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Long-term behavior of nonautonomous neutral compartmental systems

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Abstract. The asymptotic behavior of the trajectories of compartmental systems with a general set of admissible initial data is studied. More precisely, these systems are described by families of monotone nonautonomous neutral functional differential equations with nonautonomous operator. We show that the solutions asymptotically exhibit the same recurrence properties as the transport functions and the coefficients of the neutral operator. Conditions for the cases in which the delays in the neutral and non neutral parts are different, as well as for other cases unaddressed in the previous literature are also obtained.

Keywords: nonautonomous dynamical systems, monotone skew-product semiflows, neutral functional differential equations, infinite delay, compartmental systems.

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1 Introduction

In this work, the long-term behavior of the solutions of neutral compartmental systems is studied. Some interesting results as to the convergence of the solutions of such systems to their omega-limit sets are presented. These results allow for the inclusion of a significantly wider set of possible initial data than those in the previous literature.

Compartmental systems are widely used as successful models in different fields, ranging from physics and biology, to economics or sociology. These models appear naturally when dealing with processes involving a local balance of mass (see e.g. Haddad and Chellaboina [7], Jacquez [9], and Jacquez and Simon [10]).

Compartmental models with finite and infinite delay were initially studied by Győri [4], and Győri and Eller [5]. Later on, Arino and Bourad [1], and Arino and Haourigui [2] found almost periodic solutions for finite delay compartmental systems given by functional differential equations (FDEs for short) or neutral FDEs (NFDEs for short). Győri and Wu [6] also

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studied the dynamics of infinite delay neutral compartmental systems, where the neutral term represents the creation and destruction of material within the compartments. These systems were studied by Wu and Freedman [26], and Wu [25] as well.

Regarding the monotone theory for NFDEs, the fact that the positive cone defined by the exponential ordering introduced by Smith and Thieme [22] has an empty interior, poses a special difficulty. This was overcome by Krisztin and Wu [13] in their paper on scalar NFDEs with finite delay, where they show that the solutions with Lipschitz continuous initial data are asymptotically periodic. More recently, the theory developed by Novo, Obaya, and Sanz [16] gave rise to a series of papers devoted to the study of compartmental systems defined by means of NFDEs with infinite delay, such as Novo, Obaya, and Villarragut [15, 17], and Obaya and Villarragut [19, 20]. In these papers, the neutral part consists of a nonautonomous operator, initial data are assumed to be Lipschitz continuous, and the usual ordering, the exponential ordering, and a new ad hoc exponential ordering defined by means of the neutral operator are considered. Also, these papers generalize the results included in the previous literature.

Finally, a generalization of the results on the asymptotic behavior of NFDEs was given in Novo, Obaya, and Villarragut [18]. In that paper, initial data are only required to have a uniformly bounded variation on compact subintervals of $(-\infty, 0]$, which is a weaker assumption than Lipschitz continuity. The present paper studies compartmental systems defined by NFDEs with infinite delay by applying the results in [18].

Specifically, the family of systems

$$\begin{aligned}\frac{d}{dt}\left[z_i(t) - \int_{-\infty}^{0} z_i(t+s)\, d\nu_i(\omega{\cdot}t)(s)\right]\\ = -\sum_{j=0}^{m} g_{ji}(\omega{\cdot}t, z_i(t)) + \sum_{j=1}^{m} \int_{-\infty}^{0} g_{ij}(\omega{\cdot}(t+s), z_j(t+s))\, d\mu_{ij}(s) + I_i(\omega{\cdot}t)\,,\end{aligned}$$

$i = 1, \dots, m$, where $\mathbb{R} \times \Omega \to \Omega$, $(t, \omega) \mapsto \omega{\cdot}t$ is a minimal flow, $\nu_i(\omega)$ and $\mu_{ij}(\omega)$ are regular Borel measures, and $g_{ij} : \Omega \times \mathbb{R} \to \mathbb{R}$, $I_i : \Omega \to \mathbb{R}$ are real functions, is considered. For each $\omega \in \Omega$ and each $i, j \in \{1, \dots, m\}$, the measures $\nu_i(\omega)$ represent the creation and destruction of material within each compartment, the functions g_{0i} and I_i represent the flow of material toward and from the environment, respectively, and the functions g_{ij} are the so-called transport functions, modeling the flow of material among the compartments, which is not instantaneous and is regulated by the measures $\mu_{ij}(\omega)$. Some particular cases of these systems were studied by Krisztin (see e.g. [11, 12]) under weaker conditions on the transport functions.

A more specific system of equations is also considered in this work:

$$\frac{d}{dt}[z_i(t) + \widetilde{c}_i(t)\, z_i(t-\alpha_i)] = -\sum_{j=1}^{m} \widetilde{g}_{ji}(t, z_i(t)) + \sum_{j=1}^{m} \widetilde{g}_{ij}(t-\rho_{ij}, z_j(t-\rho_{ij}))\,,$$

$i = 1, \dots, m$. This system can be included in a family of systems like the previous one by means of a hull construction. Notice that this system has finite delay and it is closed, in that there is no flow of material either toward or from the environment. Specifically, for each $\omega \in \Omega$ and each $i, j \in \{1, \dots, m\}$, $\nu_i(\omega)$ and $\mu_{ij}(\omega)$ are Dirac measures, and $g_{0i} = I_i \equiv 0$. Besides, a particular example of this system is also considered, and both theoretical and numerical results related to this example are presented.

The structure of the paper is as follows. In Section 2, some preliminaries regarding the usual concepts of topological dynamics are recalled. Section 3 addresses the study of the solutions of a family of compartmental systems with infinite delay with general initial data. An application of the results of Section 3 to the study of compartmental systems with finite delay is included in Section 4. Finally, Section 5 includes the application of the results in Section 3 and Section 4 to a particular compartmental system, together with a numerical simulation of the solutions of that system and their omega-limit sets.

2 Some preliminaries

Given a compact metric space Ω, a *flow* is a continuous mapping $\sigma : \mathbb{R} \times \Omega \to \Omega,\ (t, \omega) \mapsto \sigma_t(\omega)$ which satisfies the following conditions:

(i) $\sigma_0(\omega) = \omega$ for each $\omega \in \Omega$, and

(ii) $\sigma_{t+s} = \sigma_t \circ \sigma_s$ for each $s, t \in \mathbb{R}$.

It is customary to denote $\omega{\cdot}t = \sigma_t(\omega)$ for all $(t, \omega) \in \mathbb{R} \times \Omega$. Given $\omega \in \Omega$, its *orbit* or *trajectory* is the set $\{\sigma_t(\omega) \mid t \in \mathbb{R}\}$. A subset $A \subset \Omega$ is said to be *invariant* if $\sigma_t(A) = A$ for all $t \in \mathbb{R}$, and it is said to be *minimal* if it is compact, invariant, and it contains no proper subsets with those properties apart from the empty set. Equivalently, a subset of Ω is minimal if and only if all the trajectories are dense. Zorn's lemma guarantees that a compact and invariant subset of Ω always contains a minimal subset. If Ω is minimal, it is said that the flow σ is *minimal* or *recurrent*. For example, almost periodic and almost automorphic flows are minimal (see Ellis [3], and Shen and Yi [21] for a thorough description of almost periodic and almost automorphic flows from a topological and ergodic perspective).

Let $\mathbb{R}^+$ be the set of non-negative real numbers. Given a complete metric space X, a *semiflow* is a continuous map $\Phi : \mathbb{R}^+ \times X \to X,\ (t, x) \mapsto \Phi_t(x)$ satisfying

(i) $\Phi_0(x) = x$ for all $x \in X$, and

(ii) $\Phi_{t+s} = \Phi_t \circ \Phi_s$ for all $t, s \in \mathbb{R}^+$.

Given $x \in X$, its *semiorbit* is the set $\{\Phi_t(x) \mid t \geq 0\}$. A subset $A \subset X$ is said to be *positively invariant* if $\Phi_t(A) \subset A$ for all $t \geq 0$. Given $x \in X$ with a relatively compact semiorbit, we define its *omega-limit set*, defined by

$$\mathcal{O}(x) = \bigcap_{s \geq 0} \operatorname{closure}\{\Phi_{t+s}(x) \mid t \geq 0\}\,.$$

It is easy to check that $\mathcal{O}(x)$ is nonempty, compact, connected, and positively invariant. A useful characterization of its elements is as follows: $y \in \mathcal{O}(x)$ if and only if there exists a sequence $t_n \uparrow \infty$ such that $\Phi_{t_n}(x)$ converges to y as $n \uparrow \infty$. A subset $A \subset X$ is said to be *minimal* if it is compact, positively invariant, and it contains no proper subsets with those properties apart from the empty set. If X is minimal, it is said that the semiflow Φ is *minimal*.

We are interested in a particular class of semiflows. Specifically, if Ω is a compact metric space and X is a complete metric space, a *skew-product semiflow* is a semiflow defined on $\Omega \times X$ of the form

$$\begin{aligned} \tau\colon \mathbb{R}^+ \times \Omega \times X &\longrightarrow \Omega \times X \\ (t, \omega, x) &\longmapsto (\omega{\cdot}t, u(t, \omega, x))\,, \end{aligned}$$

where, as before, $\omega{\cdot}t = \sigma_t(\omega)$ for all $(t, \omega) \in \mathbb{R} \times \Omega$, and σ is a flow on Ω referred to as the *base flow*.

3 Transformed exponential ordering for neutral compartmental systems

In this section, we focus on compartmental models. They are primarily used to describe the transport of some material among compartments joined by pipes, which takes some non-negligible time, together with the creation and destruction of material within the compartments.

To this end, we consider the set $X = C((-\infty,0],\mathbb{R}^m)$ endowed with the compact-open topology, which turns it into a Fréchet space. The space X is metrizable; it suffices to consider the metric

$$\mathsf{d}(x,y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\|x-y\|_n}{1+\|x-y\|_n}\,, \quad x,y \in X\,,$$

where $\|x\|_n = \sup_{s\in[-n,0]} \|x(s)\|$, and $\|\cdot\|$ is the maximum norm on $\mathbb{R}^m$. We consider the following phase space:

$$BC = \{x \in X \mid x \text{ is bounded}\}\,,$$

together with the supremum norm $\|\cdot\|_\infty$. The space $(BC,\|\cdot\|_\infty)$ is a Banach space. Given $r > 0$, let B_r denote the set $\{x \in BC \mid \|x\|_\infty \leq r\}$. Besides, we consider the following subspace of BC:

$$BU = \{x \in BC \mid x \text{ is uniformly continuous}\}\,.$$

Given a, $t \in \mathbb{R}$ with $t \leq a$ and a continuous function $x : (-\infty,a] \to \mathbb{R}^m$, we consider $x_t : (-\infty,0] \to \mathbb{R}^m$, $s \mapsto x(t+s)$. Notice that $x_t \in X$. Finally, we also fix a compact metric space (Ω,d), together with a flow $\sigma : \mathbb{R}\times\Omega \to \Omega$, $(t,\omega) \mapsto \omega{\cdot}t$.

A compartmental system is a device formed by m compartments $C_1,\ldots,C_m$, connected to one another by means of pipes, and the environment. For each $i \in \{1,\ldots,m\}$, let $z_i(t)$ denote the amount of material in compartment C_i at time t. There is a flow of material among compartments. We assume that the material leaves the compartments instantaneously. Then, for each i, $j \in \{1,\ldots,m\}$, the material flows from C_j to C_i according to a transit time distribution regulated by a positive regular Borel measure μ_{ij}. The volume of material being transported is given by the *transport functions* g_{ij}, which depend on time and $z_j(t)$. Besides, there is a bidirectional flow of material from and to the environment; namely, for each $i \in \{1,\ldots,m\}$, some material enters compartment C_i from the environment instantaneously, according to a function I_i depending only on time, and some material leaves compartment C_i for the environment instantaneously, according to a transport function g_{0i}. Finally, for each $i \in \{1,\ldots,m\}$, the rate of creation and destruction of material within compartment C_i is regulated by the past amount of material in that compartment together with some regular Borel measures $\nu_i(\omega)$, $\omega \in \Omega$.

The amount of material in compartment C_i, $1 \leq i \leq m$, varies according to the difference between the incoming material to C_i and the outgoing material from C_i. This way, our model is given by the following family of NFDEs:

$$\begin{aligned}\frac{d}{dt}&\left[z_i(t) - \int_{-\infty}^{0} z_i(t+s)\,d\nu_i(\omega{\cdot}t)(s)\right]\\ &= -\sum_{j=0}^{m} g_{ji}(\omega{\cdot}t, z_i(t)) + \sum_{j=1}^{m}\int_{-\infty}^{0} g_{ij}(\omega{\cdot}(t+s), z_j(t+s))\,d\mu_{ij}(s) + I_i(\omega{\cdot}t)\,, \end{aligned} \tag{3.1}$$

$\omega \in \Omega$, $i = 1,\ldots,m$, where $g_{0i}\colon \Omega\times\mathbb{R} \to \mathbb{R}^+$, $g_{ij}\colon \Omega\times\mathbb{R}\to\mathbb{R}$, $I_i\colon \Omega\to\mathbb{R}$ and $\nu_i(\omega)$ are and μ_{ij} are regular Borel measures on $(-\infty,0]$, $i,j = 1,\ldots,m$, $\omega\in\Omega$.

Let $G\colon \Omega \times BC \to \mathbb{R}^m$ be the map defined by

$$G_i(\omega, x) = -\sum_{j=0}^{m} g_{ji}(\omega, x_i(0)) + \sum_{j=1}^{m} \int_{-\infty}^{0} g_{ij}(\omega{\cdot}s, x_j(s))\, d\mu_{ij}(s) + I_i(\omega)\,, \tag{3.2}$$

$(\omega, x) \in \Omega \times BC$, $i = 1, \dots, m$, and let $D\colon \Omega \times BC \to \mathbb{R}^m$ be the map defined by

$$D_i(\omega, x) = x_i(0) - \int_{-\infty}^{0} x_i(s)\, d\nu_i(\omega)(s)\,, \quad (\omega, x) \in \Omega \times BC\,, \quad i = 1, \dots, m\,.$$

We can now rewrite the family of equations (3.1) as

$$\frac{d}{dt} D(\omega{\cdot}t, z_t) = G(\omega{\cdot}t, z_t)\,, \quad t \geq 0\,,\ \omega \in \Omega\,. \tag{3.3}$$

It is obvious that the sign of the measures $\nu_i(\omega)$ determines different internal mechanisms in the compartment C_i with respect to z_i to produce or swallow material. Since the case in which the measures $\nu_i(\omega)$ are positive and the phase space is BU, the space of uniformly continuous functions in BC, was studied in [15], [17], [19] and [20], we will focus now in the case in which they are regular Borel negative measures. Let us make some assumptions on the family of equations (3.1):

(C1) I_i and g_{ij} are continuous, g_{ij} is nondecreasing in its second variable, $g_{ij}(\cdot, 0) \equiv 0$, and $\Omega \times \mathbb{R} \to \mathbb{R}$, $(\omega, v) \mapsto \frac{\partial g_{ij}}{\partial v}(\omega, v)$ is well-defined and continuous for $i, j \in \{1, \dots, m\}$;

(C2) μ_{ij} is a positive regular Borel measure such that $\mu_{ij}((-\infty, 0]) = 1$ and $\int_{-\infty}^{0} |s|\, d\mu_{ij}(s) < \infty$ for $i, j = 1, \dots m$;

(C3) for each $\omega \in \Omega$ and $i = 1, \dots, m$,

- $\nu_i(\omega)$ is a negative regular Borel measure with $\nu_i(\omega)(\{0\}) = 0$,
- $\sup_{i=1,\dots,m} |\nu_i(\omega)|((-\infty, 0]) < 1$, where $|\nu_i(\omega)|$ denotes the total variation of $\nu_i(\omega)$, and
- $\nu_i\colon \Omega \to \mathcal{M}$, $\omega \mapsto \nu_i(\omega)$ is continuous when the total variation is considered as a norm on the set $\mathcal{M}$ of Borel regular measures on $(-\infty, 0]$;

(C4) for each $i = 1, \dots, m$, there is a negative $a_i < 0$ such that

(i) $-L_i^+(\omega) - a_i > 0$ and

(ii) $1 + \int_{-\infty}^{0} e^{a_i s}\, d\nu_i(\omega)(s) \geq 0\,,$

for each $\omega \in \Omega$, where $L_i^+(\omega) = \sum_{j=0}^{m} \sup_{v \in \mathbb{R}} \frac{\partial g_{ij}}{\partial v}(\omega, v)$.

Let A be the diagonal matrix with the diagonal elements $a_1, \dots, a_m$ given in **(C4)**. Let us consider the partial order relation on BC:

$$x \leq_A y \iff x \leq y \ \text{ and } \ y(t) - x(t) \geq e^{A(t-s)}(y(s) - x(s))\,, -\infty < s \leq t \leq 0\,,$$

where $\leq$ denotes the componentwise partial ordering on $\mathbb{R}^m$. The interior of the positive cone

$$BC_A^+ = \{x \in BC \mid x \geq 0 \ \text{ and } \ x(t) \geq e^{A(t-s)} x(s) \quad \text{for } -\infty < s \leq t \leq 0\}\,,$$

is empty.

Before stating and proving the main theorem, we need the following result concerning D and its convolution operator

$$\begin{aligned} \widehat{D} : \Omega \times BC &\longrightarrow \Omega \times BC \\ (\omega, x) &\longmapsto (\omega, \widehat{D}_2(\omega, x))\,, \end{aligned}$$

where $\widehat{D}_2(\omega, x))$ is defined for each $(\omega, x) \in \Omega \times BC$ by

$$\begin{aligned} \widehat{D}_2(\omega, x) : (-\infty, 0] &\longrightarrow \mathbb{R}^m \\ s &\longmapsto D(\omega{\cdot}s, x_s)\,. \end{aligned}$$

Theorem 3.1. *Assume that conditions* (C3) *and* (C4)(ii) *hold. For each* $\omega \in \Omega$*, let* $\widehat{L}_\omega \colon BC \to BC$ *be the linear operator defined by*

$$(\widehat{L}_\omega(x))_i(s) = \int_{-\infty}^{0} x_i(s+u)\, d\nu_i(\omega{\cdot}s)(u)\,, \quad x \in BC\,, s \leq 0\,, i = 1, \ldots, m\,.$$

Then the following statements hold:

(i) $\widehat{D}$ *is bijective,*

$$(\widehat{D}^{-1})_2(\omega, x) = \sum_{n=0}^{\infty} \widehat{L}^n_\omega(x)\,, \quad (\omega, x) \in \Omega \times BC\,, \tag{3.4}$$

and $(\widehat{D}^{-1})_2(\omega, x) \geq 0$ *for each* $(\omega, x) \in \Omega \times BC^+_A$*;*

(ii) *for each* $r > 0$*, the map* $\Omega \times B_r \to BC$*,* $(\omega, x) \mapsto \widehat{L}_\omega(x)$ *is uniformly continuous for the compact-open topology on* BC*;*

(iii) *given* $(\omega, x) \in \Omega \times BC$ *with* $D(\omega, x) = 0$*, the solution of the difference equation*

$$\begin{cases} D(\omega{\cdot}t, z_t) = 0\,, & t \geq 0\,, \\ z_0 = x\,, \end{cases}$$

satisfies $\|z(t)\| \leq c(t)\, \|x\|_\infty$ *for all* $t \geq 0$*, where* $c \in C([0,\infty), \mathbb{R})$ *and* $\lim_{t\to\infty} c(t) = 0$*. In this situation,* D *is said to be* stable.

Proof. Clearly, $\widehat{D}_2(\omega, x) = (I - \widehat{L}_\omega)(x)$ for each $(\omega, x) \in \Omega \times BC$. From condition (C3), we deduce that $\sup_{\omega\in\Omega} \|\widehat{L}_\omega\| < 1$, whence $\widehat{D}$ is invertible and (3.4) holds. In addition, $x \geq_A 0$ implies that $x_i(s) \geq 0$ and $e^{a_i r} x_i(s) \geq x_i(s+r)$ for each r, $s \leq 0$, $i = 1, \ldots, m$, which, together with the fact that $\nu_i(\omega)$ is a negative measure for each $i = 1, \ldots, m$ and condition (C4)(ii), provides

$$\begin{aligned} x_i(s) + (\widehat{L}_\omega(x))_i(s) &= x_i(s) + \int_{-\infty}^{0} x_i(s+r)\, d\nu_i(\omega{\cdot}s)(r) \\ &\geq -\int_{-\infty}^{0} e^{a_i r}\, x_i(s)\, d\nu_i(\omega{\cdot}s)(r) + \int_{-\infty}^{0} x_i(s+r)\, d\nu_i(\omega{\cdot}s)(r) \geq 0\,. \end{aligned}$$

As a consequence, since $\widehat{L}^{2n}_\omega(x) \geq 0$ for each $x \geq 0$, we deduce that

$$(\widehat{D}^{-1})_2(\omega, x) = \sum_{n=0}^{\infty} \widehat{L}^{2n}_\omega \left(x + \widehat{L}_\omega(x)\right) \geq 0$$

for each $x \geq_A 0$, and the proof of (i) is finished. An adaptation of the arguments in Theorem 3.9(iii) and Theorem 5.2(iv)–(v) of [19] to the phase space BC yield (ii) and (iii). □

Let us define $M : \Omega \times BC \to \mathbb{R}$ as follows:

$$M(\omega, x) = \sum_{i=1}^{m} D_i(\omega, x) + \sum_{i=1}^{m} \sum_{j=1}^{m} \int_{-\infty}^{0} \left(\int_{s}^{0} g_{ji}(\omega{\cdot}u, x_i(u))\, du \right) d\mu_{ji}(s)\,, \tag{3.5}$$

for each $(\omega, x) \in \Omega \times BC$. M is said to be the *total mass* of the family of equations (3.1). Thanks to conditions (C1)–(C2), for all $x \in BC$ and all $i, j \in \{1, \dots, m\}$,

$$\left| \int_{s}^{0} g_{ji}(\omega{\cdot}u, x_i(u))\, du \right| \le c_{ji}\, |s|\,,$$

where $c_{ji} = \sup_{\Omega \times [-\|x\|_\infty, \|x\|_\infty]} g_{ji}$. Hence, M is well-defined.

A key property of the total mass is established by our next result. Similar results can be found in [15], [17], [19], and [26].

Proposition 3.2. *Under assumptions (C1)–(C3), for each $r > 0$, the total mass is uniformly continuous on $\Omega \times B_r$ when the compact-open topology is considered on B_r. In addition, for all $(\omega, x) \in \Omega \times BC$ and all $t \ge 0$ where the solution is defined,*

$$M(\tau(t, \omega, x)) = M(\omega, x) + \sum_{i=1}^{m} \int_{0}^{t} (I_i(\omega{\cdot}s) - g_{0i}(\omega{\cdot}s, z_i(s, \omega, x)))\, ds\,. \tag{3.6}$$

Proof. From (C3) and the definition of D, it is easy to deduce that D is linear and continuous in its second variable for the norm $\|\cdot\|_\infty$, the map $\Omega \to \mathcal{L}(BC, \mathbb{R}^m)$, $\omega \mapsto D(\omega, \cdot)$ is continuous, and the restriction of D to $\Omega \times B_r$ is continuous when we take the restriction of the compact-open topology to B_r for all $r > 0$.

Hence, a natural generalization of Theorem 3.9 in [19] together with properties (C1)–(C2) implies that $\widehat{D}$ is uniformly continuous on $\Omega \times B_r$. The proof of Proposition 5.5 of Muñoz-Villarragut [14] can be adapted to show the uniform continuity of M on $\Omega \times B_r$. Finally, a computation similar to the one given in [26] proves the variation formula (3.6). □

Let us recall a regularity condition introduced in [18] concerning a class of initial data $x \in BC$:

(R) for each $i \in \{1, \dots, n\}$, x_i is of bounded variation on every compact subinterval of $(-\infty, 0]$, and

$$\sup \left\{ V_{[-k,-k+1]}(x_i) \mid i \in \{1, \dots, m\},\, k \ge 1 \right\} < \infty\,,$$

where $V_{[-k,-k+1]}(x_i)$ denotes the total variation of x_i on the interval $[-k, -k+1]$.

Notice that property (R) is satisfied by all the Lipschitz continuous elements of BC, but not all the elements of BC satisfying (R) are Lipschitz continuous. Besides, the subspace $\mathcal{R}$ of BC determined by (R) is a Banach space for the norm

$$\|x\|_\infty + \sup \left\{ V_{[-k,-k+1]}(x_i) \mid i \in \{1, \dots, m\},\, k \ge 1 \right\}, \quad x \in \mathcal{R}\,.$$

Theorem 3.3. *Assume conditions (C1)–(C4). Given $(\omega_0, x_0) \in \Omega \times BC$ with $\widehat{D}_2(\omega_0, x_0)$ satisfying property (R), if $z(\cdot, \omega_0, x_0)$ is a bounded solution of $(3.3)_{\omega_0}$, then the omega-limit set $\mathcal{O}(\omega_0, x_0) = \{(\omega, c(\omega)) \mid \omega \in \Omega\}$ is a copy of the base and*

$$\lim_{t \to \infty} \mathsf{d}(u(t, \omega_0, x_0), c(\omega_0{\cdot}t)) = 0\,,$$

where $c : \Omega \to BU$ is a continuous equilibrium, i.e. $u(t, \omega, c(\omega)) = c(\omega{\cdot}t)$ for each $\omega \in \Omega$ and $t \ge 0$, and it is continuous for the compact-open topology on BU.

Proof. As above, we consider the exponential ordering $\leq_A$. Thanks to Theorem 3.1, we are in a position to consider the *transformed exponential ordering* $\leq_{D,A}$ introduced in [19]. It is defined on each fiber of the product $\Omega \times BC$ as follows: for each (ω, x), $(\omega, y) \in \Omega \times BC$,

$$(\omega, x) \leq_{D,A} (\omega, y) \iff \widehat{D}_2(\omega, x) \leq_A \widehat{D}_2(\omega, y).$$

Our aim is to apply Theorem 4.12 of [18] in order to complete this proof. First, as seen in Theorem 3.1(iii) and in the proof of Proposition 3.2, $D(\omega, \cdot)$ is linear and continuous for the norm for all $\omega \in \Omega$, the mapping $\Omega \to \mathcal{L}(BC, \mathbb{R}^m)$, $\omega \mapsto D(\omega, \cdot)$ is continuous, the restriction of D to $\Omega \times B_r$ is continuous when we consider the compact-open topology on B_r for all $r > 0$, and D is stable, so conditions (D1)–(D4) of [18] hold.

Besides, from assumptions **(C1)**–**(C2)**, it follows that the function G defined by (3.2) is continuous on $\Omega \times BC$. Moreover, its restriction to $\Omega \times B_r$ is Lipschitz continuous in its second variable when the norm is considered on B_r, and it is continuous when the compact-open topology is considered on B_r for each $r > 0$. This implies conditions (N1) and (N2) of [18].

Let (ω, x), $(\omega, y) \in \Omega \times BC$ with $(\omega, x) \leq_{D,A} (\omega, y)$. From Theorem 3.1(i) we deduce that $x \leq y$. Then, from the nondecreasing character of g_{ij}, stated in **(C1)**, the fact that $\mu_{ij}(\omega)$ are positive and $\nu_i(\omega)$ are negative measures, as assumed in **(C2)** and **(C3)** respectively, and $a_i < 0$, we deduce that

$$-\sum_{j=0}^{m} \big(g_{ji}(\omega, y_i(0)) - g_{ji}(\omega, x_i(0))\big) \geq -L_i^+(\omega)\big(y_i(0) - x_i(0)\big)\,,$$

$$\sum_{j=1}^{m} \int_{-\infty}^{0} \big(g_{ij}(\omega{\cdot}s, y_j(s)) - g_{ij}(\omega{\cdot}s, x_j(s))\big)\, d\mu_{ij}(\omega)(s) \geq 0\,,$$

$$c_i(x, y) := a_i \int_{-\infty}^{0} \big(y_i(s) - x_i(s)\big)\, d\nu_i(\omega)(s) \geq 0\,, \tag{3.7}$$

and, from **(C4)**(i), we conclude that

$$\begin{aligned} G_i(\omega, y) - G_i(\omega, x) - a_i\big(D_i(\omega, y) - D_i(\omega, x)\big) \\ \geq \big(-L_i^+(\omega) - a_i\big)\big(y_i(0) - x_i(0)\big) + c_i(x, y) \geq 0\,. \end{aligned} \tag{3.8}$$

This guarantees that condition (N3) of [18] is satisfied.

Fix (ω, x), $(\omega, y) \in \Omega \times BC$ with $(\omega, x) \leq_{D,A} (\omega, y)$. Suppose that (ω, x) and (ω, y) admit a backward orbit extension and there is a subset $J \subset \{1, \ldots, m\}$ such that the following conditions hold

$$\begin{aligned} \widehat{D}_2(\omega, x)_i &= \widehat{D}_2(\omega, y)_i \quad \text{for each } i \notin J\,, \\ \widehat{D}_2(\omega, x)_i(s) &< \widehat{D}_2(\omega, y)_i(s) \quad \text{for each } i \in J \text{ and } s \leq 0\,. \end{aligned}$$

If $i \in J$, then we have $\widehat{D}_2(\omega, x)_i(s) < \widehat{D}_2(\omega, y)_i(s)$, that is, $D_i(\omega{\cdot}s, x_s) < D_i(\omega{\cdot}s, y_s)$ for each $s \leq 0$. In particular, if $s = 0$, we obtain

$$\int_{-\infty}^{0} \big(y_i(s) - x_i(s)\big)\, d\nu_i(\omega)(s) < y_i(0) - x_i(0)\,.$$

As before, from $(\omega, x) \leq_{D,A} (\omega, y)$, we deduce that $x \leq y$ and, since $\nu_i(\omega)$ is a negative measure, we have two options:

$$\int_{-\infty}^{0} \big(y_i(s) - x_i(s)\big)\, d\nu_i(\omega)(s) < 0 \quad \text{or} \quad y_i(0) - x_i(0) > 0\,.$$

In the first case, since $a_i < 0$, we deduce from (3.7) that $c_i(x,y) > 0$ and, in the second case, from **(C4)**(i), we have $\big(-L_i^+(\omega) - a_i\big)(y_i(0) - x_i(0)) > 0$. Therefore, inequality (3.8) is strict in both cases. As a result, condition (N4) of [18] holds.

Let us check that, if (ω,x), $(\omega,y) \in \Omega \times BC$ with $(\omega,x) \le_{D,A} (\omega,y)$ and $z(t,\omega,x)$, $z(t,\omega,y)$ are defined, then

$$0 \le D_i(\tau(t,\omega,y)) - D_i(\tau(t,\omega,x)) \le M(\omega,y) - M(\omega,x)\,, \quad i = 1,\dots,m\,. \tag{3.9}$$

Thanks to Theorem 4.8 of [18], the skew-product semiflow τ is monotone. As a result, $\tau(t,\omega,x) \le_{D,A} \tau(t,\omega,y)$ whenever they are defined, i.e $\widehat{D}_2(\tau(t,\omega,x)) \le_A \widehat{D}_2(\tau(t,\omega,y))$. Thus, Theorem 3.1(i) provides $z_t(\omega,x) \le z_t(\omega,y)$, and **(C1)** implies

$$g_{ij}(\omega{\cdot}t, z_j(t,\omega,x)) \le g_{ij}(\omega{\cdot}t, z_j(t,\omega,y))\,.$$

Moreover, from $\widehat{D}_2(\tau(t,\omega,x)) \le_A \widehat{D}_2(\tau(t,\omega,y))$, it follows that $\widehat{D}_2(\tau(t,\omega,x)) \le \widehat{D}_2(\tau(t,\omega,y))$, and we deduce that $D_i(\tau(t,\omega,x)) \le D_i(\tau(t,\omega,y))$ for $i = 1,\dots,m$.

As a consequence, **(C2)** and the total mass variation formula (3.6) yield

$$\begin{aligned}
0 &\le D_i(\tau(t,\omega,y)) - D_i(\tau(t,\omega,x)) \le \sum_{i=1}^{m} [D_i(\tau(t,\omega,y)) - D_i(\tau(t,\omega,x))] \\
&\le M(\tau(t,\omega,y)) - M(\tau(t,\omega,x)) = M(\omega,y) - M(\omega,x) \\
&\quad + \sum_{i=1}^{m} \int_0^t (g_{0i}(\omega{\cdot}s, z_i(s,\omega,x)) - g_{0i}(\omega{\cdot}s, z_i(s,\omega,y)))\, ds \le M(\omega,y) - M(\omega,x)\,,
\end{aligned}$$

as claimed.

Finally, thanks to Proposition 3.2, given $\varepsilon > 0$ and $r > 0$, there exists $\delta > 0$ such that, if x, $y \in B_r$ with $(\omega,x) \le_{D,A} (\omega,y)$ and $\mathsf{d}(x,y) < \delta$, then $0 \le M(\omega,y) - M(\omega,x) < \varepsilon$. Consequently, if x, $y \in B_r$ and $(\omega,x) \le_{D,A} (\omega,y)$, from (3.9), it follows that $0 \le D_i(\omega{\cdot}t, z_t(\omega,y)) - D_i(\omega{\cdot}t, z_t(\omega,x))) < \varepsilon$. Hence, the fact that $\nu_i(\omega)$ is a negative measure implies

$$0 \le z_i(t,\omega,y) - z_i(t,\omega,x) \le \varepsilon + \int_{-\infty}^{0} (z_i(t+s,\omega,y) - z_i(t+s,\omega,x))\, d\nu_i(\omega)(s) \le \varepsilon\,,$$

whence, for all x, $y \in B_r$ with $(\omega,x) \le_{D,A} (\omega,y)$ and $\mathsf{d}(x,y) < \delta$, $\|z(t,\omega,y) - z(t,\omega,x)\| \le \varepsilon$ whenever they are defined, so condition (N5) of [18] is satisfied. This property is usually referred to as *uniform stability for the ordering* $\le_{D,A}$ of B_r and it was introduced in [19]. An application of Theorem 4.12 of [18] finishes the proof, as expected. □

Remark 3.4. In compartmental systems, one is interested in initial conditions (ω,x) for which the solutions are bounded and positive, i.e. $u(t,\omega,x) \ge 0$ for each $t \ge 0$. If 0 is a solution, for instance when the compartmental systems (3.1) are closed, the set $\mathcal{P} = \{(\omega,x) \in \Omega \times BC \mid (\omega,x) \ge_{D,A} 0\}$ satisfies this property. Indeed, if $(\omega,x) \in \mathcal{P}$, then $u(t,\omega,x) \ge_{D,A} 0$ for all $t \ge 0$, whence $\widehat{u}(t,\omega,x) \ge_A 0$ for all $t \ge 0$, and Theorem 3.1(i) provides $u(t,\omega,x) \ge 0$ for all $t \ge 0$. The boundedness follows from the uniform stability on each B_r for the ordering $\le_{D,A}$. In addition, note that $(\omega,x) \ge_{D,A} 0$ means that $\widehat{D}_2(\omega,x) \ge_A 0$, which implies, as seen in Proposition 3.8 of [18], that $\widehat{D}_2(\omega,x)$ satisfies **(R)**. Summing up, the conclusions of Theorem 3.3 state that, for closed compartmental systems, $\mathcal{O}(\omega,x)$ is a copy of the base for all initial data in $\mathcal{P}$, those we are interested in.

Under some conditions, in order to verify that $\widehat{D}_2(\omega, x)$ satisfies the property **(R)**, it is sufficient to prove that x does. The following definition is a natural generalization of Definition 6.1 of [19]. As before, $\mathcal{M}$ denotes the set of Borel regular measures on $(-\infty, 0]$.

Definition 3.5. A map $\nu : \Omega \to \mathcal{M}$ is said to be *Lipschitz continuous along the flow* σ if, for each $\omega \in \Omega$, the map $\mathbb{R} \to \mathcal{M}$, $t \mapsto \nu(\omega{\cdot}t)$ is Lipschitz continuous, when the norm given by the total variation is considered on $\mathcal{M}$.

Remark 3.6. Thanks to the minimal character of the base flow on Ω, if the map $\mathbb{R} \to \mathcal{M}$, $t \mapsto \nu(\omega_0{\cdot}t)$ is Lipschitz continuous with Lipschitz constant $L > 0$ for one $\omega_0 \in \Omega$, the same holds for all the maps $\mathbb{R} \to \mathcal{M}$, $t \mapsto \nu(\omega{\cdot}t)$, $\omega \in \Omega$, whence ν is Lipschitz continuous along the flow.

Proposition 3.7. *Assume that ν_i is Lipschitz continuous along the flow and there exists a positive measure $\overline{\nu}$ with finite total variation such that $|\nu_i(\omega)| \leq \overline{\nu}$ for all $i \in \{1, \dots, m\}$ and all $\omega \in \Omega$. If $(\omega, x) \in \Omega \times BC$ and x satisfies property* **(R)**, *then $\widehat{D}_2(\omega, x)$ also satisfies property* **(R)**.

Proof. Let $L_\nu > 0$ be a Lipschitz constant valid for ν_i for all $i \in \{1, \dots, m\}$ and

$$V = \sup \left\{ V_{[-k,-k+1]}(x_i) \mid i \in \{1, \dots, m\},\, k \geq 1 \right\} .$$

Fix $i \in \{1, \dots, m\}$, $k \in \mathbb{N}$ and let $-k = t_0 < t_1 < \cdots < t_n = -k+1$ be a partition of the interval $[-k, -k+1]$. Then, for all $s \leq 0$, the set $\{t_j + s \mid j \in \{0, \dots, n\}\}$ is included in a partition of an interval of the form $[-l, -l+2]$ for some $l \in \mathbb{N}$ with $l \geq 2$. From this fact, it follows that

$$\begin{aligned}
\sum_{j=1}^{n} \left| \widehat{D}_2(\omega, x)_i(t_j) - \widehat{D}_2(\omega, x)_i(t_{j-1}) \right| &= \sum_{j=1}^{n} \left| D_i(\omega{\cdot}t_j, x_{t_j}) - D_i(\omega{\cdot}t_{j-1}, x_{t_{j-1}}) \right| \\
&\leq \sum_{j=1}^{n} |x_i(t_j) - x_i(t_{j-1})| \\
&\quad + \sum_{j=1}^{n} \left| \int_{-\infty}^{0} x_i(t_j + s)\, d\nu_i(\omega{\cdot}t_j)(s) - \int_{-\infty}^{0} x_i(t_{j-1} + s)\, d\nu_i(\omega{\cdot}t_{j-1})(s) \right| \\
&\leq V + \sum_{j=1}^{n} \int_{-\infty}^{0} |x_i(t_j + s) - x_i(t_{j-1} + s)|\, d|\nu_i(\omega{\cdot}t_j)|(s) \\
&\quad + \sum_{j=1}^{n} \int_{-\infty}^{0} |x_i(t_{j-1} + s)|\, d|\nu_i(\omega{\cdot}t_j) - \nu_i(\omega{\cdot}t_{j-1})|(s) \\
&\leq V + \int_{-\infty}^{0} \sum_{j=1}^{n} |x_i(t_j + s) - x_i(t_{j-1} + s)|\, d\overline{\nu}(s) + \sum_{j=1}^{n} |t_j - t_{j-1}|\, L_\nu\, \|x\|_\infty \\
&\leq V + 2\, V\, \overline{\nu}((-\infty, 0]) + L_\nu\, \|x\|_\infty\, ,
\end{aligned}$$

which is a bound independent of k and n, and $\widehat{D}_2(\omega, x)$ satisfies **(R)**, as claimed. □

Finally, we remind that [19] contains a dynamical study of the compartmental systems (3.3) on BU when the measures $\nu_i(\omega)$ are positive for $i = 1, \dots, n$ and $\omega \in \Omega$. It provides technical assumptions under which the omega-limit set of every initial datum (ω, x), with x Lipschitz continuous and $z(\cdot, \omega, x)$ bounded, is a copy of the base. From the conclusions of this paper, it follows that this result remains also valid when x satisfies condition **(R)**.

4 Neutral compartmental systems with finite delay

In this section, we include a nonautonomous neutral compartmental system with finite delay satisfying some recurrence conditions on the temporal variation into a family of the form (3.3), and we apply the conclusions of the previous section to the study of the long-term behavior of the solutions of the initial system.

We consider the system of NFDEs with finite delay

$$\frac{d}{dt}[z_i(t)+\widetilde{c}_i(t)\,z_i(t-\alpha_i)] = -\sum_{j=1}^{m}\widetilde{g}_{ji}(t,z_i(t)) + \sum_{j=1}^{m}\widetilde{g}_{ij}(t-\rho_{ij},z_j(t-\rho_{ij})) \qquad (4.1)$$

$i=1,\dots,m$. Let us denote by $\widetilde{g}=(\widetilde{g}_{ij})_{i,j}\colon \mathbb{R}\times\mathbb{R}\to\mathbb{R}^{m\times m}$, $\widetilde{c}=(c_i)_i\colon\mathbb{R}\to\mathbb{R}^m$ and assume that

- **(c1)** $\widetilde{g}$ is C^1 in its second variable and $\widetilde{g}$, $\frac{\partial}{\partial v}\widetilde{g}$ are uniformly continuous and bounded on $\mathbb{R}\times\{v_0\}$ for all $v_0\in\mathbb{R}$;
- **(c2)** $\widetilde{g}_{ij}(t,0)=0$ and $\widetilde{g}_{ij}(t,\cdot)$ is nondecreasing for all $t\in\mathbb{R}$ and $i,j=1,\dots,m$;
- **(c3)** $\widetilde{c}$ is Lipschitz continuous, nonnegative, and $\sup_{s\in\mathbb{R}}\widetilde{c}_i(s)<1$, $i=1,\dots m$;
- **(c4)** letting $(t,(\mathfrak{c},\mathfrak{g}))\mapsto(\mathfrak{c}_t,\mathfrak{g}_t)$ denote the translation flow, that is, $\mathfrak{c}_t(s)=\mathfrak{c}(t+s)$ and $\mathfrak{g}_t(s,v)=\mathfrak{g}(t+s,v)$ for all $(s,v)\in\mathbb{R}^2$, the closure of the set $\{(\widetilde{c}_t,\widetilde{g}_t)\mid t\in\mathbb{R}\}$ for the compact-open topology, referred to as the *hull* of $(\widetilde{c},\widetilde{g})$, is minimal (in this situation, $(\widetilde{c},\widetilde{g})$ is said to be *recurrent*);
- **(c5)** for each $i=1,\dots,m$, there is a negative $a_i<0$ such that
  - (i) $-L_i^+-a_i>0$ and
  - (ii) $\widetilde{c}_i(t)\le e^{a_i\alpha_i}$, for each $t\in\mathbb{R}$,

  where $L_i^+=\sum_{j=1}^m\sup_{(t,v)\in\mathbb{R}^2}\frac{\partial\widetilde{g}_{ij}}{\partial v}(t,v)$.

Remark 4.1. Notice that no relation among the delays ρ_{ij} and α_i, $i,j=1,\dots,m$, is assumed. Conditions of the same type for the direct exponential ordering and negative coefficients $\widetilde{c}_i$ were obtained in [13], [17] and [20].

In this situation, we may include the system (4.1) in a family of nonautonomous NFDEs. Specifically, let Ω be the hull of $(\widetilde{c},\widetilde{g})$, as defined in **(c4)**. Ω is a compact metric space thanks to **(c1)** and **(c3)** (see Hino, Murakami, and Naito [8]). Let $\sigma:\mathbb{R}\times\Omega\to\Omega$, $(t,\omega)\mapsto\omega{\cdot}t$ be the flow defined on Ω by translation, which is minimal, as specified in **(c4)**. It is noteworthy that the almost periodic and almost automorphic cases are included in this formulation.

Let us consider the continuous map $(c,g):\Omega\times\mathbb{R}\to\mathbb{R}^m\times\mathbb{R}^{m\times m}$, $(\omega,v)\mapsto\omega(0,v)$. Let us denote $c=(c_i)_i$, $g=(g_{ij})_{i,j}$, and define $G:\Omega\times BC\to\mathbb{R}^m$ by

$$G_i(\omega,x) = -\sum_{j=1}^{m} g_{ji}(\omega,x_i(0)) + \sum_{j=1}^{m} g_{ij}(\omega{\cdot}(-\rho_{ij}),x_j(-\rho_{ij}))\,,\quad (\omega,x)\in\Omega\times BC\,,$$

and $D\colon\Omega\times BC\to\mathbb{R}^m$ by

$$D_i(\omega,x)=x_i(0)+c_i(\omega)\,x_i(-\alpha_i)\,,\quad (\omega,x)\in\Omega\times BC\,,\ i=1,\dots,m\,.$$

Hence, the family

$$\frac{d}{dt}D(\omega{\cdot}t,z_t) = G(\omega{\cdot}t,z_t)\,, \quad t\geq 0\,,\ \omega\in\Omega\,, \tag{4.2}$$

is of the form (3.1) for the negative Borel regular measure $\nu_i(\omega) = -c_i(\omega)\,\delta_{-\alpha_i}$ and the positive Borel regular measure $\mu_{ij} = \delta_{-\rho_{ij}}$, $i, j = 1,\dots,m$, and includes system (4.1) when $\omega = (\widetilde{c},\widetilde{g})$.

Notice that the system (4.2) represents a *closed* compartmental system for each $\omega\in\Omega$, that is, there is no flow of material from or to the environment. Consequently, from **(c2)**, it follows that zero is a bounded solution of all the systems of the family and, thanks to (3.6), the total mass is constant along the trajectories.

Theorem 4.2. *Assume that* **(c1)**–**(c5)** *hold and fix* $(\omega_0,x_0)\in\Omega\times BC$ *such that* x_0 *satisfies property* **(R)**. *Then the solution* $z(\cdot,\omega_0,x_0)$ *of* $(4.2)_{\omega_0}$ *with initial value* x_0 *is bounded, the omega-limit set* $\mathcal{O}(\omega_0,x_0) = \{(\omega,x(\omega))\mid\omega\in\Omega\}$ *is a copy of the base and*

$$\lim_{t\to\infty}\mathsf{d}(u(t,\omega_0,x_0),x(\omega_0{\cdot}t)) = 0\,,$$

where $x:\Omega\to BU$ *is a continuous equilibrium.*

Proof. From **(c1)**–**(c5)**, $\nu_i(\omega) = -c_i(\omega)\,\delta_{-\alpha_i}$ and $\mu_{ij} = \delta_{-\rho_{ij}}$ for i, $j = 1,\dots m$, it is easy to check that **(C1)**–**(C4)** hold. Notice in particular that, from Remark 3.6, if $\widetilde{c}$ is Lipschitz continuous, then ν_i is Lipschitz continuous along the flow for all $i\in\{1,\dots,m\}$. In addition, since x_0 satisfies property **(R)** and the conditions of Proposition 3.7 are clearly satisfied, then $\widehat{D}_2(\omega,x_0)$ satisfies property **(R)** as well. Therefore, the result follows from Theorem 3.3, once the boundedness of $z(\cdot,\omega_0,x_0)$ is proved.

As shown in Theorem 3.3, each B_r is uniformly stable for the ordering $\leq_{D,A}$. This and the fact that zero is a solution of (4.2) for all $\omega\in\Omega$ allows us to deduce that each solution $z(\cdot,\omega,x)$ with $(\omega,x)\geq_{D,A}(\omega,0)$ or $(\omega,x)\leq_{D,A}(\omega,0)$ is globally defined and bounded. Since $\widehat{D}_2(\omega,x_0)$ satisfies property **(R)**, thanks to Proposition 3.8 of [18], there exists $h\in BC$ such that $h\geq_A 0$ and $h\geq_A\widehat{D}_2(\omega,x)$, i.e $(\omega,\widehat{h})\geq_{D,A}(\omega,0)$ and $(\omega,\widehat{h})\geq_{D,A}(\omega,x_0)$ for $\widehat{h} = (\widehat{D}^{-1})_2(\omega,h)$. Consequently, from the first inequality we deduce that $z(t,\omega,\widehat{h})$ is globally defined and bounded and, hence, from the second inequality and again the uniform stability of each B_r for the ordering $\leq_{D,A}$, we conclude that $z(\cdot,\omega,x_0)$ is globally defined and bounded, as desired. □

Regarding the solutions of the system (4.1), we obtain the following result, stated in the almost periodic case, but similar conclusions hold changing almost periodicity for constancy, periodicity, almost automorphy or recurrence. All solutions with initial data x_0 satisfying property **(R)**, in particular those with Lipschitz continuous initial data, are asymptotically of the same type as the transport functions and the coefficients of the neutral part.

Theorem 4.3. *Under assumptions* **(c1)**–**(c5)**, *if both* $\widetilde{c}$ *and* $\widetilde{g}$ *are almost periodic, then there exist infinitely many almost periodic solutions of system* (4.1). *Moreover, all the solutions with initial data* $x_0\in BC$ *satisfying property* **(R)** *are asymptotically almost periodic.*

Proof. Fix $\omega_0 = (\widetilde{c},\widetilde{g})$ and $x_0\in BC$ satisfying property **(R)**. The omega-limit set $\mathcal{O}(\omega_0,x_0)$ is a copy of the base, whence, $t\mapsto z(t,\omega_0,x(\omega_0)) = x(\omega_0{\cdot}t)(0)$ is an almost periodic solution of (4.1) and

$$\lim_{t\to\infty}\|z(t,\omega_0,x_0) - z(t,\omega_0,x(\omega_0))\| = 0\,.$$

Let us check that there are infinitely many minimal subsets. Let $x^k \in BC$ denote the constant function $x^k \equiv k\,(1,\dots,1)$. The total mass associated to (ω_0, x^k) is

$$M(\omega_0, x^k) = \sum_{i=1}^{m} k\,(1 + c_i(\omega_0)) + \sum_{i=1}^{m}\sum_{j=1}^{m}\int_{-\infty}^{0}\left(\int_{s}^{0} g_{ji}(\omega_0{\cdot}\tau, k)\,d\tau\right) d\mu_{ji}(s)\,,$$

which diverges to infinity as $k \to \infty$. Therefore, for all $r > 0$, there exists $k_r > 0$ such that $M(\omega_0, x^{k_r}) = r$. Finally, since the total mass is constant along the trajectories, $\mathcal{O}(\omega_0, x^{k_r})$ provides a different minimal set and, hence, a different almost periodic solution for each $r > 0$. □

5 Numerical experiments

In this section, we apply the results included in Section 4 and perform numerical experiments on the neutral compartmental system:

$$\begin{aligned}
&\frac{d}{dt}\big(z_1(t) + 0.1\,(1+\cos(\sqrt{2}\,(t-0.5)))\,z_1(t-0.5)\big)\\
&\qquad = -\,0.1\,(1+\cos(\sqrt{2}\,t))\,\tanh(z_1(t)) - 0.2\,(1+\cos(t))\,\tanh(z_1(t))\\
&\qquad\quad + 0.1\,(1+\cos(\sqrt{2}\,(t-1)))\,\tanh(z_1(t-1))\\
&\qquad\quad + 0.2\,(1+\sin(\sqrt{2}\,(t-2)))\,\tanh(z_2(t-2))\,,\\
&\frac{d}{dt}\big(z_2(t) + 0.05\,(1+\sin((t-1)))\,z_2(t-1)\big)\\
&\qquad = -\,0.2\,(1+\sin(\sqrt{2}\,t))\,\tanh(z_2(t)) - 0.1\,(1+\sin(t))\,\tanh(z_2(t))\\
&\qquad\quad + 0.2\,(1+\cos((t-0.3)))\,\tanh(z_1(t-0.3))\\
&\qquad\quad + 0.1\,(1+\sin((t-0.5)))\,\tanh(z_2(t-0.5))\,,
\end{aligned} \tag{5.1}$$

for $t \geq 0$. Clearly, the system of equations (5.1) is of the form given in (4.1). In this case,

$$\begin{aligned}
&\widetilde{g}_{11}(t,x) = 0.1\,(1+\cos(\sqrt{2}\,t))\,\tanh(x)\,, && \widetilde{g}_{12}(t,x) = 0.2\,(1+\sin(\sqrt{2}\,t))\,\tanh(x)\,,\\
&\widetilde{g}_{21}(t,x) = 0.2\,(1+\cos(t))\,\tanh(x)\,, && \widetilde{g}_{22}(t,x) = 0.1\,(1+\sin(t))\,\tanh(x)\,,\\
&\rho_{11} = 1\,, && \rho_{12} = 2\,,\\
&\rho_{21} = 0.3\,, && \rho_{22} = 0.5\,,\\
&\widetilde{c}_1(t) = 0.1\,(1+\cos(\sqrt{2}\,t))\,, && \widetilde{c}_2(t) = 0.05\,(1+\sin(t))\,,\\
&\alpha_1 = 0.5\,, && \alpha_2 = 1\,.
\end{aligned}$$

The following result shows that the system of equations (5.1) also satisfies the conditions assumed throughout Section 4.

Proposition 5.1. *The system of equations* (5.1) *satisfies conditions* **(c1)**–**(c5)**.

Proof. Conditions **(c1)**–**(c4)** are immediate. As for condition **(c5)**, it is trivial to check that $l_{11}^+ \leq 0.2$ and $l_{21}^+ \leq 0.4$, whence $L_1^+ \leq 0.6$. As a result, by choosing $a_1 = -0.7$, we obtain $e^{a_1\alpha_1} > 0.7$, which is an upper bound of $\widetilde{c}_1$. Analogously, $l_{12}^+ \leq 0.4$ and $l_{22}^+ \leq 0.2$, whence $L_2^+ \leq 0.6$. By choosing $a_2 = -0.7$, we obtain $e^{a_2\alpha_2} > 0.4$, which is an upper bound of $\widetilde{c}_2$, as wanted. □

It is noteworthy that, as in Section 4, we can define Ω as the hull of the coefficients of the system of equations (5.1). It is well-known that the flow defined on Ω by translation is isomorphic to the Kronecker flow defined on the 2-torus $\mathbb{T}^2 = (\mathbb{R}/[0,2\pi])^2$, which is defined by

$$\begin{aligned}\zeta : \mathbb{R}^+ \times \mathbb{T}^2 &\longrightarrow \mathbb{T}^2\\ (t,\theta_1,\theta_2) &\longmapsto (\theta_1+\sqrt{2}\,t,\theta_2+t)\,.\end{aligned}$$

Notice that ζ is a minimal flow (see e.g. Walters [23] for further details). This fact allows us to include the system of equations (5.1) in the family of equations

$$\begin{aligned}&\frac{d}{dt}\big(z_1(t)+0.1\,(1+\cos(\theta_1+\sqrt{2}\,(t-0.5)))\,z_1(t-0.5)\big) =\\ &\quad= -\,0.1\,(1+\cos(\theta_1+\sqrt{2}\,t))\,\tanh(z_1(t)) - 0.2\,(1+\cos(\theta_2+t))\,\tanh(z_1(t))\\ &\qquad + 0.1\,(1+\cos(\theta_1+\sqrt{2}\,(t-1)))\,\tanh(z_1(t-1))\\ &\qquad + 0.2\,(1+\sin(\theta_1+\sqrt{2}\,(t-2)))\,\tanh(z_2(t-2))\,,\\ &\frac{d}{dt}\big(z_2(t)+0.05\,(1+\sin(\theta_2+(t-1)))\,z_2(t-1)\big) =\\ &\quad= -\,0.2\,(1+\sin(\theta_1+\sqrt{2}\,t))\,\tanh(z_2(t)) - 0.1\,(1+\sin(\theta_2+t))\,\tanh(z_2(t))\\ &\qquad + 0.2\,(1+\cos(\theta_2+(t-0.3)))\,\tanh(z_1(t-0.3))\\ &\qquad + 0.1\,(1+\sin(\theta_2+(t-0.5)))\,\tanh(z_2(t-0.5))\,,\end{aligned} \tag{5.2}$$

for $t \geq 0$, where $(\theta_1,\theta_2) \in \mathbb{T}^2$.

In this situation, we can apply the results obtained in Section 4 to the system of equations (5.1).

Theorem 5.2. *Fix $(\theta_1,\theta_2,x_0) \in \mathbb{T}^2 \times BC$ such that x_0 satisfies property* **(R)**. *Then the solution $z(\cdot,\theta_1,\theta_2,x_0)$ of (5.2)$_{(\theta_1,\theta_2)}$ with initial value x_0 is bounded, the omega-limit set $\mathcal{O}(\theta_1,\theta_2,x_0) = \{(\varphi_1,\varphi_2,x(\varphi_1,\varphi_2)) \mid (\varphi_1,\varphi_2) \in \mathbb{T}^2\}$ is a copy of the base and*

$$\lim_{t\to\infty} \mathrm{d}(u(t,\theta_1,\theta_2,x_0), x(\zeta_t(\theta_1,\theta_2))) = 0\,,$$

where $x : \mathbb{T}^2 \to BU$ is a continuous equilibrium. Moreover, there are infinitely many almost periodic solutions of system (5.1) and all the solutions with initial data x_0 satisfying property **(R)** *are asymptotically almost periodic.*

Proof. It follows immediately from Proposition 5.1, Theorem 4.2, and Theorem 4.3. □

In what follows, some numerical simulations of the neutral compartmental system (5.1) will be presented. Numerical methods for neutral differential equations with delay are well-known (see e.g. Wen and Yu [24] and the references therein). All our computations were carried out with the Matlab function `ddensd`, with a relative tolerance of 10^{-5} and an absolute tolerance of 10^{-10}.

Figure 5.1 shows the solutions of system (5.1) with the following initial data $x_0^i : (-\infty,0] \to \mathbb{R}^2$, $i = 1, 2, 3$, which satisfy property **(R)**:

(i) $x_0^1(t) = (-\sin(t),\, \max\{2, \sqrt{-t}\})$ for all $t \in (-\infty,0]$;

(ii) $x_0^2(t) = (0.5,\, 1)$ for all $t \in (-\infty,0]$;

(iii) $x_0^3(t) = (0.3\,e^t,\, 0.3\,(1+\cos(2\,t)))$ for all $t \in (-\infty,0]$.

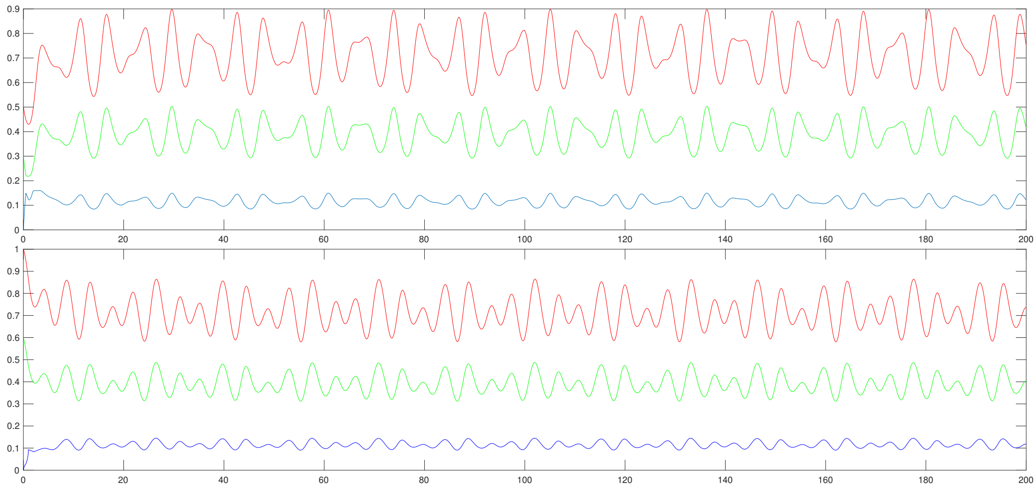


Figure 5.1: First and second components of the solutions of system (5.1) with initial data x_0^1 (blue), x_0^2 (red), and x_0^3 (green) for $t \in [0, 200]$.

Note that x_0^1 is *not* Lipschitz continuous.

In order to approximate the omega-limit of the solution of system (5.1) with initial datum x_0^1, $z : \mathbb{R} \to \mathbb{R}^2$, the system of equations (5.1) was integrated for $t \in [0,\, 5000]$. We considered the time subintervals $J_k = [500\,k,\, 500\,(k+1)]$, $k = 0, \dots, 9$, and restricted the solution z to those subintervals. Then, the graphs over the 2-torus of both components of $z|_{J_k}$,

$$\{(\sqrt{2}\,t \pmod{2\,\pi},\, t \pmod{2\,\pi}, z_i(t)) \mid t \in J_k\}, \quad i = 1, 2,$$

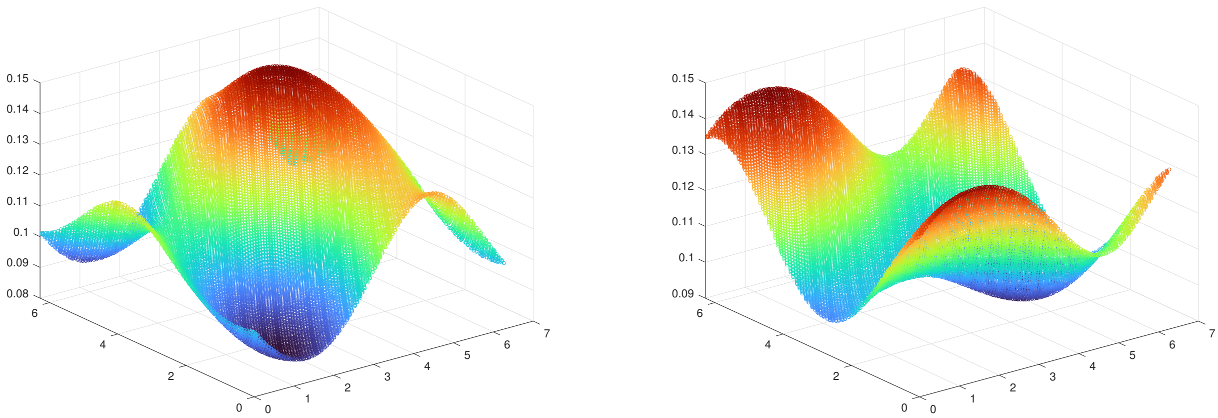
can be approximated by two surfaces S_k^1 and S_k^2, respectively, for $k = 0, \dots, 9$. These approximated surfaces were computed by means of a cubic interpolation of the scattered data provided by the numerical integration of z on J_k , for $k = 0, \dots, 9$. Finally, the distance between any two consecutive surfaces in the finite sequence $S_0^i, \dots, S_9^i$, $i = 1, 2$, was calculated by comparing them on a fixed uniform grid of 32×32 points on the 2-torus $\mathbb{T}^2$, and by taking the supremum over the whole grid. The results for both components of the solution are included in Table 5.1.

Surfaces	Distance ($i = 1$)	Distance ($i = 2$)
S_0^i, S_1^i	$8.76{\cdot}10^{-2}$	$1.21{\cdot}10^{-1}$
S_1^i, S_2^i	$4.32{\cdot}10^{-6}$	$4.81{\cdot}10^{-6}$
S_2^i, S_3^i	$4.36{\cdot}10^{-6}$	$5.02{\cdot}10^{-6}$
S_3^i, S_4^i	$4.40{\cdot}10^{-6}$	$6.21{\cdot}10^{-6}$
S_4^i, S_5^i	$4.52{\cdot}10^{-6}$	$4.93{\cdot}10^{-6}$
S_5^i, S_6^i	$4.32{\cdot}10^{-6}$	$5.03{\cdot}10^{-6}$
S_6^i, S_7^i	$4.28{\cdot}10^{-6}$	$7.92{\cdot}10^{-6}$
S_7^i, S_8^i	$4.32{\cdot}10^{-6}$	$4.58{\cdot}10^{-6}$
S_8^i, S_9^i	$4.38{\cdot}10^{-6}$	$5.17{\cdot}10^{-6}$

Table 5.1: Distance between consecutive surfaces in the finite sequence $S_0^i, \dots, S_9^i$, $i = 1, 2$.

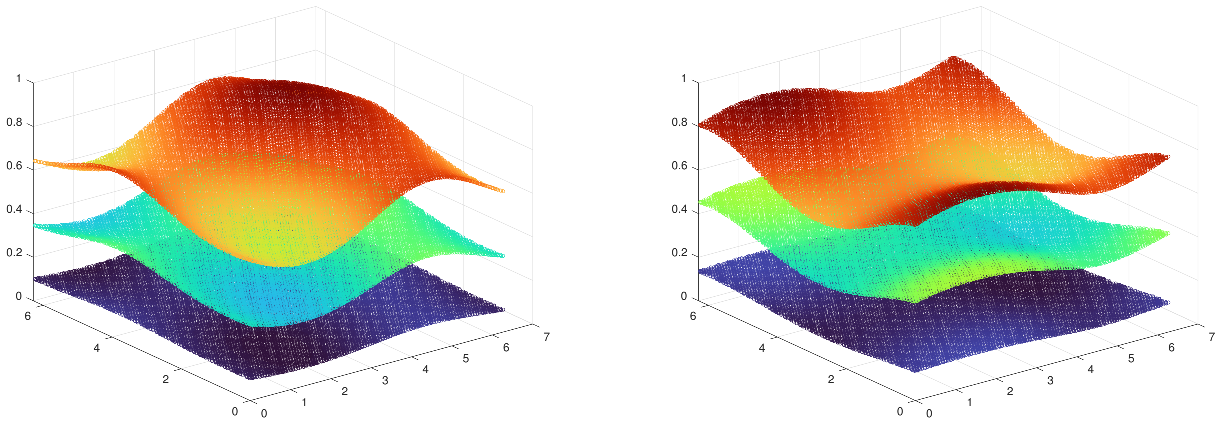
Clearly, the distance between two consecutive surfaces S^i_k and S^i_{k+1}, $i = 1, 2$, $k = 0, \dots, 8$, gets close to zero very rapidly, but then stabilizes. This fact is not surprising due to the mere uniform stability that was checked in the proof of Theorem 3.3, which does not necessarily imply a fast convergence.

Figure 5.2 contains the omega-limit set of the solution of system (5.1) with initial datum x^1_0. Its components have been approximated by the surfaces S^1_9 and S^2_9, respectively. This omega-limit set is clearly the graph of a continuous function, as proved in Theorem 5.2. Moreover, Figure 5.3 shows the omega-limit sets of the solutions of system (5.1) with initial data x^1_0, x^2_0, and x^3_0. Thanks to Proposition 3.2, a similar argument to that in the proof of Theorem 4.3 implies that there is a different omega-limit set for each value of the total mass of system (5.1) (see e.g. [15] for further results as to the comparison of different omega-limit sets in closed compartmental systems).



(a) First component. (b) Second component.

Figure 5.2: Omega-limit of the solution of system (5.1) with initial data x^1_0.



(a) First components. (b) Second components.

Figure 5.3: First and second components of the omega-limits of the solutions of system (5.1) with initial data x^1_0, x^2_0, and x^3_0.

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