t\ge 2) \tag{2.4}$$

with the continuous function $r:[2,\infty)\to\mathbb{R}$ satisfying

$$|r(t)| \le Ke^{-ct} \qquad (t\ge 2). \tag{2.5}$$

From equation (2.3)

$$\begin{aligned}u(t) &= e^{-a(t-1)}u(1) + \int_1^t (a+\varepsilon)e^{-a(t-s)}e^{-a(s-1)}\,ds\\ &= e^{-at}\left[1+(a+\varepsilon)e^a(t-1)\right] \qquad (t\in[1,2])\end{aligned}$$

and

$$u'(t) = e^{-at}\left[-a - a(a+\varepsilon)e^{a}(t-1) + (a+\varepsilon)e^{a}\right] \qquad (t \in (1,2]).$$

Define

$$t_0 = t_0(\varepsilon) = 1 + \frac{1}{a} - \frac{1}{(a+\varepsilon)e^{a}}.$$

Choose $\varepsilon_3 \in (0, \varepsilon_2]$ so that

$$\varepsilon_3 < \frac{a}{1-a}\left(e^{-a} - 1 + a\right)$$

provided $a \in (0,1)$, and let $\varepsilon_3 = \varepsilon_2$ if $a \geq 1$.

Suppose $\varepsilon \in (0, \varepsilon_3)$. Then $t_0 = t_0(\varepsilon) \in (1,2)$ is the unique zero of u' in $(1,2)$, and it is easy to see that

$$\max_{t\in[1,2]} u(t) = u(t_0) = e^{-at_0}\left[1 + (a+\varepsilon)e^{a}(t_0 - 1)\right] = \frac{a+\varepsilon}{a}\exp\left[\frac{ae^{-a}}{a+\varepsilon} - 1\right]. \tag{2.6}$$

Step 4. In this step we show the following

CLAIM:

(i) For each $k \in \mathbb{N}$

$$\max_{t\in[k+1,k+2]} u(t) \leq \left(1 + \frac{\varepsilon}{a}\right) \max_{t\in[k,k+1]} u(t),$$

and

(ii) for each $N \in \mathbb{N}$

$$\max_{t\in[N+1,N+2]} u(t) \leq \left(1 + \frac{\varepsilon}{a}\right)^{N} \max_{t\in[1,2]} u(t).$$

Let $k \in \mathbb{N}$ be given. If $\max_{t\in[k+1,k+2]} \leq \max_{t\in[k,k+1]} u(t)$ then the stated inequality obviously holds for k. If $\max_{t\in[k+1,k+2]} u(t) > \max_{t\in[k,k+1]} u(t)$, then there exists a $t_1 \in (k+1, k+2]$ such that $u'(t_1) \geq 0$ and $u(t_1) = \max_{t\in[k+1,k+2]} u(t)$. Equation (2.3) at $t = t_1$ and $u'(t_1) \geq 0$ imply the inequality $-au(t_1) + (a+\varepsilon)u(t_1 - 1) \geq 0$. Hence

$$\max_{t\in[k+1,k+2]} u(t) = u(t_1) \leq \frac{a+\varepsilon}{a} u(t_1 - 1) \leq \left(1 + \frac{\varepsilon}{a}\right) \max_{t\in[k,k+1]} u(t),$$

that is, the stated inequality is satisfied. This proves (i).

A repeated application of (i) gives (ii):

$$\begin{aligned}\max_{t\in[N+1,N+2]} u(t) &\leq \left(1 + \frac{\varepsilon}{a}\right) \max_{t\in[N,N+1]} u(t) \leq \left(1 + \frac{\varepsilon}{a}\right)^{2} \max_{t\in[N-1,N]} u(t) \\ &\leq \cdots \leq \left(1 + \frac{\varepsilon}{a}\right)^{N} \max_{t\in[1,2]} u(t).\end{aligned}$$

Step 5. Choose $\xi_0 \in (\exp(e^{-a} - 1), 1)$. The function

$$(0, \infty) \ni \varepsilon \mapsto \frac{a+\varepsilon}{a}\exp\left[\frac{ae^{-a}}{a+\varepsilon} - 1\right] \in \mathbb{R}$$

strictly increases and its limit is $\exp(e^{-a} - 1)$ as $\varepsilon \to 0+$. Therefore there exists an $\varepsilon_4 \in (0, \varepsilon_3)$ such that

$$\frac{a+\varepsilon}{a}\exp\left[\frac{ae^{-a}}{a+\varepsilon} - 1\right] < \xi_0$$

for all $\varepsilon \in (0, \varepsilon_4)$.

By the equality (2.6) in Step 3 and the choice of ε_4, for all $\varepsilon \in (0, \varepsilon_4)$, the inequality $\max_{t\in[1,2]} u(t) < \xi_0$ holds. Then by the CLAIM in Step 4

$$\max_{t\in[1,N+2]} u(t) < \left(1 + \frac{\varepsilon}{a}\right)^N \xi_0 \tag{2.7}$$

follows for all $N \in \mathbb{N}$.

For a given $N \in \mathbb{N}$, from (2.7) one gets

$$\max_{t\in[1,N+2]} u(t) < 1$$

provided $\varepsilon \in (0, \varepsilon_4)$ is so small that

$$\varepsilon < a\left[(1/\xi_0)^{1/N} - 1\right]. \tag{2.8}$$

Step 6. Let $N \in \mathbb{N} \setminus \{1, 2\}$ be given. We look for a condition on $\varepsilon \in (0, \varepsilon_4)$ to guarantee

$$u'(t) > 0 \quad \text{for all } t > N. \tag{2.9}$$

Equation (2.3) gives that

$$au(t) < (a + \varepsilon)u(t - 1) \quad \text{for all } t > N \tag{2.10}$$

is sufficient to yield (2.9). By the representation (2.4) condition (2.10) is equivalent to

$$\frac{a}{1 + a + \lambda_0} e^{\lambda_0 t}\left[\left(1 + \frac{\varepsilon}{a}\right) e^{-\lambda_0} - 1\right] > ar(t) - (a + \varepsilon)r(t - 1) \qquad (t > N),$$

that is

$$\left(1 + \frac{\varepsilon}{a}\right) e^{-\lambda_0} - 1 > \frac{1 + a + \lambda_0}{a} e^{-\lambda_0 t}\left[ar(t) - (a + \varepsilon)r(t - 1)\right] \qquad (t > N).$$

From $\varepsilon < 1$, $0 < \lambda_0(\varepsilon) < 1$ and (2.5) one obtains

$$\begin{aligned} &\frac{1 + a + \lambda_0}{a} e^{-\lambda_0 t}\left[ar(t) - (a + \varepsilon)r(t - 1)\right] \\ &\quad < \frac{(a + 2)(2a + 1)}{a} K e^{-c(t-1)} \\ &\quad < \frac{(a + 2)(2a + 1)}{a} K e^{c} e^{-cN} \qquad (t > N). \end{aligned}$$

Recall that, by the choice of ε_1 in Step 1,

$$\lambda_0(\varepsilon) < \frac{2\varepsilon}{2a + 1}.$$

Hence

$$e^{-\lambda_0(\varepsilon)} > 1 - \lambda_0(\varepsilon) > 1 - \frac{2\varepsilon}{2a + 1}.$$

Thus, by using $\varepsilon_1 < 1/4$ as well,

$$\begin{aligned} \left(1 + \frac{\varepsilon}{a}\right) e^{-\lambda_0(\varepsilon)} - 1 &> \left(1 + \frac{\varepsilon}{a}\right)\left(1 - \frac{2\varepsilon}{2a + 1}\right) - 1 \\ &= \frac{\varepsilon - 2\varepsilon^2}{a(2a + 1)} > \frac{\varepsilon}{2a(2a + 1)}. \end{aligned}$$

Consequently, (2.9) holds if, in addition to $\varepsilon \in (0, \varepsilon_4)$,

$$\varepsilon > \xi_1 e^{-cN} \tag{2.11}$$

with $\xi_1 = 2(a+2)(2a+1)^2 K e^c$.

Step 7. In order to satisfy conditions (2.8) and (2.11) simultaneously consider $a\left[(1/\xi_0)^{1/N} - 1\right]$ and $\xi_1 e^{-cN}$. By L'Hospital's rule

$$\lim_{N\to\infty} \frac{\xi_1 e^{-cN}}{a\left[(1/\xi_0)^{1/N} - 1\right]} = 0.$$

Therefore there exists an integer $N_0 > 2$ such that

$$\frac{\xi_1 e^{-cN}}{a\left[(1/\xi_0)^{1/(N+1)} - 1\right]} < 1 \quad \text{for all integers } N \geq N_0. \tag{2.12}$$

Define $\varepsilon_* \in (0, \varepsilon_4)$ so that

$$\varepsilon_* < a\left[(1/\xi_0)^{1/N_0} - 1\right].$$

Let $\varepsilon \in (0, \varepsilon_*)$ be fixed. By $\varepsilon < \varepsilon_*$ and $\lim_{N\to\infty} a\left[(1/\xi_0)^{1/N} - 1\right] = 0$ there exists a maximal integer $N(\varepsilon) \geq N_0$ so that

$$\varepsilon < a\left[(1/\xi_0)^{1/N(\varepsilon)} - 1\right]. \tag{2.13}$$

The maximality of $N(\varepsilon) \geq N_0$ and inequality (2.12) imply

$$\xi_1 e^{-cN(\varepsilon)} < a\left[(1/\xi_0)^{1/(N(\varepsilon)+1)} - 1\right] \leq \varepsilon.$$

Therefore, we arrive at the inequality

$$\xi_1 e^{-cN(\varepsilon)} < \varepsilon < a\left[(1/\xi_0)^{1/N(\varepsilon)} - 1\right], \tag{2.14}$$

that is, for every $\varepsilon \in (0, \varepsilon_*)$ inequalities (2.11) and (2.8) hold with $N = N(\varepsilon)$.

Step 8. By Steps 5–7, for each $\varepsilon \in (0, \varepsilon_*)$ there exists an integer $N = N(\varepsilon) > 2$ such that the unique continuous function $u = u(\varepsilon) : [0, \infty) \to \mathbb{R}$ satisfying $u(t) = e^{-at}$ for $t \in [0, 1]$, and equation (2.3) on $(1, \infty)$ has the properties

$$\begin{aligned} 1 = u(0) > u(t) > 0 &\quad \text{for all } t \in (0, N+2), \\ u'(t) > 0 &\quad \text{for all } t > N, \\ u(t) \to \infty &\quad \text{as } t \to \infty. \end{aligned} \tag{2.15}$$

The last property is clear from $\lambda_0(\varepsilon) > 0$, (2.4) and (2.5).

From (2.15) it follows that there exits a unique $\sigma(\varepsilon) > N(\varepsilon) + 2 > 4$ so that $u(\sigma(\varepsilon)) = 1$ and $u'(\sigma(\varepsilon)) > 0$. From $u'(\sigma(\varepsilon)) > 0$ it is clear that $u(\sigma(\varepsilon) - 1) \neq a/(a+\varepsilon)$. The maximality of $N(\varepsilon)$ in inequality (2.13) implies that $N(\varepsilon) \to \infty$, $\sigma(\varepsilon) \to \infty$ as $\varepsilon \to 0+$.

Let $\omega(\varepsilon) = \sigma(\varepsilon) + 1 + (1/a)\log u(\sigma(\varepsilon) + 1) > 5$. Define the function $w : [0, \omega(\varepsilon)] \to \mathbb{R}$ by

$$w(t) = \begin{cases} u(t) & \text{if } t \in [0, \sigma(\varepsilon) + 1], \\ u(\sigma(\varepsilon) + 1) e^{-a(t - \sigma(\varepsilon) - 1)} & \text{if } t \in [\sigma(\varepsilon) + 1, \omega(\varepsilon)]. \end{cases}$$

Then $w(t) > 1$ for all $t \in (\sigma, \omega)$, and $w(\omega) = 1$. Let $p : \mathbb{R} \to \mathbb{R}$ be the $\omega(\varepsilon)$-periodic extension of w to $\mathbb{R}$.

For the fixed $a > 0$ set $\varepsilon_0 = \varepsilon_*$. Observe that c, K, and consequently ξ_0, ξ_1, depend only on a. Then relation (2.12) shows that N_0 is also a function of a . Therefore, ε_0 depends only on a.

If $b \in (a, a + \varepsilon_0)$ then the above constructed $p(\varepsilon)$ with $\varepsilon = b - a \in (0, \varepsilon_*)$ is clearly an $\omega(\varepsilon)$-periodic solution of equation (E_∞) satisfying (H). Setting $\omega(a, b) = \omega(\varepsilon)$ and $\sigma(a, b) = \sigma(\varepsilon)$, we see that all statements of Theorem 1.2 are satisfied, and the proof is complete.

The typical shape of the periodic solutions obtained in this paper for (E_∞) is shown in Figure 2.1 with $a = 9$, $b = 9.7$.

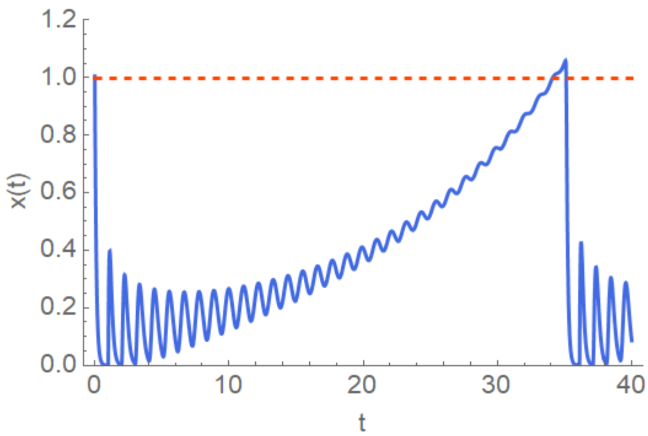

Figure 2.1: The periodic solution of (E_∞) for $a = 9$, $b = 9.7$

Acknowledgements

This research was supported by the grants NKFIH-K-129322, NKFIH-1279-2/2020 of the Ministry for Innovation and Technology, Hungary, and by the EU-funded Hungarian Grant EFOP-3.6.2-16-2017-0015.

The author thanks the reviewer for the relevant comments that contributed to improving the paper.

References

[1] P. Amil, C. Cabeza, A. C. Marti, Exact discrete-time implementation of the Mackey–Glass delayed model, *IEEE Transactions on Circuits and Systems II: Express Briefs* **62**(2015), 681–685. https://doi.org/10.1109/TCSII.2015.2415651

[2] F. Á. Bartha, T. Krisztin, A. Vígh, Stable periodic orbits for the Mackey–Glass equation, preprint.

[3] J. B. van den Berg, C. Groothedde, J.-P. Lessard, A general method for computer-assisted proofs of periodic solutions in delay differential problems, *J. Dynam. Differential Equations* (2020). https://doi.org/10.1007/s10884-020-09908-6

[4] O. Diekmann, S. A. van Gils, S. M. Verduyn Lunel, H.-O. Walther, *Delay equations. Functional, complex, and nonlinear analysis*, Applied Mathematical Sciences, Vol. 110, Springer-Verlag, New York, 1995. https://doi.org/10.1007/978-1-4612-4206-2; MR1345150

[5] K. Gopalsamy, S. I. Trofimchuk, N. R. Bantsur, A note on global attractivity in models of hematopoiesis, *Ukraïn. Mat. Zh.* **50**(1998), 5–12; reprinted in *Ukrainian Math. J.* **50**(1998), 3–12. https://doi.org/10.1007/BF02514684; MR1669243

[6] U. an der Heiden, H.-O. Walther, Existence of chaos in control systems with delayed feedback, *J. Differential Equations* **47**(1983), 273–295. https://doi.org/10.1016/0022-0396(83)90037-2; MR688106

[7] L. Junges, J. A. C. Gallas, Intricate routes to chaos in the Mackey–Glass delayed feedback system, *Phys. Lett. A* **376**(2012), 2109–2116. https://doi.org/10.1016/j.physleta.2012.05.022

[8] G. Karakostas, Ch. G. Philos, Y. G. Sficas, Stable steady state of some population models, *J. Dynam. Differential Equations* **4**(1992), 161–190. https://doi.org/10.1007/BF01048159; MR1150401

[9] G. Kiss, G. Röst, Controlling Mackey–Glass chaos, *Chaos* **27**(2017), 114321. https://doi.org/10.1063/1.5006922; MR3716183

[10] T. Krisztin, M. Polner, G. Vas, Periodic solutions and hydra effect for delay differential equations with nonincreasing feedback, *Qual. Theory Dyn. Syst.* **16**(2017), 269–292. https://doi.org/10.1007/s12346-016-0191-2; MR3671725

[11] T. Krisztin, G. Vas, Large-amplitude periodic solutions for differential equations with delayed monotone positive feedback, *J. Dynam. Differential Equations* **23**(2011), 727–790. https://doi.org/10.1007/s10884-011-9225-2; MR2859940

[12] T. Krisztin, G. Vas, The unstable set of a periodic orbit for delayed positive feedback, *J. Dynam. Differential Equations* **28**(2016), 805–855. https://doi.org/10.1007/s10884-014-9375-0; MR3537356

[13] M. C. Mackey, L. Glass, Oscillation and chaos in physiological control systems, *Science* **197**(1977), 278–289. https://doi.org/10.1126/science.267326

[14] M. C. Mackey, C. Ou, L. Pujo-Menjouet, J. Wu, Periodic oscillations of blood cell population in chronic myelogenous leukemia, *SIAM J. Math. Anal.* **38**(2006), 166–187. https://doi.org/10.1137/04061578X; MR2217313

[15] B. Lani-Wayda, Erratic solutions of simple delay equations, *Trans. Am. Math. Soc.* **351**(1999), 901–945. https://doi.org/10.1090/S0002-9947-99-02351-X; MR1615995

[16] A. Lasota, Ergodic problems in biology, *Astérisque* **50**(1977), 239–250. MR0490015

[17] E. Liz, G. Röst, Dichotomy results for delay differential equations with negative Schwarzian derivative, *Nonlinear Anal. Real World Appl.* **11**(2010), 1422–1430. https://doi.org/10.1016/j.nonrwa.2009.02.030; MR2646558

[18] E. Liz, E. Trofimchuk, S. Trofimchuk, Mackey–Glass type delay differential equations near the boundary of absolute stability, *J. Math. Anal. Appl.* **275**(2002), 747–760. https://doi.org/10.1016/S0022-247X(02)00416-X; MR1943777

[19] G. Röst, J. Wu, Domain-decomposition method for the global dynamics of delay differential equations with unimodal feedback, *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.* **463**(2007), 2655–2669. https://doi.org/10.1098/rspa.2007.1890; MR2352875

[20] W. Rudin, *Real and complex analysis*, McGraw-Hill Book Co., New York–Toronto, Ont.-London, 1966. MR0210528

[21] A. L. Skubachevskii, H.-O. Walther, On the Floquet multipliers of periodic solutions to non-linear functional differential equations, *J. Dynam. Differential Equations* **18**(2006), 257–355. https://doi.org/10.1007/s10884-006-9006-5; MR2229980

[22] R. Szczelina, A computer assisted proof of multiple periodic orbits in some first order non-linear delay differential equations, *Electron. J. Qual. Theory Differ. Equ.* **2016**, No. 83, 1–19. https://doi.org/10.14232/ejqtde.2016.1.83; MR3547459

[23] R. Szczelina, P. Zgliczynski, Algorithm for rigorous integration of delay differential equations and the computer-assisted proof of periodic orbits in the Mackey–Glass equation, *Found. Comput. Math.* **18**(2018), 1299–1332. https://doi.org/10.1007/s10208-017-9369-5; MR3875841

[24] G. Vas, Configurations of periodic orbits for equations with delayed positive feedback, *J. Differential Equations* **262**(2017), 1850–1896. https://doi.org/10.1016/j.jde.2016.10.031; MR3582215

[25] H.-O. Walther, Contracting return maps for some delay differential equations, in: T. Faria, P. Freitas (eds.), *Topics in functional differential and difference equations (Lisbon, 1999)*, Fields Inst. Commun., Vol. 29, Amer. Math. Soc., Providence, RI, 2001, pp. 349–360. MR1821790

[26] H.-O. Walther, Contracting return maps for monotone delayed feedback, *Discrete Contin. Dyn. Syst.* **7**(2001), 259–274. https://doi.org/10.3934/dcds.2001.7.259; MR1808399