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Existence and multiplicity of eigenvalues for some double-phase problems involving an indefinite sign reaction term

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Abstract.
We study the following class of double-phase nonlinear eigenvalue problems

$$-\operatorname{div}\left[\phi(x,|\nabla u|)\nabla u+\psi(x,|\nabla u|)\nabla u\right]=\lambda f(x,u)$$

in Ω, $u=0$ on $\partial\Omega$, where Ω is a bounded domain from $\mathbb{R}^N$ and the potential functions ϕ and ψ have $(p_1(x);p_2(x))$ variable growth. The primitive of the reaction term of the problem (the right-hand side) has indefinite sign in the variable u and allows us to study functions with slower growth near $+\infty$, that is, it does not satisfy the Ambrosetti–Rabinowitz condition. Under these hypotheses we prove that for every parameter $\lambda\in\mathbb{R}_+^*$, the problem has an unbounded sequence of weak solutions. The proofs rely on variational arguments based on energy estimates and the use of Fountain Theorem.

Keywords: double-phase differential operator, continuous spectrum, variable exponent, multiplicity of eigenvalues, infinitely many solutions.
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1 Introduction

The study of variational problems with nonstandard growth conditions has been developed extensively over the last years. Moreover as the technology development in some important areas like robotics, aircraft and airspace and the image restoration was very intensive, and in order to obtain important results, new mathematical models arose.

The $p(x)$-growth conditions can be regarded as a key factor in the modelling of some fluids which have different inhomogeneities, for instance we can mention here the lithium polymetachrylate , which is an electrorheological fluid. The main characteristic of these types of fluids is the fact that their viscosity depends on the electric field in the fluid, that is the viscosity of the fluid is inverse proportional to the strength of the electric field.

As new types of materials arose in the domains that we mentioned before, new problems arose also in the field of variable exponent analysis and partial differential equations which

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involve several variable exponents. Therefore in the last years, double-phase problems which involve several variable exponents and some nonstandard $(p_1(x), p_2(x))$-growth behavior for potential functions have been extensively studied.

In this paper we are concerned with the study of a class of non-autonomous eigenvalue problem with variable $(p_1(x); p_2(x))$-growth rate condition in the left hand side of the problem and a general reaction term (that is in the right-hand side of the problem), which is p_2^+-superlinear at infinity and whose primitive may be sign changing. An important characteristic of the above mentioned problem is the fact that the associated energy density changes its ellipticity according to the point.

The research in this paper in based on some new type of differential operators, which have been introduced by I. H. Kim and Y. H. Kim [8], which enables us to solve some problems which imply the possible lack of uniform convexity. In this paper we extend the results of I. H. Kim and Y. H. Kim by studying a double-phase problem and we use a new type of reaction term which require weaker conditions than the Ambrosetti–Rabinowitz condition (for the sake of simplicity we will denote this condition as the (AR)-condition) and allows us to study functions that have a p_2^+-superlinear growth near infinity but the growth is too slow to satisfy the (AR)-condition. Also,the primitive of the reaction term is allowed to be sign-changing. An example of this type of reaction term will be presented in the last section of this paper together with some important examples and new directions of research. Furthermore, for the best of our knowledge for this type of operators even in the simpler cases, when the differential operator is driven by only one potential function the possibility that the primitive of the reaction function to be sign-changing has not been considered.

This paper also aim to extend some spectral results for some simpler cases studied in the following works: S. Baraket, S. Chebbi, N. Chorfi, V. Rădulescu [2], M. Rodrigues [19], V. F. Uță [22] and K. Q. Wang, M. Zhou [23]. A comparison between these results will be made later in this paper.

Hence, we consider the following double-phase nonlinear eigenvalue problem:

$$\begin{cases} -\operatorname{div}\left[\phi(x,|\nabla u|)\nabla u\right] - \operatorname{div}\left[\psi(x,|\nabla u|)\nabla u\right] = \lambda f(x,u), \text{ in } \Omega, \\ u = 0, \text{ on } \partial\Omega, \end{cases} \tag{P}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with Lipschitz boundary and $\lambda \in \mathbb{R}$ is a real parameter.

These types of problems generalize a broad variety of models. We will briefly describe the most important ones.

For instance if we may need to model a composite that changes its hardening point exponent according to the point. To this end we refer to the work of M. Colombo, G. Mingione [3], where the associated energies are of type:

$$u \mapsto \int_\Omega |\nabla u|^{p_1(x)} dx + \int_\Omega a(x)|\nabla u|^{p_2(x)} dx \tag{1.1}$$

and

$$u \mapsto \int_\Omega |\nabla u|^{p_1(x)} dx + \int_\Omega a(x)|\nabla u|^{p_2(x)} \log(e+|x|) dx, \tag{1.2}$$

where $p_1(x) \le p_2(x)$, $p_1 \ne p_2$, for all $x \in \Omega$ and $a(x) \ge 0$.

Also a comprehensive study of this variety of models is presented in the following survey paper of G. Mingione, V. Rădulescu [9]. For the regularity of the minimizers functionals for double phase operator we recommend for more details the paper of M. Ragusa, A. Tachikawa [15].

The models presented above describe the behavior of two materials with variable power hardening exponents $p_1(x)$ and $p_2(x)$, with the geometry of a composite for one of the materials described by the coefficient $a(x)$.

As the potentials that drive our nonhomogeneous double-phase operator are very general we will consider the following special cases:

(C_1) The potential functions ϕ and ψ may describe a weighted $p(x)$-Laplacian-like operator

$$-\operatorname{div}[\phi(x,|\nabla u|)\nabla u] - \operatorname{div}[\psi(x,\nabla u)\nabla u] = -\operatorname{div}\left[a(x)|\nabla u|^{p_1(x)-2}\nabla u\right] - \operatorname{div}\left[b(x)|\nabla u|^{p_2(x)-2}\nabla u\right],$$

where the functions $a(x), b(x) \in L^\infty(\Omega)$, and there exist some constant α_0 such that $a(x) \geq \alpha_0$, $b(x) \geq \alpha_0$ for almost all $x \in \Omega$;

(C_2) The potential functions ϕ and ψ may describe the generalized mean curvature operator, thus we obtain the following differential operator:

$$-\operatorname{div}[\phi(x,|\nabla u|)\nabla u] - \operatorname{div}[\psi(x,\nabla u)\nabla u] = -\operatorname{div}\left[\left(1+|\nabla u|^2\right)^{\frac{p_1(x)-2}{2}}\nabla u\right] - \operatorname{div}\left[\left(1+|\nabla u|^2\right)^{\frac{p_2(x)-2}{2}}\nabla u\right]$$

(C_3) The potential functions ϕ and ψ may describe the differential operator that describe the capillary phenomenon:

$$\begin{aligned}
&-\operatorname{div}[\phi(x,|\nabla u|)\nabla u] - \operatorname{div}[\psi(x,\nabla u)\nabla u] \\
&= -\operatorname{div}\left[\left(|\nabla u|^{p_1(x)-2} + \frac{|\nabla u|^{2p_1(x)-2}}{\left(1+|\nabla u|^{2p_1(x)}\right)^{1/2}}\right)\nabla u\right] \\
&\quad -\operatorname{div}\left[\left(|\nabla u|^{p_2(x)-2} + \frac{|\nabla u|^{2p_2(x)-2}}{\left(1+|\nabla u|^{2p_2(x)}\right)^{1/2}}\right)\nabla u\right].
\end{aligned}$$

Remark 1.1. Also there can be considered more complex cases where the potential functions have different behavior, for example potential ϕ may describe the case (C_1), and the potential ψ could describe any of the other cases.

It is obvious that the case (C_1) generalize the relation described by (1.1). In order to obtain the case described by (1.2) we will have to study the following differential operator:

$$-\operatorname{div}[\phi(x,|\nabla u|)\nabla u] - \operatorname{div}[a(x)\psi(x,|\nabla u|)\log(e+|x|)\nabla u]. \tag{1.3}$$

The study of the case (C_3) is motivated by its important applicabilities in various fields varying from the industrial, biomedical and pharmaceutical to the microfluidic systems. In order to describe the capillarity phenomenon we must consider the effects of two opposing forces: adhesion, that is, the attractive (or repulsive) force between the molecules of the liquid and those of the container; and cohesion, that is, the attractive force between the molecules of the liquid.

Problems involving this type of differential operator were intensely studied in the last years. For example we consider the following works: [8, 17–19, 22]. Also, more closely related results for anisotropic problems with unbalanced growth may be found in [1] and for the double phase operators with lack of compactness we refer to [20].

The main results of this paper consist in two theorems which ensures us that for every $\lambda > 0$, $\lambda \in \mathbb{R}$, the problem (P) admits an unbounded sequence of solutions with higher and higher energies. Both of the proofs are based on variational arguments, energy estimates and the use of the Fountain Theorem.

High energy solutions for similar problems were studied under more restrictive hypotheses in the following works: [19, 22], where the reaction function is supposed to satisfy the so called (AR)-condition, or in [23] where the differential operator enables us to study some simple case, where in order to make connections to our problem the potential function ϕ is supposed to verify just the case (C_2) and the potential function $\psi \equiv 0$, but the nonlinearity in the right-hand side of the problem is more general than the one used in [19] and [22]. This generality comes at a cost, that is, the parameter λ is allowed to take values just in a bounded interval near the origin.

In the last section of this work we give some striking examples and some remarks in order to illustrate the validity of our results. Moreover, we draw a parallel between previous results and the new results presented in this paper as well as some future perspectives of research in this direction.

2 The functional framework

Through this section we will introduce the basic properties of variable exponent spaces, that will constitute necessary the functional framework that we need in the study of problem (P).

These results are described in the following books: J. Musielak [10], L. Diening, P. Hästö, P. Harjulehto, M. Růžička [4], V. Rădulescu and D. Repovš [17]. We also refer to the survey paper by V. Rădulescu [16].

Let Ω be a bounded domain in $\mathbb{R}^N$.

For a measurable function $p : \overline{\Omega} \to \mathbb{R}$ we define:

$$p^+ = \sup_{x\in\Omega} p(x) \quad \text{and} \quad p^- = \inf_{x\in\Omega} p(x).$$

Set:

$$C_+(\Omega) = \left\{ p \in C(\overline{\Omega}) : \; p(x) > 1, \text{ for all } x \in \overline{\Omega} \right\}.$$

The variable exponent Lebesgue space $L^{p(x)}(\Omega)$ is defined

$$L^{p(x)}(\Omega) = \left\{ u; \;\; u : \Omega \to \mathbb{R} \text{ a measurable function} : \int_\Omega |u|^{p(x)} dx < \infty \right\},$$

and with the norm:

$$|u|_{p(x)} = \inf \left\{ \mu > 0 : \int_\Omega \left| \frac{u(x)}{\mu} \right|^{p(x)} dx \leq 1 \right\},$$

$L^{p(x)}(\Omega)$ becomes a Banach space whose dual is the space $L^{p'(x)}(\Omega)$, where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$.

Remark 2.1. If $1 < p(x) < \infty$, $L^{p(x)}(\Omega)$ is reflexive Banach space. Moreover, if p is measurable and bounded, then $L^{p(x)}(\Omega)$ is also separable.

Remark 2.2. If $0 < |\Omega| < \infty$ and $h(x)$, $r(x)$ with $h(x) < r(x)$ almost everywhere in Ω, are two variable exponents then the following continuous embedding holds

$$L^{r(x)}(\Omega) \hookrightarrow L^{h(x)}(\Omega).$$

Let $L^{p'(x)}(\Omega)$ denotes the dual space of $L^{p(x)}(\Omega)$. For all $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$ the following Hölder type inequality holds:

$$\left| \int_\Omega uv\, dx \right| \leq \left(\frac{1}{p^-} + \frac{1}{p'^-} \right) |u|_{p(x)} |v|_{p'(x)} \leq 2|u|_{p(x)} |v|_{p'(x)}. \tag{2.1}$$

A key role in the studies which imply the variable exponent Lebesgue spaces is played by the modular of $L^{p(x)}(\Omega)$, which is $\rho_{p(x)} : L^{p(x)}(\Omega) \to \mathbb{R}$ and is defined by

$$\rho_{p(x)}(u) = \int_\Omega |u(x)|^{p(x)} dx.$$

Remark 2.3. If $p(x) \not\equiv$ *constant* in Ω, for u, $(u_n) \in L^{p(x)}(\Omega)$, the following relations hold true:

$$|u|_{p(x)} < 1 \Rightarrow \ |u|_{p(x)}^{p^+} \leq \rho_{p(x)}(u) \leq |u|_{p(x)}^{p^-}, \tag{2.2}$$

$$|u|_{p(x)} > 1 \Rightarrow \ |u|_{p(x)}^{p^-} \leq \rho_{p(x)}(u) \leq |u|_{p(x)}^{p^+}, \tag{2.3}$$

$$|u|_{p(x)} = 1 \Rightarrow \ \rho_{p(x)}(u) = 1, \tag{2.4}$$

$$|u_n - u|_{p(x)} \to 0 \Leftrightarrow \ \rho_{p(x)}(u_n - u) \to 0. \tag{2.5}$$

The variable exponent Sobolev space $W^{1,p(x)}(\Omega)$ is defined by

$$W^{1,p(x)}(\Omega) = \left\{ u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega) \right\}.$$

On $W^{1,p(x)}(\Omega)$ we may consider the following equivalent norms:

$$\|u\|_{p(x)} = |u|_{p(x)} + |\nabla u|_{p(x)}$$

and

$$\|u\| = \inf \left\{ \mu : \int_\Omega \left(\left| \frac{\nabla u(x)}{\mu} \right|^{p(x)} + \left| \frac{u(x)}{\mu} \right|^{p(x)} \right) dx \leq 1 \right\}.$$

We define $W_0^{1,p(x)}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{p(x)}$ or

$$W_0^{1,p(x)}(\Omega) = \left\{ u; u|_{\partial\Omega} = 0, u \in L^{p(x)}(\Omega), |\nabla u| \in L^{p(x)}(\Omega) \right\}.$$

Taking account of [8] for $p \in C_+(\overline{\Omega})$ we have the $p(\cdot)$-*Poincaré type inequality*

$$|u|_{p(x)} \leq C|\nabla u|_{p(x)}, \tag{2.6}$$

where $C > 0$ is a constant which depends on p and Ω.

For $\Omega \subset \mathbb{R}^N$ a bounded domain and p a global log-Hölder continuous function, on $W_0^{1,p(x)}(\Omega)$ we can work with the norm $|\nabla u|_{p(x)}$ equivalent with $\|u\|_{p(x)}$.

Remark 2.4. If $p, q : \Omega \to (1, \infty)$ are Lipschitz continuous, $p^+ < N$ and $p(x) \le q(x) \le p^*(x)$, for any $x \in \Omega$, where $p^*(x) = \frac{Np(x)}{N-p(x)}$, the embedding

$$W_0^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$$

is compact and continuous.

Remark 2.5. If $0 < |\Omega| < \infty$, and $p_2(x) < p_1(x)$ in Ω, then there holds the following continuous embedding

$$W_0^{1,p_1(x)}(\Omega) \hookrightarrow W_0^{1,p_2(x)}(\Omega).$$

Remark 2.6 ([5]). Let $p(x)$ and $q(x)$ be measurable functions such that $p(x) \in L^\infty(\Omega)$ and $1 \le p(x)q(x) \le \infty$ almost everywhere in Ω. Let $u \in L^{q(x)}(\Omega)$, $u \ne 0$. Then

$$|u|_{p(x)q(x)} \ge 1 \Rightarrow |u|^{p^-}_{p(x)q(x)} \le \left||u|^{p(x)}\right|_{q(x)} \le |u|^{p^+}_{p(x)q(x)}$$
$$|u|_{p(x)q(x)} \le 1 \Rightarrow |u|^{p^+}_{p(x)q(x)} \le \left||u|^{p(x)}\right|_{q(x)} \le |u|^{p^-}_{p(x)q(x)}.$$

In particular, if $p(x) = p$ is a constant, then $||u|^p|^p_{pq(x)}$.

3 Basic hypotheses and auxiliary results

In this section we will give the basic properties of the potential functions ϕ and ψ which drive us to the differential operator described in the first section. Also we impose the new conditions on the reaction function and the theoretical auxiliary results we need in order to achieve the solutions of problem (P).

Therefore, we assume that the reaction function $f(x,z)$ satisfies the following conditions:

(R_1) $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, that is:
$\to f(\cdot, z)$ is measurable for all $z \in \mathbb{R}$;
$\to f(x, \cdot)$ is continuous for almost all $x \in \Omega$.

(R_2) There exists $C > 0$, a nonnegative constant such that

$$|f(x,z)| \le C\left(1 + |z|^{q(x)-1}\right)$$

for all $x \in \Omega$ and $z \in \mathbb{R}$, where $q \in C_+(\overline{\Omega})$.

Define

$$F(x,z) = \int_0^z f(x,t)dt. \tag{3.1}$$

(R_3) $\lim_{|z|\to\infty} \frac{|F(x,z)|}{|z|^{p_2^+}} = +\infty$ uniformly in x, and there exists $q_0 > 0$, such that

$$F(x,z) \ge 0 \quad \text{for all } x \in \Omega \text{ and } z \in \mathbb{R},$$

with $|z| > q_0$.

(R_4) Define:

$$\mathcal{R}(x,z) := \frac{1}{p_2^+} f(x,z)z - F(x,z) \geq 0$$

and let $C_1 > 0$, a nonnegative constant and $\mu \in C_+(\Omega)$ with $\mu^- > \max\{1, \frac{N}{p_1^-}\}$ such that

$$|F(x,z)|^{\mu(x)} \leq C_1 |z|^{p_1^- \mu(x)} \mathcal{R}(x,z),$$

for all $x \in \Omega$ and $z \in \mathbb{R}$, with $|z| \geq q_0$.

(R_5) Let $\omega > p_2^+$ and $\eta > 0$ two constants such that

$$\omega F(x,z) \leq f(x,z)z + \eta |z|^{p_1^-},$$

for all $x \in \Omega$, $z \in \mathbb{R}$.

(R_6) $f(x,-z) = -f(x,z)$, for all $x \in \Omega$ and $z \in \mathbb{R}$.

Hypotheses on the potential functions that generates the double-phase differential operator are the following:

(HS_1) $\phi, \psi : \Omega \times [0,\infty) \to [0,\infty)$ and

- $\to$ $\phi(\cdot, z)$, $\psi(\cdot, z)$ are measurable on Ω for all $z \geq 0$;
- $\to$ $\phi(x, \cdot)$, $\psi(x, \cdot)$ are locally absolutely continuous on $[0,\infty)$ for almost all $x \in \Omega$.

(HS_2) For some functions $\alpha_1 \in L^{p_1'(x)}(\Omega)$ and $\alpha_2 \in L^{p_2'(x)}(\Omega)$ and a nonnegative constant ξ we have that

- $\to$ $|\phi(x,|z|)z| \leq \alpha_1(x) + \xi |z|^{p_1(x)-1}$;
- $\to$ $|\psi(x,|z|)z| \leq \alpha_2(x) + \xi |z|^{p_2(x)-1}$.

for almost all $x \in \Omega$, and all $z \in \mathbb{R}^N$.

(HS_3) For some constant $C_{\phi,\psi} > 0$, all $x \in \Omega$ and all $z > 0$ we have that:

- $\to$ $\phi(x,z) \geq C_{\phi,\psi} z^{p_1(x)-2}$ and $z\frac{\partial \phi}{\partial z} + \phi(x,z) \geq C_{\phi,\psi} z^{p_1(x)-2}$
- $\to$ $\psi(x,z) \geq C_{\phi,\psi} z^{p_2(x)-2}$ and $z\frac{\partial \psi}{\partial z} + \psi(x,z) \geq C_{\phi,\psi} z^{p_2(x)-2}$.

Let $S_0(x,z) = \int_0^z \phi(x,t)t\,dt + \int_0^z \psi(x,t)t\,dt$, we define

$$S(u) = \int_\Omega S_0(x, |\nabla u|)dx. \tag{3.2}$$

An important role in our variational approach is played by the fact that the following assumption holds true for the potentials ϕ and ψ:

(HS_4) For all $x \in \overline{\Omega}$, all $z \in \mathbb{R}^N$, the following estimate is true:

$$0 \leq [\phi(x,z) + \psi(x,z)]\, |z|^2 \leq p_2^+ S_0(x,|z|).$$

In order to obtain our results we must state the growth behavior of the variable exponents:

$$\begin{cases} 1 < p_1^- \leq p_1(x) \leq p_1^+ < p_2^- \leq p_2(x) \leq p_2^+ < q^- \leq q(x) \leq q^+ < p_1^*(x); \\ p_1^*(x) = \frac{Np_1(x)}{N - p_1(x)}. \end{cases} \tag{3.3}$$

Remark 3.1. Taking account on the relation (3.3) and the embedding theorems for variable exponent Lebesgue and Sobolev spaces we will choose $W_0^{1,p_2(x)}(\Omega)$ as functional space for the solutions of problem (P), and for the simplicity of the writing by $\|\cdot\|$ we will denote the norm associated to $W_0^{1,p_2(x)}(\Omega)$ ($\|\cdot\|_{p_2(x)}$).

We can now define the weak solution for the problem (P).

Definition 3.2. We say that $u \in W_0^{1,p_2(x)}(\Omega) \setminus \{0\}$ is a nontrivial weak solution of the problem (P) if

$$\int_\Omega [\phi(x,|\nabla u|) + \psi(x,|\nabla u|)] \nabla u \nabla v dx = \lambda \int_\Omega f(x,u) v dx$$

for all $v \in W_0^{1,p_2(x)}(\Omega)$.

In order to point out the existence and multiplicity results for our problem we define the following energy functional associated to the problem (P) as it follows:

$$\begin{aligned} &E_\lambda : W_0^{1,p_2(x)}(\Omega) \to \mathbb{R} \\ &E_\lambda(z) = S(z) - \lambda T(z), \end{aligned}$$

where $S(z)$ is defined by relation (3.2) and $T(z) = \int_\Omega F(x,z)dx$, with $F(x,z)$ defined as in relation (3.1).

Taking account of [8, Lemmas 3.2, 3.4], some details from [2, Section 4] and of [23, Lemma 3.1] it is easily to observe that E_λ is of class $C^1\big(W_0^{1,p_2(x)}(\Omega), \mathbb{R}\big)$.

In order to reveal the existence and multiplicity of eigenvalues associated to our problem, we will point out that the critical points of the energy functional E_λ. We can observe that the critical points of E_λ are weak solutions for the problem (P):

$$\begin{aligned} \langle E_\lambda(u), \varphi \rangle = &\int_\Omega [\phi(x,|\nabla u|) + \psi(x,|\nabla u|)] \nabla u \nabla \varphi dx \\ &- \lambda \int_\Omega f(x,u) \varphi dx, \quad \text{for all } \varphi \in W_0^{1,p_2(x)}(\Omega). \end{aligned}$$

Definition 3.3. We say that $E_\lambda \in C^1\big(W_0^{1,p_2(x)}(\Omega), \mathbb{R}\big)$ fulfills the $(C)_c$-condition if for any sequence $(u_n)_n \subset W_0^{1,p_2(x)}(\Omega)$ the following relation holds true:

$$E_\lambda(u_n) \to c \quad \text{and} \quad \|E_\lambda'(u_n)\|_{W^{-1,p_2'(x)}(\Omega)}(1 + \|u_n\|) \to 0$$

we can find a convergent subsequence.

A central role in the proof of the main results of this paper is played by the Fountain Theorem. As we have seen in the Section 2, the variable exponent Sobolev spaces are reflexive and separable Banach spaces. Therefore, taking account of the Remark 3.1, we consider that for $W_0^{1,p_2(x)}(\Omega)$ we have $(e_j)_j \subset W_0^{1,p_2(x)}(\Omega)$ and $(e_j^*) \subset W^{-1,p_2'(x)}(\Omega)$ such that

$$\begin{aligned} W_0^{1,p_2(x)}(\Omega) &= \overline{\text{span}\{e_j : j = 1,2,\dots\}} \\ W^{-1,p_2'(x)}(\Omega) &= \overline{\text{span}\{e_j^* : j = 1,2,\dots\}} \end{aligned}$$

and

$$\langle e_i, e_j^* \rangle = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j, \end{cases}$$

where $\langle \cdot, \cdot \rangle$ represents the duality product between $W_0^{1,p_2(x)}(\Omega)$ and $W^{-1,p_2'(x)}(\Omega)$. We define

$$\begin{cases} X_j = \operatorname{span}\{e_j\}, \\ Y_k = \bigoplus_{j=1}^{k} X_j, \\ Z_k = \overline{\bigoplus_{j=k}^{\infty} X_j}. \end{cases} \tag{3.4}$$

Theorem 3.4 (Fountain Theorem [18]). *Let $E \in C^1(X)$ be an even functional, where $(X, \|\cdot\|)$ is a separable and reflexive Banach space. Suppose that for every $k \in \mathbb{N}$ large enough, there exists $\rho_k > r_k > 0$ such that*

(i) $\inf\{E(u) : u \in Z_k, \|u\| = r_k\} \to +\infty$ *as* $k \to +\infty$,

(ii) $\max\{E(u) : u \in Y_k, \|u\| = \rho_k\} \leq 0$,

(iii) *E satisfies the Palais–Smale condition for every $c > 0$.*

Then E has a sequence of critical values tending to $+\infty$.

For more details and applications on the Fountain Theorem we refer to X. Fan, Q. Zhang [6], D. Repovš [18] and V. F. Uță [22]. A comprehensive study for various forms of this theorem and its extensions can be found in the following works of Y. Jabri [7], P. Pucci, V. Rădulescu [11], P. Pucci, J. Serrin [14] and P. Pucci, J. Serrin [13]. Also for double phase problems we recommend the following work of P. Pucci, V. Rădulescu [12], M. Ragusa, A. Tachikawa [15], X. Shi, V. Rădulescu, D. Repovš, Q. Zhang [20].

We proceed now to prove some helpful propositions.

Proposition 3.5. *Suppose that conditions (HS_1)–(HS_4), (R_2)–(R_4) hold true, then every $(C)_c$ sequence associated to the energy functional E_λ is bounded.*

Proof. Let $(u_n)_n \subset W_0^{1,p_2(x)}(\Omega)$ be a $(C)_c$ sequence. In order to prove that it is bounded we argue by contradiction and suppose that

$$\|u_n\| \to +\infty \quad \text{as } n \to \infty. \tag{3.5}$$

Using the above relation and taking n large enough we obtain that:

$$\begin{aligned} c + 1 &\geq E_\lambda(u_n) - \frac{1}{p_2^+}\langle E_\lambda'(u_n), u_n\rangle \\ &= \int_\Omega S_0(x, |\nabla u_n|)dx - \frac{1}{p_2^+}\int_\Omega [\phi(x, |\nabla u_n|) + \psi(x, |\nabla u_n|)]\,|\nabla u_n|^2 dx \\ &\quad - \lambda \int_\Omega F(x, u_n)dx + \frac{\lambda}{p_2^+}\int_\Omega f(x, |\nabla u_n|)u_n dx \end{aligned} \tag{3.6}$$

Now using hypothesis (HS_4) we get that:

$$\begin{aligned} c + 1 &\geq \int_\Omega \left(1 - \frac{p_2^+}{p_2^+}\right) S_0(x, |\nabla u_n|)dx - \lambda \int_\Omega F(x, u_n)dx + \frac{\lambda}{p_2^+}\int_\Omega f(x, u_n)u_n dx \\ &\geq -\lambda \int_\Omega F(x, u_n)dx + \frac{\lambda}{p_2^+}\int_\Omega f(x, u_n)u_n dx. \end{aligned}$$

By assumption (R_4) we obtain that

$$c+1 \geq \lambda \int_\Omega \mathcal{R}(x,u_n)dx.$$

As we supposed, the relation (3.5) holds true, then for n sufficiently large we have that $\|u_n\|>1$. Hence by the fact that $(u_n)_n$ is a $(C)_c$-sequence we obtain that:

$$\begin{aligned} 0 = \lim_{n\to\infty} \frac{c+o(1)}{\|u_n\|^{p_1^-}} &= \lim_{n\to\infty} \frac{E_\lambda(u_n)}{\|u_n\|^{p_1^-}} \\ &\geq \frac{\displaystyle\int_\Omega S_0(x,|\nabla u_n|)dx - \lambda \int_\Omega F(x,u_n)dx}{\|u_n\|^{p_1^-}}. \end{aligned} \tag{3.7}$$

Now using (HS_4), and (HS_3) we obtain that

$$\begin{aligned} 0 &\geq \frac{\displaystyle\frac{1}{p_2^+}\int_\Omega \left[\phi(x,|\nabla u_n|)+\psi(x,|\nabla u_n|)\right]|\nabla u_n|^2dx - \lambda\int_\Omega F(x,u_n)dx}{\|u_n\|^{p_1^-}} \\ &\geq \frac{\displaystyle\frac{1}{p_2^+}\int_\Omega C_{\phi,\psi}\left(|\nabla u_n|^{p_1(x)}+|\nabla u_n|^{p_2(x)}\right)dx - \lambda\int_\Omega F(x,u_n)dx}{\|u_n\|^{p_1^-}}. \end{aligned}$$

Now using the modular properties (2.3), (2.4) we obtain that

$$S(u) \geq \frac{C_{\phi,\psi}}{p_2^+}\left(\|u_n\|^{p_1^-}_{p_1(x)} + \|u_n\|^{p_2^-}\right)$$

Now taking account of the fact that by relation (3.3) $p_1^- < p_2^-$, we have that

$$S(u) \geq \frac{C_{\phi,\psi}}{p_2^+}\|u_n\|^{p_1^-}.$$

Hence from (3.7) we obtain that

$$0 \leq \frac{C_{\phi,\psi}}{p_2^+}\frac{\|u_n\|^{p_1^-}}{\|u_n\|^{p_1^-}} - \frac{\lambda\displaystyle\int_\Omega F(x,u_n)dx}{\|u_n\|^{p_1^-}},$$

which yields to

$$\frac{C_{\phi,\psi}}{p_2^+\lambda} \leq \limsup_{n\to\infty}\int_\Omega \frac{|F(x,u_n)|}{\|u_n\|}dx. \tag{3.8}$$

Let $0 \leq a \leq b$ and $D_n^{a,b} = \{z\in\Omega:\ a \leq |u_n(z)| < b\}$.

Consider in what follows $w_n = \frac{u_n}{\|u_n\|}$. It is obvious that $\|w_n\|=1$ and there exist a nonnegative constant C_w such that $|w_n|_{q(x)} \leq C_2\|w_n\| = C_w$.

As a consequence of the above facts (passing eventually to a subsequence), we can find an element w_0 such that $w_n \rightharpoonup w_0$ in $W_0^{1,p_2(x)}(\Omega)$.

Moreover,

$$\begin{aligned} w_n \to w_0 \quad &\text{in } L^{r(x)}(\Omega), \qquad 1\leq r(x) < p_1^*(x) \\ w_n(x) \to w_0(x) \quad &\text{a.e. on } \Omega. \end{aligned} \tag{3.9}$$

In what follows we have to split the proof in two cases:

(I) $w_0 = 0$;

(II) $w_0 \neq 0$.

Let firstly assume that $w_0 = 0$.

We obtain that

$$\begin{cases} w_n \to 0 \text{ in } L^{r(x)}(\Omega) \\ w_n(x) \to 0 \text{ a.e. on } \Omega \end{cases}$$

and by assumption (R_2) we have that

$$\int_{D_n^{0,\rho}} \frac{|F(x,u_n)|}{\|u_n\|^{p_1^-}} dx \leq \frac{C(\rho + \rho^{\overline{q}})|\Omega|}{\|u_n\|^{p_1^-}} \to 0, \tag{3.10}$$

where $\overline{q} = q^+$ if $\rho \geq 1$ and $\overline{q} = q^-$ if $\rho < 1$.

Let $\mu'(x)$ be the conjugate exponent for $\mu(x)$, i.e., $\mu'(x) = \frac{\mu(x)}{\mu(x)-1}$, by hypothesis (R_4) we have that $\mu^- > \max\left\{1, \frac{N}{p_1^-}\right\}$, hence $1 < p_1^- \mu'(x) < p_1^*(x)$. Therefore we get that $w_n \to 0$ in $L^{p_1^- \mu'(x)}(\Omega)$ as $n \to \infty$.

Using Remark 2.6, assumption (R_4), relation (3.6) and (3.9) one have that

$$\begin{aligned}
\int_{D_n^{\rho,+\infty}} \frac{|F(x,u_n)|}{|u_n|^{p_1^-}} |w_n|^{p_1^-} dx &\leq 2 \left| \frac{|F(x,u_n)|}{|u_n|^{p_1^-}} \right|_{L^{\mu(x)}(D_n^{\rho,+\infty})} \left| |w_n|^{p_1^-} \right|_{L^{\mu'(x)}(D_n^{\rho,+\infty})} \\
&\leq 2 \max\left\{ \left(\int_{D_n^{\rho,+\infty}} \frac{|F(x,u_n)|^{\mu(x)}}{|u_n|^{p_1^- \mu(x)}} dx \right)^{\frac{1}{\mu^+}}, \left(\int_{D_n^{\rho,+\infty}} \frac{|F(x,u_n)|^{\mu(x)}}{|u_n|^{p_1^- \mu(x)}} dx \right)^{\frac{1}{\mu^-}} \right\} \\
&\quad \cdot \max\left\{ \left(\int_{D_n^{\rho,+\infty}} |w_n|^{p_1^- \mu'(x)} dx \right)^{\frac{1}{(\mu')^+}}, \left(\int_{D_n^{\rho,+\infty}} |w_n|^{p_1^- \mu'(x)} dx \right)^{\frac{1}{(\mu')^-}} \right\} \\
&\leq 2 \max\left\{ \left(\int_{D_n^{\rho,+\infty}} \mathcal{R}(x,u_n) dx \right)^{\frac{1}{\mu^+}}, \left(\int_{D_n^{\rho,+\infty}} \mathcal{R}(x,u_n) dx \right)^{\frac{1}{\mu^-}} \right\} \\
&\quad \cdot \max\left\{ \left(\int_{D_n^{\rho,+\infty}} |w_n|^{p_1^- \mu'(x)} dx \right)^{\frac{1}{(\mu')^-}}, \left(\int_{D_n^{\rho,+\infty}} |w_n|^{p_1^- \mu'(x)} dx \right)^{\frac{1}{(\mu')^+}} \right\} \\
&\leq 2 \max\left\{ \left(\frac{C_1}{\lambda}(c+1)^{\frac{1}{\mu^+}} \right), \left(\frac{C_1}{\lambda}(c+1)^{\frac{1}{\mu^-}} \right) \right\} \\
&\quad \cdot \max\left\{ \left(\int_{D_n^{\rho,+\infty}} |w_n|^{p_1^- \mu'(x)} dx \right)^{\frac{1}{(\mu')^-}}, \left(\int_{D_n^{\rho,+\infty}} |w_n|^{p_1^- \mu'(x)} dx \right)^{\frac{1}{(\mu')^-}} \right\} \\
&\to 0 \quad \text{as } n \to +\infty.
\end{aligned} \tag{3.11}$$

By relations (3.10) and (3.11), one have that

$$\begin{aligned}
\int_\Omega \frac{|F(x,u_n)|}{\|u_n\|^{p_1^-}} dx &= \int_{D_n^{0,\rho}} \frac{|F(x,u_n)|}{\|u_n\|^{p_1^-}} dx + \int_{D_n^{\rho,+\infty}} \frac{|F(x,u_n)|}{\|u_n\|^{p_1^-}} dx \\
&= \int_{D_n^{0,\rho}} \frac{|F(x,u_n)|}{\|u_n\|^{p_1^-}} dx + \int_{D_n^{\rho,+\infty}} \frac{|F(x,u_n)|}{|u_n|^{p_1^-}} |w_n|^{p_1^-} dx \\
&\to 0 \quad \text{as } n \to +\infty.
\end{aligned} \tag{3.12}$$

which is a contradiction with the fact that $\limsup_{n\to\infty}\int_\Omega \frac{|F(x,u_n)|}{\|u_n\|^{p_1^-}}dx > 0$.

We proceed now to prove the second case, and assume that $w_0 \neq 0$. Therefore there exists D^* such that $D^* := \{z \in \Omega : w_0(z) \neq 0\}$, with $|D^*| > 0$, where $|D^*|$ is the Lebesgue measure of D^*.

So, for almost every $z \in D^*$, we have that

$$\lim_{n\to\infty} |u_n(z)| = +\infty. \tag{3.13}$$

Therefore, by (3.13) we get that $D^* \subset D_n^{\rho,+\infty}$ for $n \in \mathbb{N}$ sufficiently large.

With similar arguments as above (see relation (3.10)) it yields that

$$\int_{D_n^{0,\rho}} \frac{|F(x,u_n)|}{\|u_n\|^{p_2^+}}dx \leq \frac{C(\rho+\rho^{\overline{q}})|\Omega|}{\|u_n\|^{p_2^+}} \to 0 \quad \text{as } n \to \infty \tag{3.14}$$

By hypotheses (R_2), (R_3), relation (3.14) and taking use of the Fatou's Lemma one have that

$$\begin{aligned} 0 = \lim_{n\to\infty}\frac{c+o(1)}{\|u_n\|^{p_2^+}} &= \lim_{n\to\infty}\frac{E_\lambda(u_n)}{\|u_n\|^{p_2^+}} \\ \leq \lim_{n\to\infty}&\left[\frac{\int_\Omega S_0(x,|\nabla u_n|)}{\|u_n\|^{p_2^+}}dx - \frac{\lambda}{\|u_n\|^{p_2^+}}\int_\Omega F(x,u_n)dx\right]. \end{aligned} \tag{3.15}$$

In order to complete our proof and obtain the desired contradiction we will compute the term of the energy functional driven by the double-phase operator and the term driven by the reaction function separately.

We firstly compute the part driven by the differential operator. So, taking use of the fact that $\|u_n\| \to \infty$ as $n \to \infty$, using hypothesis (HS_2), Hölder's inequality and the fact that $W_0^{1,p_2(x)}(\Omega) \hookrightarrow W_0^{1,p_1(x)}(\Omega)$ continuously (and one have $\|u\|_{p_1(x)} \leq C_{p_1}\|u\|$, for some nonnegative constant C_{p_1}) we obtain that

$$\begin{aligned} S(x,|\nabla u_n|) &= \int_\Omega S_0(x,|\nabla u_n|)dx \\ &\leq C_\phi|\alpha_1|_{p_1'(x)}\|u_n\|_{p_1(x)}^{p_1^+} + \frac{\xi}{p_1^-}\|u_n\|_{p_1(x)}^{p_1^+} + C_\psi|\alpha_2|_{p_2'(x)}\|u_n\|^{p_2^+} + \frac{\xi}{p_2^+}\|u_n\|^{p_2^+} \\ &\leq C_M\|u_n\|^{p_2^+} \end{aligned} \tag{3.16}$$

where $C_M = \left(C_\phi|\alpha_1|_{p_1'(x)}\cdot C_{p_1} + \frac{\xi}{p_1^-}C_{p_1}\right) + \left(C_\psi|\alpha_2|_{p_2'(x)} + \frac{\xi}{p_2^-}\right)$, and C_ϕ, C_ψ are two nonnegative constants, $C_\phi, C_\psi > 0$, which depend on the potential functions ϕ, ψ and on the continuous embeddings: $W_0^{1,p_1(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega)$, $W_0^{1,p_2(x)}(\Omega) \hookrightarrow W_0^{1,p_1(x)}(\Omega)$.

We proceed now to compute the second part of the energy functional driven by our reaction function term of the problem (P).

Combining relations (3.15) and (3.16)

$$\begin{aligned} 0 &\leq \lim_{n\to\infty}\left[\frac{C_M\|u_n\|^{p_2^+}}{\|u_n\|^{p_2^+}} - \lambda\left(\int_{D_n^{0,\rho}}\frac{F(x,u_n)}{\|u_n\|^{p_2^+}}dx + \int_{D_n^{\rho,+\infty}}\frac{F(x,u_n)}{\|u_n\|^{p_2^+}}dx\right)\right] \\ &= \lim_{n\to\infty}\left[C_M - \frac{\lambda}{\|u_n\|^{p_2^+}}\left(\int_{D_n^{0,\rho}}F(x,u_n)dx + \int_{D_n^{\rho,+\infty}}F(x,u_n)dx\right)\right] \end{aligned} \tag{3.17}$$

$$
\begin{aligned}
&\leq \lim_{n\to\infty}\left[C_M - \frac{\lambda}{\|u_n\|^{p_2^+}}\int_{D_n^{\rho,+\infty}} F(x,u_n)dx\right]\\
&\leq \limsup_{n\to\infty}\left[C_M - \frac{\lambda}{\|u_n\|^{p_2^+}}\int_{D_n^{\rho,+\infty}} F(x,u_n)dx\right]\\
&= C_M - \liminf_{n\to\infty}\lambda\int_{D_n^{\rho,+\infty}} \frac{F(x,u_n)}{|u_n|^{p_2^+}}|w_n|^{p_2^+}dx\\
&= C_M - \liminf_{n\to\infty}\lambda\int_{\Omega} \frac{F(x,u_n)}{|u_n|^{p_2^+}}\chi_{D_n^{\rho,+\infty}}(x)|w_n|^{p_2^+}dx\\
&\leq C_M - \lambda\int_{\Omega} \liminf_{n\to\infty}\frac{F(x,u_n)}{|u_n|^{p_2^+}}\chi_{D_n^{\rho,+\infty}}(x)|w_n|^{p_2^+}dx\\
&\to -\infty, \quad \text{as } n\to\infty,
\end{aligned}
$$

which contradicts relation (3.17).

Therefore we obtained the fact that any $(C)_c$-sequence is bounded, so our proof is complete. □

Proposition 3.6. *If assumptions (R_1)–(R_4) hold true, then for every $(C)_c$-sequence of E_λ we can find a convergent subsequence in $W_0^{1,p_2(x)}(\Omega)$.*

Proof. Suppose that $(v_n)_n \subset W_0^{1,p_2(x)}(\Omega)$ is a $(C)_c$-sequence for E_λ. Using Proposition 3.5 we have that $(v_n)_n$ is bounded in $W_0^{1,p_2(x)}(\Omega)$, so, passing eventually to a subsequence we obtain the fact that $v_n \rightharpoonup v_0$ in $W_0^{1,p_2(x)}(\Omega)$. Using Remark 2.6 it yields that $(v_n)_n$ is bounded in $L^{q(x)}(\Omega)$ and by the continuous and compact embedding $W_0^{1,p_2(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$, we get that $v_n \to v_0$ in $L^{q(x)}(\Omega)$ as $n\to\infty$.

By straightforward computations we obtain that

$$
\begin{aligned}
&\int_\Omega |f(x,v_n)-f(x,v_0)|\,|v_n-v_0|dx\\
&\quad\leq \int_\Omega (|f(x,v_n)|+|f(x,v_0)|)\,|v_n-v_0|dx\\
&\quad\leq \int_\Omega \left[C(1+|v_n|^{q(x)-1})+C(1+|v_0|^{q(x)-1})\right]|v_n-v_0|dx\\
&\quad\leq 2C\int_\Omega |v_n-v_0|dx + C\int_\Omega |v_n|^{q(x)-1}|v_n-v_0|dx + \int_\Omega |v_0|^{q(x)-1}|v_n-v_0|dx \qquad (3.18)\\
&\quad\leq 2C|v_n-v_0|_{L^1(\Omega)} + 2C\left||v_n|^{q(x)-1}\right|_{q'(x)}\cdot|v_n-v_0|_{q(x)} + 2C\left||v_0|^{q(x)-1}\right|_{q'(x)}|v_n-v_0|_{q(x)}\\
&\quad\leq 2C|v_n-v_0|_{L^1(\Omega)} + 2C\max\left\{|v_n|_{q(x)}^{q^+-1},|v_n|_{q(x)}^{q^--1}\right\}\cdot|v_n-v_0|_{q(x)}\\
&\qquad + 2C\max\left\{|v_0|_{q(x)}^{q^+-1},|v_0|_{q(x)}^{q^--1}\right\}|v_n-v_0|_{q(x)} \to 0 \quad \text{as } n\to\infty,
\end{aligned}
$$

where $q'(x)$ is the conjugate exponent of $q(x)$, i.e. $\frac{1}{q(x)}+\frac{1}{q'(x)}=1$.

Now taking account of [8, Lemma 3.2] one have that:

$$
\begin{aligned}
&\langle S'(v_n)-S'(v_0), v_n-v_0\rangle\\
&\qquad = \langle E_\lambda'(v_n)-E_\lambda'(v_0), v_n-v_0\rangle + \int_\Omega [f(x,v_n)-f(x,v_0)]\,(v_n-v_0)dx \qquad (3.19)
\end{aligned}
$$

and by Definition 3.3, keeping in mind that $(v_n)_n$ is a $(C)_c$-sequence of the energy functional E_λ, we get that:

$$\lim_{n\to\infty} \langle E'_\lambda(v_n) - E'_\lambda(v_0), v_n - v_0\rangle = 0. \tag{3.20}$$

Now by relations (3.18), (3.19), (3.20) and taking account of [8, Lemma 3.4] we obtain the fact that

$$\lim_{n\to\infty} \langle S'(v_n) - S'(v_0), v_n - v_0\rangle = 0,$$

and by the fact that S is of type $(S)_+$ (see also [8, Lemma 3.4]) it yields that $v_n \to v_0$ in $W_0^{1,p_2(x)}(\Omega)$, and so, our proof is complete. □

Proposition 3.7. *If assumptions (R_1)–(R_3) and (R_5) hold true, then for every $(C)_c$-sequence of E_λ, we can find a convergent subsequence in $W_0^{1,p_2(x)}(\Omega)$.*

Proof. Taking use of Proposition 3.5, and keeping in mind the proof of Proposition 3.6 we only have to prove that our sequence is bounded in $W_0^{1,p_2(x)}(\Omega)$.

Let $(v_n)_n \subset W_0^{1,p_2(x)}(\Omega)$ be a $(C)_c$-sequence for E_λ. Arguing by contradiction we suppose that $\|v_n\| \to \infty$ as $n \to \infty$. Now, taking $w_n = \frac{v_n}{\|v_n\|}$, we get that $\|w_n\| = 1$, for all $n \in \mathbb{N}$, futhermore we obtain that $|w_n|_{q(x)} \le C_w\|w_n\|$, where $C_w > 0$ is a constant.

By the above facts and passing eventually to a subsequence we may find w_0 such that

$$w_n \rightharpoonup w_0 \quad \text{in } W_0^{1,p_2(x)}(\Omega), \tag{3.21}$$

and by the compact embedding $W_0^{1,p_2(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$ we obtain that

$$\begin{aligned} w_n &\to w_0 \quad \text{in } L^{q(x)}(\Omega) \\ w_n(x) &\to w_0(x) \quad \text{a.e. on } \Omega. \end{aligned} \tag{3.22}$$

Now by the definition of E_λ it yields that:

$$\begin{aligned} c+1 &\ge E_\lambda(v_n) - \frac{1}{\omega}\langle E'_\lambda(v_n), v_n\rangle \\ &= \int_\Omega S_0(x,|\nabla v_n|) - \frac{1}{\omega}\left[\phi(x,|\nabla v_n|)\nabla v_n + \psi(x,|\nabla v_n|)\nabla v_n\right] dx \\ &\quad + \lambda\int_\Omega \left[\frac{1}{\omega} f(x,v_n)v_n - F(x,v_n)\right] dx. \end{aligned}$$

Now by hypothesis (HS_4) combined with (HS_3) we get that

$$\begin{aligned} c+1 &\ge \int_\Omega \left(1 - \frac{p_2^+}{\omega}\right) S_0(x,|\nabla v_n|)dx + \lambda\int_\Omega \left[\frac{1}{\omega} f(x,v_n)v_n - F(x,v_n)\right] dx \\ &\ge \int_\Omega \left(1 - \frac{p_2^+}{\omega}\right)\cdot\frac{C_{\phi,\psi}}{p_2^+}\left[|\nabla v_n|^{p_1(x)-2} + |\nabla v_n|^{p_2(x)-2}\right]|\nabla v_n|^2 dx \\ &\quad + \lambda\int_\Omega \left[\frac{1}{\omega} f(x,v_n)v_n - F(x,v_n)\right] dx \\ &\ge \left(1 - \frac{p_2^+}{\omega}\right)\cdot\frac{C_{\phi,\psi}}{p_2^+}\int_\Omega |\nabla v_n|^{p_2(x)}dx + \lambda\int_\Omega \left[\frac{1}{\omega} f(x,v_n)v_n - F(x,v_n)\right] dx. \end{aligned}$$

Using assumption (R_5) we obtain that:

$$c+1 \geq \left(1-\frac{p_2^+}{\omega}\right)\cdot\frac{C_{\phi,\psi}}{p_2^+}\|v_n\|^{p_2^-}-\frac{\lambda}{\omega}\eta\int_\Omega |v_n|^{p_1^-}dx$$

for any $n \geq 0$.

Hence passing to $(w_n)_n$ we obtain that:

$$\begin{aligned}\frac{\lambda}{\omega}\eta\int_\Omega |w_n|^{p_1^-}dx &\geq \left(1-\frac{p_2^+}{\omega}\right)\cdot\frac{C_{\phi,\psi}}{p_2^+}\\ &\Rightarrow \frac{\lambda\eta}{\omega}\cdot\frac{\omega p_2^+}{(\omega-p_2^+)C_{\phi,\psi}}\int_\Omega |w_n|^{p_1^-}dx \geq 1\\ &\Rightarrow \frac{\lambda\eta p_2^+}{(\omega-p_2^+)C_{\phi,\psi}}|w_n|_{p_1^-}^{p_1^-}\geq 1\\ &\Rightarrow \frac{\lambda\eta p_2^+}{(\omega-p_2^+)C_{\phi,\psi}}\limsup_{n\to\infty}|w_n|_{p_1^-}^{p_1^-}\geq 1 \qquad (3.23)\end{aligned}$$

Now, keeping in mind relations (3.21) and (3.22) we have that $w_n \to w_0$ in $L^{p_1^-}(\Omega)$, moreover by (3.23) we get that $w_0 \neq 0$.

In order to obtain the desired contradiction we apply the same technique as in the case (II) from the proof of Proposition 3.5 and the contradiction is obtained.

Therefore we have the fact that $(v_n)_n$ is bounded in $W_0^{1,p_2(x)}(\Omega)$. In order to complete the proof we only have to repeat the steps taken in the proof of Proposition 3.6 and the work is accomplished. □

4 Main results

In this section using the Fountain Theorem we will reveal the fact that the problem (P) has an unbounded sequence of weak solutions with higher and higher energies.

We are now ready to enunciate and prove our main results.

Theorem 4.1. *If assumptions (HS_1)–(HS_4), (R_1)–(R_4), (R_6) and (3.3) hold true, then for every $\lambda > 0$ the problem (P) possesses an infinite sequence of nontrivial weak solutions.*

Theorem 4.2. *If assumptions (HS_1)–(HS_4), (R_1)–(R_3), (R_5), (R_6) and (3.3) hold true, then for every $\lambda > 0$ the problem (P) possesses an infinite sequence of nontrivial weak solutions.*

Proof of Theorem 4.1. As we have seen in the previous section, as $W_0^{1,p_2(x)}(\Omega)$ is separable, reflexive Banach space, let us consider Y_k and Z_k denoted by relation (3.4).

Firstly we check if condition (i) from the Theorem 3.4 holds true.

Let $a_k := \sup\left\{|u|_{q(x)} : \|u\| = 1, u \in Z_k\right\}$. It is easily to observe the fact that $a_k \to 0$ as $k \to \infty$. The reasoning behind the above statement is the following. By the definition of $(a_k)_k$ we get that $a_k > a_{k+1} \geq 0$, therefore $a_k \to a \geq 0$, as $k \to \infty$. By the reflexivity of $W_0^{1,p_2(x)}(\Omega)$, and taking $u_k \in Z_k$, $\|u_k\| = 1$ for each $k \in \mathbb{N}$ such that

$$0 \leq a_k - |u_k|_{q(x)} \leq \frac{1}{k},$$

we get that $(u_k)_k$ has a convergent subsequence and suppose $u_k \rightharpoonup u_1$ in $W_0^{1,p_2(x)}(\Omega)$. Keeping in mind the definition of Z_k we obtain that $u_1 = 0$. Taking account of [8, Lemma 3.4] we have that $u_k \to 0$ in $L^{q(x)}(\Omega)$, so it yields that $a = 0$.

Now let $u \in Z_k$ with $\|u\| = \rho_k > 1$, where ρ_k will be specified later.

Using hypotheses (HS_3) and (HS_4) and (2.3) we have that

$$\begin{aligned} E_\lambda(u) &= \int_\Omega S_0(x, |\nabla u|)dx - \lambda \int_\Omega F(x,u)dx \\ &\geq \frac{C_{\phi,\psi}}{p_2^+}\left(\int_\Omega |\nabla u|^{p_1(x)}dx + \|u\|^{p_2^-}\right) - \lambda \int_\Omega F(x,u)dx \\ &\geq \frac{C_{\phi,\psi}}{p_2^+}\|u\|^{p_2^-} - \lambda \int_\Omega F(x,u)dx. \end{aligned} \tag{4.1}$$

Now using assumption (R_2) we get that

$$F(x,z) \leq C(|z| + |z|^{q(x)}) \leq 2C(1 + |z|^{q(x)}) \tag{4.2}$$

for all $(x,z) \in \Omega \times \mathbb{R}$.

Using (4.1) and (4.2) we obtain that

$$\begin{aligned} E_\lambda(u) \geq &\frac{C_{\phi,\psi}}{p_2^+}\|u\|^{p_2^+} - 2\lambda C \int_\Omega \left(1 + |u|^{q(x)}\right)dx \\ \geq &\frac{C_{\phi,\psi}}{p_2^+}\|u\|^{p_2^+} - 2\lambda C\left[|\Omega| + \max\left\{|u|_{q(x)}^{q^-}, |u|_{q(x)}^{q^+}\right\}\right] \\ &\text{(where } |\Omega| \text{ represents the Lebesgue measure of } \Omega\text{).} \end{aligned}$$

Taking account of the continuous embedding $W_0^{1,p_2(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$ we have $|u|_{q(x)} \leq C_3\|u\|$ and then the above inequality becomes:

$$\begin{aligned} E_\lambda(u) &\geq \frac{C_{\phi,\psi}}{p_2^-}\|u\|^{p_2^-} - 2\lambda C\left[|\Omega| + \max\left\{C_3^{q^-}\|u\|^{q^-}, C_3^{q^+}\|u\|^{q^+}\right\}\right] \\ &\geq \frac{C_{\phi,\psi}}{p_2^-}\|u\|^{p_2^-} - 2\lambda C\tilde{C}^q\|u\|^{q^+} - 2\lambda C|\Omega| \\ &\quad \text{(where } \tilde{C}^q = \max\left\{C_3^{q^+}, C_3^{q^-}\right\}\text{)} \\ &\geq \frac{C_{\phi,\psi}}{p_2^-}\|u\|^{p_2^-} - 2\lambda C\tilde{C}^q a_k^{q^+}\|u\|^{q^+} - 2\lambda C|\Omega|. \end{aligned}$$

It can be easily checked that if we choose

$$\rho_k = \left(\frac{2\lambda C\tilde{C}^q}{C_{\phi,\psi}} \cdot p_2^- a_k^{p_2^+}\right)^{\frac{1}{p_2^- - q^+}} \tag{4.3}$$

combined with the fact that $p_2^- < q^+$ and $a_k \to 0$ as $k \to +\infty$, we obtain that $\rho_k \to +\infty$ as $k \to +\infty$.

Taking $\|u\| = \rho_k$ with ρ_k as stated in relation (4.3) we obtain that

$$E_\lambda(u) \to +\infty \quad \text{as } k \to +\infty,$$

and so, the validity of condition (i) is proved.

We check now if the condition (ii) from the Fountain Theorem holds true.
Assume that $u \in Y_k$ and $\|u\| = \tau_k > 1$, where τ_k will be specified later.
By hypothesis (HS_2) we have that

$$E_\lambda(u) \leq 2C_4|\alpha_1|_{p_1'(x)} \max\left\{\|u\|_{p_1(x)}^{p_1^-}, \|u\|_{p_1(x)}^{p_1^+}\right\} + \frac{\xi}{p_1^-} \max\left\{\|u\|_{p_1(x)}^{p_1^-}, \|u\|_{p_1(x)}^{p_1^+}\right\}$$
$$+ 2C_5|\alpha_2|_{p_2'(x)}\|u\|^{p_2^+} + \frac{\xi}{p_2^-}\|u\|^{p_2^+} - \lambda \int_\Omega F(x,u)dx,$$

where $C_4,\ C_5$ are some strictly nonnegative constants.

Taking account of the continuous embedding described in Remark 2.5 we obtain that

$$E_\lambda(u) \leq C_6\|u\|^{p_2^+} - \lambda \int_\Omega F(x,u)dx \tag{4.4}$$

where $C_6 = \left(2C_4|\alpha_1|_{p_1'(x)}C_{p_1} + \frac{\xi}{p_1^-}C_{p_1}\right) + \left(2C_5|\alpha_2|_{p_2'(x)} + \frac{\xi}{p_2^-}\right)$, and $C_{p_1} = \max\left\{C_2^{p_1^-}, C_2^{p_2^+}\right\}$.

In order to complete the proof of condition (ii), we argue by contradiction and assume that (ii) is not true for some given n. Hence we can find a sequence $(v_n)_n \subset Y_n$ such that

$$\|v_n\| \to +\infty \quad \text{as } n \to +\infty \quad \text{and} \quad E_\lambda(v_n) \geq 0. \tag{4.5}$$

Suppose now that $w_n = \frac{v_n}{\|v_n\|}$, therefore $\|w_n\| = 1$. As $\dim Y_k < +\infty$, then we can find some $w_0 \in Y_k \setminus \{0\}$ such that, passing eventually to a subsequence we get that

$$\begin{cases} w_n \to w_0, \\ w_n(x) \to w_0(x) \text{ a.e. } x \in \Omega \end{cases} \quad \text{as } n \to +\infty.$$

As $w(x) \neq 0$, we get that $|v_n(x)| \to +\infty$ as $n \to +\infty$. Taking account of hypothesis (R_3) we obtain that

$$\lim_{n\to+\infty} \frac{F(x,|v_n(x)|)}{\|v_n\|^{p_2^+}} = \lim_{n\to+\infty} \frac{F(x,v_n(x))}{|v_n(x)|^{p_2^+}}|w_n(x)|^{p_2^+} = +\infty$$

for all $x \in D_0 := \{x \in \Omega : \ w(x) \neq 0\}$. With the same arguments as in the proof of Proposition 3.6 we get that

$$\int_{D_0} \frac{F(x,v_n)}{\|v_n\|^{p_2^+}}dx \to +\infty \quad \text{as } n \to +\infty.$$

Taking $n \in \mathbb{N}$, large enough we have that $D_0 \subset D_n^{\rho,+\infty}$ (the domain considered in the proof of Proposition 3.5), and so the following estimates hold true:

$$\begin{aligned}
E_\lambda(v_n) &\leq C_6\|v_n\|^{p_2^+} - \lambda\left[\int_{D_n^{0,\rho}} F(x,v_n)dx + \int_{D_n^{\rho,+\infty}} F(x,v_n)dx\right] \\
&\leq C_6\|v_n\|^{p_2^+} + C_7\int_{D_n^{0,\rho}} (\rho+\rho^q)\,dx - \int_{D_n^{\rho,+\infty}} F(x,v_n)dx \\
&\quad \text{(where } C_7 > 0 \text{ is some nonnegative constant)} \\
&\leq C_6\|v_n\|^{p_2^+} + C_7\,(\rho+\rho^q)\,|\Omega| - \int_{D_n^{\rho,+\infty}\cap D_0} F(x,v_n)dx \\
&\leq \|v_n\|^{p_2^+}\left(C_6 + \frac{C\,(\rho+\rho^q)\,|\Omega|}{\|v_n\|^{p_2^+}} - \int_{D_n^{\rho,+\infty}\cap D_0} \frac{F(x,v_n)}{\|v_n\|^{p_2^+}}dx\right) \\
&\to -\infty \quad \text{as } n \to +\infty,
\end{aligned}$$

which is a contradiction with relation (4.5) and so we have completed the proof that condition (ii) holds true.

As in Proposition 3.6 we have proved that the energy functional E_λ verifies the $(C)_c$-condition and by hypothesis (R_6) the function that gives the reaction term of our equation is odd we can conclude the proof of Theorem 4.1 by simply applying the Fountain Theorem. □

Remark 4.3. Taking account of the above theorem we have proved that for every $\lambda > 0$ we have an unbounded sequence of solutions obtained for higher and higher energies.

Proof of Theorem 4.2. With the same arguments as in the proof of Theorem 4.1, we can point out that condition (i) of the Fountain Theorem is checked (as the assumptions (R_4) and (R_5) plays no role in this part of the proof).

To check the validity of condition (ii) from the Fountain Theorem we combine the arguments from the verification of condition (ii) of the proof to Theorem 4.1 with similar arguments as in the proof of Proposition 3.7 and the condition is checked. Therefore, as in the Proposition 3.7 we have verified the fact that our energy functional satisfies the $(C)_c$-condition and by (R_6) the reaction term of our problem is an odd function, and as $E_\lambda(z) = E_\lambda(-z)$ we only have to apply the Fountain Theorem.

Hence for the energy functional E_λ we have obtained an unbounded sequence of critical values $(u_n)_n \subset W_0^{1,p_2(x)}(\Omega)$ such that $E'_\lambda(u_n) \to 0$ and $E_\lambda(u_n) \to c$ as $n \to +\infty$. □

5 Some examples and final remarks

As the definitions of our double phase-operator and of our reaction term are very general, in what follows we will give some specific examples in order to illustrate the validity of our results.

Example 5.1. Consider the following weight coefficient functions $a, b : \Omega \to \mathbb{R}$, with $a, b \in L^\infty(\Omega)_+$ for all $x \in \Omega$. Suppose there exist a constant $C_{a,b} > 0$ such that $a(x), b(x) \geq C_{a,b}$ for all $x \in \Omega$. Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function which satisfy the assumptions $(R_1) - (R_6)$, (3.3) then the results of Theorems 4.1, 4.2 hold true for the following class of Dirichlet problems:

$$\begin{cases} -\operatorname{div}\left[a(x)|\nabla u|^{p_1(x)-2}\nabla u + b(x)|\nabla u|^{p_2(x)-2}\nabla u\right] = \lambda f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

It is easy to check the fact that our differential operator satisfy hypotheses (HS_1)–(HS_4).

Example 5.2. As we stated in the first section of this paper our potential functions ϕ and ψ generalize the following type of differential operator

$$A(x,z) = \left(1 + \frac{z^{p(x)}}{\sqrt{1 + z^{2p(x)}}}\right) z^{p(x)-2} \tag{5.1}$$

corresponding to the differential operator which describes the capillary phenomenon, so we obtain the following class of double-phase problems:

$$\begin{cases} -\operatorname{div}\left[\left(|\nabla u|^{p_1(x)-2}+\frac{|\nabla u|^{2p_1(x)-2}}{\left(1+|\nabla u|^{2p_1(x)}\right)^{1/2}}\right)\nabla u \right. \\ \qquad\qquad \left. +\left(|\nabla u|^{p_2(x)-2}+\frac{|\nabla u|^{2p_2(x)-2}}{\left(1+|\nabla u|^{2p_2(x)}\right)^{1/2}}\right)\nabla u\right]=\lambda f(x,u) & \text{in } \Omega, \\ u=0 & \text{on } \partial\Omega, \end{cases}$$

If hypotheses (3.3), (R_1)–(R_6) hold true, then the results of Theorems 4.1 and 4.2 hold true for this class of problems, i.e., this class of problems admits infinitely many nontrivial weak solutions with high and higher energies.

By simple computations we could verify that the potential function of type A from relation (5.1) satisfy the assumptions (HS_1)–(HS_4). For a thorough proof of the validity of our example we can associate the following energy functional to our problem $E_\lambda : W_0^{1,p_2(x)}(\Omega) \to \mathbb{R}$ defined by

$$\begin{aligned} E_\lambda(u) = & \int_\Omega \frac{1}{p_1(x)}\left[|\nabla u|^{p_1(x)}+\left(1+|\nabla u|^{2p_1(x)}\right)^{1/2}\right]dx+ \\ & + \int_\Omega \frac{1}{p_2(x)}\left[|\nabla u|^{p_2(x)}+\left(1+|\nabla u|^{2p_2(x)}\right)^{1/2}\right]dx-\lambda\int_\Omega F(x,u)dx \end{aligned}$$

and recalculate the computations for this functional energy.

In what follows we will give some examples and remarks on the reaction function and the results of this paper.

In order to prove the boundedness of the Palais–Smale sequence it is very popular in the literature to use the (AR)-condition, i.e.,

(AR) There exist some constants $A>0$, $\omega>p_2^+$ such that for $|z|>A$ and for almost every $x\in\Omega$

$$0<\omega F(x,z)\le zf(x,z)$$

where $F(x,z)=\int_0^z f(x,t)dt$.

Remark 5.3. The (AR)-condition described above implies the fact that our reaction function $f(x,\cdot)$ must have at least $(\omega-1)$-polynomial growth near $+\infty$.

Remark 5.4. There exists an entire class of functions that are superlinear at infinity, but does not satisfy the (AR)-condition for any $\omega>p_2^+$.

An example of this type of function is

$$f(x,z)=p_2^+|z|^{p_2^+-2}z\ln(1+z^2), \tag{5.2}$$

and we obtain that

$$F(x,z)=|z|^{p_2^+}\ln\left(1+z^2\right)-\frac{2|z|^{p_2^+}z}{1+z^2}. \tag{5.3}$$

Remark 5.5. It is easily to observe that the function defined in relation (5.2) does not satisfy the (AR)-condition, but it satisfies conditions (R_3) and (R_4), therefore the results of Theorems 4.1 and 4.2 hold true.

Remark 5.6. (i) Similar results but under the stronger hypothesis (i.e. (AR)-condition is to be satisfied by the reaction term of the problem) where obtained for this problem in [22] and in [8](where furthermore the differential operator is driven only by the potential function ϕ).
(ii) Some spectral results for this type of problem which does not use the (AR)-condition where obtained in [22], but with the price of taking the real parameter λ in a small interval near the origin, and the growth of the reaction function to be more general, i.e., $q^- < p_1^-$, but in this case it is not know the behavior of the quantity $\sup_{x\in\Omega} q(x)$.

Remark 5.7. Also for the coercive case of the problem we refer to [2,21], for the double-phase differential operator and to [8] for the simpler case were the differential operator is driven by only one potential term.

Remark 5.8. According to the terminology used in this paper the study of integral functionals described by relations (1.1), (1.2) correspond to differential operators described by (C_1) and relation (1.3). An interesting extension of the results obtained in this paper can be realized by studying this problems in a more general framework of Musielak–Orlicz spaces. To this end we refer to some results described in [17, Chapter 4].

Remark 5.9. An important role in obtaining our results is played by assumptions (3.3) which indicates the fact that we are in the subcritical framework in the sense of Sobolev variable exponents. No results are known in the critical or supercritical framework. Moreover, no results are known even in the "almost critical" case with lack of compactness where (3.3) is replaced by

$$p_1(x) < p_2(x) < q^- \leq q(x) \leq q^+ \preceq p_1^*(x) \quad \text{for all } x \in \overline{\Omega},$$

where $q(x) \preceq p_1^*(x)$ means that there exists $z \in \Omega$ such that $q(z) = p_1^*(z)$ and $q(x) < p_1^*(x)$ for all $x \in \overline{\Omega} \setminus \{z\}$.

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