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NON-TRIVIAL SOLUTIONS TO CERTAIN MATRIX EQUATIONS*

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Abstract. The existence of non-trivial solutions X to matrix equations of the form $F(\mathbf{X}, \mathbf{A}_1, \mathbf{A}_2, \cdots, \mathbf{A}_s) = G(\mathbf{X}, \mathbf{A}_1, \mathbf{A}_2, \cdots, \mathbf{A}_s)$ over the real numbers is investigated. Here F and G denote monomials in the $(n \times n)$-matrix $\mathbf{X} = (x_{ij})$ of variables together with $(n \times n)$-matrices $\mathbf{A}_1, \mathbf{A}_2, \cdots, \mathbf{A}_s$ for $s \geq 1$ and $n \geq 2$ such that F and G have different total positive degrees in $\mathbf{X}$. An example with $s = 1$ is given by $F(\mathbf{X}, \mathbf{A}) = \mathbf{X}^2\mathbf{A}\mathbf{X}$ and $G(\mathbf{X}, \mathbf{A}) = \mathbf{A}\mathbf{X}\mathbf{A}$ where $deg(F) = 3$ and $deg(G) = 1$. The Borsuk-Ulam Theorem guarantees that a non-zero matrix $\mathbf{X}$ exists satisfying the matrix equation $F(\mathbf{X}, \mathbf{A}_1, \mathbf{A}_2, \cdots, \mathbf{A}_s) = G(\mathbf{X}, \mathbf{A}_1, \mathbf{A}_2, \cdots, \mathbf{A}_s)$ in $(n^2 - 1)$ components whenever F and G have different total odd degrees in $\mathbf{X}$. The Lefschetz Fixed Point Theorem guarantees the existence of special orthogonal matrices $\mathbf{X}$ satisfying matrix equations $F(\mathbf{X}, \mathbf{A}_1, \mathbf{A}_2, \cdots, \mathbf{A}_s) = G(\mathbf{X}, \mathbf{A}_1, \mathbf{A}_2, \cdots, \mathbf{A}_s)$ whenever $deg(F) > deg(G) \geq 1$, $\mathbf{A}_1, \mathbf{A}_2, \cdots, \mathbf{A}_s$ are in $SO(n)$, and $n \geq 2$. Explicit solution matrices $\mathbf{X}$ for the equations with $s = 1$ are constructed. Finally, nonsingular matrices $\mathbf{A}$ are presented for which $\mathbf{X}^2\mathbf{A}\mathbf{X} = \mathbf{A}\mathbf{X}\mathbf{A}$ admits no non-trivial solutions.

Key words. Polynomial equation, Matrix equation, Non-trivial solution.

AMS subject classifications. 39B42, 15A24, 55M20, 47J25, 39B72

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