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TOEPLITZ BAND MATRICES WITH EXPONENTIALLY GROWING CONDITION NUMBERS*

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Abstract. The paper deals with the spectral condition numbers $\kappa(T_n(b))$ of sequences of Toeplitz matrices $T_n(b) = (b_{j-k})_{j,k=1}^n$ as n goes to infinity. The function $b(e^{i\theta}) = \sum_k b_k e^{ik\theta}$ is referred to as the symbol of the sequence $\{T_n(b)\}$. It is well known that $\kappa(T_n(b))$ may increase exponentially if the symbol b has very strong zeros on the unit circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$, for example, if b vanishes on some subarc of $\mathbb{T}$. If b is a trigonometric polynomial, in which case the matrices $T_n(b)$ are band matrices, then b cannot have strong zeros unless it vanishes identically. It is shown that the condition numbers $\kappa(T_n(b))$ may nevertheless grow exponentially or even faster to infinity. In particular, it is proved that this always happens if b is a trigonometric polynomial which has no zeros on $\mathbb{T}$ but nonzero winding number about the origin.

The techniques employed in this paper are also applicable to Toeplitz matrices generated by rational symbols b and to the condition numbers associated with l^p norms $(1 \le p \le \infty)$ instead of the l^2 norm.

Key words. Toeplitz matrix, band matrix, matrix norm, condition number

AMS subject classifications. 15A12, 15A60, 47B35, 65F35

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