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BOUNDS FOR GRAPH EXPANSIONS VIA ELASTICITY*

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Abstract. In two recent papers, one by Friedland and Schneider, the other by Förster and Nagy, the authors used polynomial matrices to study the effect of graph expansions on the spectral radius of the adjacency matrix. Here it is shown that the notion of the elasticity of the entries of a nonnegative matrix coupled with the Variational Principle for Pressure from symbolic dynamics can be used to derive sharper bounds than existing estimates. This is achieved for weighted and unweighted graphs, and the case of equality is characterized. The work is within the framework of studying measured graphs where each edge is assigned a positive length as well as a weight.

Key words. Graph expansions, Nonnegative matrices, Elasticity, Matrix polynomials.

AMS subject classifications. 05C50, 15A48, 37B10, 92D25.

1. Introduction. In two recent papers Förster and Nagy [6] and Friedland and Schneider [7] consider the effect of an expansion of unweighted and weighted directed graphs, respectively, on the spectral radius of its adjacency matrix. A graph expansion is obtained from a given graph when an edge is replaced by a path. One of the key results is that the spectral radius strictly decreases under such an operation on an unweighted irreducible graph.

The Perron–Frobenius theory does not directly apply in this situation since one is comparing spectral radii of adjacency matrices of different sizes. However, as demonstrated in [7] and [6], one can write equal size adjacency matrices for a graph and its expansion if one allows polynomial entries in the matrix. Polynomial matrices have also arisen as important tools for realization and classification problems in symbolic dynamics; see [1, 3, 8].

More formally, let $G = (\mathcal{V}, \mathcal{E})$ be an *irreducible* directed graph with a finite vertex set $\mathcal{V}$ and a finite edge set $\mathcal{E}$ (we allow for multiple edges between two vertices). Let $w : \mathcal{E} \to \mathbb{R}^+$ (for $X \subset \mathbb{R}$, we use the notation X^+ to denote the set of elements of X which are strictly positive). We call the function w a *weight function* and the pair (G, w) a *weighted graph.* Let $W = (w(i, j))$ be the adjacency matrix for (G, w), i.e., $w(i, j)$ is the sum of weights of edges from vertex i to vertex j. Now suppose that for each edge $e \in \mathcal{E}$, we assign a positive integer $\alpha(e)$ and modify the graph by the following process: introduce $\alpha(e) - 1$ new vertices to the graph, remove the edge e and replace e with a path $\mathcal{P}(e)$ of length $\alpha(e)$ which originates at the initial vertex of e, passes through the $\alpha(e) - 1$ new vertices, and ends at the terminal vertex of e. If we assign the weight of $w(e)$ to the first edge in $\mathcal{P}(e)$ and 1 to all others, we obtain a new weighted graph which we will denote (G_α, w_α).

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Förster and Nagy show that if $m = \min \alpha(e)$ and $M = \max \alpha(e)$, then $\rho(G_\alpha, w_\alpha)$, the spectral radius of the weighted graph adjacency matrix for the graph expansion, lies between the numbers $\rho(G, w)^{1/m}$ and $\rho(G, w)^{1/M}$ [6, Proposition 3.1] (this is the weighted graph version of Friedland and Schneider [7, Corollary 3.7]). The main goal of this paper is to refine these inequalities. Of particular interest is the case where only a single edge is expanded, a situation where the bounds in [6, 7] are generally not sharp.

We achieve our results through the use of a symbolic dynamic characterization of the spectral radius of an irreducible matrix and using the notion of the *elasticity* of the Perron root of A with respect to the (i, j)–th entry which is the quantity given by

$$e_{i,j}(A) \;=\; \frac{A(i, j)}{\rho(A)} \frac{\partial \rho(A)}{\partial A(i, j)}. \tag{1.1}$$

We comment that it is known (see De Kroon, Plaisier, van Groenendael, and Caswell [5] and Caswell [4]) that

$$\sum_{i,j=1}^{n} e_{i,j}(A) \;=\; 1.$$

Thus all our bounds turn out to include a weighted sum over edges of contributions to the spectral radius for the graph expansion (see, for instance, (3.7) and (3.8)). We are also able to characterize the situation where our bounds are sharp.

For this work it is convenient for us to associate a length function $\alpha : \mathcal{E} \to \mathbb{R}^+$ with a weighted graph (G, w) and consider the triple $\mathcal{G} = (G, w, \alpha)$. We will call such a triple a *measured graph* and we define the spectral radius of any such object (Definition 3.4). When α is a positive integer valued function, $\rho(G, w, \alpha)$ coincides with $\rho(G_\alpha, w_\alpha)$, the spectral radius of the adjacency matrix for a graph expansion (Theorem 3.7). As such we are able to obtain results about graph expansions as special cases of results on measured graphs.

In our new terminology, both [6] and [7] consider $\lim_{n\to\infty} \rho(G, w, \alpha_n)$ where $\{\alpha_n\}$ is a sequence of length functions on G, where $\alpha_n \to \infty$ on some subgraph $G' \subset G$. It is just as natural for us to consider the inverse situation, namely, the behavior of $\lim_{n\to\infty} \rho(G, w, \alpha_n)$, where $\{\alpha_n\}$ is a sequence of length functions tending to zero on a subgraph of G.

The outline of the paper is as follows. In Section 2 we state and prove a version of the Variational Principle for Pressure, which involves elasticities of the entries of irreducible nonnegative matrices. In Section 3 we apply our version of the Variational Principle for Pressure and establish bounds on the change in spectral radius induced by a change in the length function for an irreducible measured graph. As a consequence, our results sharpen bounds from [6, 7] on the change in spectral radius arising from a graph expansion (Subsection 3.1). In Subsection 3.2 we examine the special case of the expansion of a single edge. In Subsection 3.3 we study limiting behavior of spectral radii with decreasing sequences of length functions.

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ELA
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2. The Variational Principle for Pressure. In this section we apply a theorem from topological dynamics to obtain a lower bound on the quotient of spectral radii of two nonnegative irreducible matrices (Theorem 2.2). We first recall the following fact about elasticity which follows, for example, from Stewart [12, Exer.1, p.305], when it is applied to the definition of elasticity as given in (1.1):

PROPOSITION 2.1. *Let A be an $n \times n$ nonnegative irreducible matrix. Let $u = (u(1), u(2), \ldots, u(n))$ and $v = (v(1), v(2), \ldots, v(n))^T$ be left and right Perron eigenvectors of A, respectively. Then*

$$e_{i,j}(A) \;=\; \frac{u(i)A(i,j)v(j)}{\rho(A)uv}.$$

Next we state the main theorem of this section, which follows easily from the Variational Principle for Pressure applied to shifts of finite type in dynamical systems (see Parry and Tuncel [11] or Walters [13] for an introduction). The version below is only modified slightly from the version appearing in the paper of Kirkland, Neumann, Ormes, and Xu [9]. We mention that for an $n \times n$ nonnegative matrix A, the matrix $\operatorname{sgn}(A)$ below is the $n \times n$ $(0,1)$–matrix in which $\operatorname{sgn}(A)(i,j) = 1 \Leftrightarrow A(i,j) > 0$.

THEOREM 2.2. *Let A and B be $n \times n$ nonnegative irreducible matrices with $\operatorname{sgn}(B) = \operatorname{sgn}(A)$. Then*

$$\log(\rho(B)) \;\geq\; \log(\rho(A)) + \sum_{A(i,j)>0} \log\left(\frac{B(i,j)}{A(i,j)}\right) e_{i,j}(A). \tag{2.1}$$

Moreover, the following are equivalent:
(i) *Equality holds in* (2.1)*.*
(ii) *$B = cDAD^{-1}$, for some scalar $c > 0$ and positive diagonal matrix D.*
(iii) *$e_{i,j}(A) = e_{i,j}(B)$, for all $1 \leq i, j \leq n$.*

Proof. Let C be any nonnegative, irreducible matrix. Let $\mathcal{M}_C$ denote the set of all stochastic matrices P with $\operatorname{sgn}(P) = \operatorname{sgn}(C)$. The Variational Principle for Pressure implies

$$\log(\rho(C)) \;=\; \sup_{P \in \mathcal{M}_C} \sum_{C(i,j)>0} e_{i,j}(P) \log\left(\frac{C(i,j)}{P(i,j)}\right). \tag{2.2}$$

(An equivalent formulation appears in [9, Theorem 3.1].)

Furthermore, one can easily check that the supremum in equation (2.2) is achieved at the stochastic matrix P_C where

$$P_C(i,j) = \rho(C)^{-1} C(i,j) v(j)/v(i)$$

and $v = [v(1), \ldots, v(n)]^T$ is a right Perron vector for C.

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Now, since $\mathrm{sgn}(A) = \mathrm{sgn}(B)$, we have that $P_A \in \mathcal{M}_B$ and hence

$$\begin{aligned}\log(\rho(B)) &\geq \sum_{B(i,j)>0} e_{i,j}(P_A)\log\left(\frac{B(i,j)}{P_A(i,j)}\right)\\ &= \sum_{A(i,j)>0} e_{i,j}(P_A)\log\left(\frac{B(i,j)}{A(i,j)}\right) + \sum_{A(i,j)>0} e_{i,j}(P_A)\log\left(\frac{A(i,j)}{P_A(i,j)}\right) \qquad (2.3)\\ &= \log\rho(A) + \sum_{A(i,j)>0} e_{i,j}(P_A)\log\left(\frac{B(i,j)}{A(i,j)}\right).\end{aligned}$$

Using Proposition 2.1, we see that $e_{i,j}(P_A) = e_{i,j}(A)$, for all $1 \leq i,j \leq n$, and the main inequality (2.1) is established.

Turning next to the case of equality, we refer to [11, Theorem 25] for the following fact: For any irreducible matrix C, if the supremum in (2.2) is achieved at a stochastic matrix P, then $P = P_C$. Therefore equality in (2.1) is achieved if and only if $P_A = P_B$.

Assume now that $P_A = P_B$. Let v and w be right Perron eigenvectors for A and B, respectively. Then for all pairs i,j:

$$\frac{A(i,j)v(j)}{\rho(A)v(i)} = \frac{B(i,j)w(j)}{\rho(B)w(i)}.$$

Let $c = \rho(B)/\rho(A)$ and let $D = D_1(D_2)^{-1}$, where D_1 is diagonal matrix with $D_1(i,i) = v(i)$ and D_2 is a diagonal matrix with $D_2(i,i) = w(i)$. Then one can compute $cDAD^{-1} = B$ and therefore (i) implies (ii).

Now assume that $B = cDAD^{-1}$, for some scalar $c > 0$ and a positive diagonal matrix D. Then $c = \rho(B)/\rho(A)$. Let u and v be left and right Perron eigenvectors for A, respectively. Then uD^{-1} and Dv are left and right Perron eigenvectors for B, respectively. Since $B(i,j) = cD(i,i)A(i,j)D(j,j)^{-1}$, for all $1 \leq i,j \leq n$, it follows from Proposition 2.1 that $e_{i,j}(A) = e_{i,j}(B)$. Therefore (ii) implies (iii).

Finally, assume that $e_{i,j}(A) = e_{i,j}(B)$, for all i,j. Then, by reversing the roles of A and B in inequality (2.1), we obtain

$$\log(\rho(A)) \geq \log(\rho(B)) + \sum_{A(i,j)>0} \log\left(\frac{A(i,j)}{B(i,j)}\right) e_{i,j}(B). \qquad (2.4)$$

Now using $e_{i,j}(A) = e_{i,j}(B)$, we obtain

$$\begin{aligned}\log(\rho(A)) &\geq \log(\rho(B)) + \sum_{A(i,j)>0} \log\left(\frac{A(i,j)}{B(i,j)}\right) e_{i,j}(A)\\ &= \log(\rho(B)) - \sum_{A(i,j)>0} \log\left(\frac{B(i,j)}{A(i,j)}\right) e_{i,j}(A).\end{aligned} \qquad (2.5)$$

This is precisely the reverse of the inequality (2.1). Therefore (iii) implies (i). ◻

Below is one more property of elasticity that will be useful to us in the subsequent section.

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PROPOSITION 2.3.
Let A be an $n \times n$ nonnegative and irreducible matrix. Let k, l, m be fixed indices such that $A(i,l) > 0$ if and only if $i = k$, and $A(l,j) > 0$ if and only if $j = m$. Then $e_{k,l}(A) = e_{l,m}(A)$.

Proof. Let $u = (u(1), u(2), \ldots, u(n))$ and $v = (v(1), v(2), \ldots, v(n))^T$ be left and right Perron eigenvectors for A, respectively. Then we have that

$$\rho(A)u(l) \;=\; \sum_{i=1}^{n} u(i)A(i,l) \;=\; u(k)A(k,l)$$

and

$$\rho(A)v(l) \;=\; \sum_{j=1}^{n} A(l,j)v(j) \;=\; A(l,m)v(m).$$

Therefore,

$$e_{k,l}(A) \;=\; \frac{u(k)A(k,l)v(l)}{\rho(A)uv} \;=\; \frac{u(l)v(l)}{uv}$$

and

$$e_{l,m}(A) \;=\; \frac{u(l)A(l,m)v(m)}{\rho(A)uv} \;=\; \frac{u(l)v(l)}{uv}. \;\square$$

3. Graph Expansions And The Spectral Radius. The main goal of this section is to refine bounds on changes in spectral radii arising from graph expansions on irreducible matrices. First we review some preliminaries and remark that Theorem 2.2 has a direct interpretation in terms of changes in spectral radii corresponding to changes in weighting functions for a directed graph.

Let $G = (\mathcal{V}, \mathcal{E})$ be an *irreducible* directed graph and let $w : \mathcal{E} \to \mathbb{R}^+$ be a weight function. Note that the graph G need not be simple, i.e., there may be more than one edge from one vertex to another. Let $w(i,j)$ denote the sum of the weights of edges from vertex i to vertex j and let $W = (w(i,j))$ be the usual adjacency matrix of the weighted graph (G, w). We define $\rho(G, w) = \rho(W)$.

Given two different weighting functions $w, w' : \mathcal{E} \to \mathbb{R}^+$, Theorem 2.2 offers the following bound on differences in the spectral radii of (G, w) and (G, w').

COROLLARY 3.1. *Let G be an irreducible directed graph and let w, w' be two weight functions on the edges in G. Then*

$$\log(\rho(G, w')) \;\geq\; \log(\rho(G, w)) + \sum_{w(i,j)>0} \left[\log(w'(i,j)) - \log(w(i,j))\right] e_{i,j}(W).$$

In particular,

$$\prod\nolimits_{w(i,j)>0} \left(\frac{w'(i,j)}{w(i,j)}\right)^{e_{i,j}(W)} \geq 1 \quad \Longrightarrow \quad \rho(G, w') \geq \rho(G, w).$$

Electronic Journal of Linear Algebra ISSN 1081-3810
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Volume 10, pp. 163-178, July 2003

The converse of this statement fails as the following example shows.

EXAMPLE 3.2. Let $\mathcal{V} = \{1, 2\}$, let $W = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, and $W' = \begin{pmatrix} 1/3 & 2 \\ 1 & 1 \end{pmatrix}$. The elasticity of W is a constant; $e_{i,j}(W) = 1/4$, for $1 \le i, j \le 2$. Therefore, $\prod \left(\frac{w'(i,j)}{w(i,j)}\right)^{e_{i,j}(W)} = \left(\frac{2}{3}\right)^{1/4} < 1$, but $\rho(W') = (2 + \sqrt{19})/3 > 2 = \rho(W)$.

Next we consider the effect of changes of the lengths of edges in an irreducible graph. That is, we will study graph expansions which are essentially the same as those in [6]. The only difference being that two different edges from i to j may be replaced by two paths of the same length here. It will be convenient for us to consider, at least formally, a more general situation, and then apply results to graph expansions and their inverses. Let (G, w) be a weighted graph. Consider a map $\alpha : \mathcal{E} \to \mathbb{R}^+$ which we will call a *length function*. We will call the triple $\mathcal{G} = (G, w, \alpha)$ a *measured graph*. We define the adjacency matrix of a measured graph $\mathcal{G}$ to be the matrix $A(t)$ of functions of t, where $A(t)(i, j) = \sum_{e \in [i,j]} w(e) t^{\alpha(e)}$ and $[i, j]$ denotes the set of all edges in $\mathcal{E}$ which have initial vertex i and terminal vertex j.

REMARK 3.3. In the case where the length of each edge is 1, i.e., $\alpha \equiv 1$, $A(t)$ is the usual adjacency matrix for the weighted graph (G, w), multiplied by the scalar t.

DEFINITION 3.4. The *spectral radius of a measured graph* $\mathcal{G} = (G, w, \alpha)$, which we denote $\rho(\mathcal{G})$, is the reciprocal of the smallest positive root of the function $\det(I - A(t))$, where $A(t)$ is the adjacency matrix for $\mathcal{G}$.

An immediate consequence of this definition is the following statement:

PROPOSITION 3.5. *Let $\mathcal{G}$ be an irreducible measured graph. Then $\rho(\mathcal{G})$ is the reciprocal of the unique positive real number t for which $\rho(A(t)) = 1$.*

We remark that there are several special cases of this setup which correspond to established notions. For example, in the case where $\alpha \equiv 1$, $\rho(\mathcal{G})$ is the usual spectral radius of the weighted graph (G, w), i.e., $\rho(G, w, 1) = \rho(G, w)$. Further, if $A(t)$ is the adjacency matrix for any measured graph (G, w, α), $\rho(A(1)) = \rho(G, w)$.

Theorem 3.7 below is the key observation. There we show that if α maps the edge set into the positive integers, then $\rho(G, w, \alpha)$ is equal to $\rho(G_\alpha, w_\alpha)$, where (G_α, w_α) is the weighted graph expanded by the function α as defined in [6, 7]. We include proofs here for completeness, but the arguments are inherent in [6, 7] and versions can be found in many works in symbolic dynamics; for example, see the survey paper [2]. The proof follows from Lemma 3.6 below.

LEMMA 3.6. *Suppose A is an $n \times n$ nonnegative matrix and $i_0, i_1, \ldots, i_{m+1}$ are indices such that*

- *for all $j = 1, 2, \ldots, m$, $A(i_j, l) > 0$ if and only if $l = i_{j+1}$,*
- *for all $j = 0, 1, \ldots, (m-1)$, $A(k, i_{j+1}) > 0$ if and only if $k = i_j$.*

Let B be the matrix $(n - m) \times (n - m)$ matrix which is obtained from A by deleting the rows and columns with indices $i_1, i_2, \ldots, i_m$ and adding the product

$$A(i_0, i_1)A(i_1, i_2) \cdots A(i_m, i_{m+1})$$

to the (i_0, i_{m+1}) entry. Then

$$\det(I - A) = \det(I - B).$$

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Proof. Let $i = i_0$ and $j = i_{m+1}$. We will assume that for $k = 1, 2, \ldots, m$, $i_k = n - m + k$ (this is true up to a permutation of A). That is, we assume A is of the form

$$A = \begin{pmatrix} X & Y \\ Z & V \end{pmatrix},$$

where X is the matrix A with rows and columns $i_1, i_2, \ldots, i_m$ deleted, Y is the $(n-m) \times m$ matrix with $A(i, i_1)$ in the $(i, 1)$ position and zeros elsewhere, Z is the $m \times (n-m)$ matrix with $A(i_m, j)$ in the (m, j) position and zeros elsewhere, V is the $m \times m$ superdiagonal matrix with elements of the form $A(i_k, i_{k+1})$ in the $(k, k+1)$ entries and zeros elsewhere. We perform the following sequence of operations on the matrix $I - A$:

1. Multiply the i_mth row by $A(i_{m-1}, i_m)$ and add the result to the i_{m-1}th row.
2. Multiply the i_{m-1}th row by $A(i_{m-2}, i_{m-1})$ and add the result to the i_{m-2}th row.

   $\vdots$
3. Multiply the i_1th row by $A(i_0, i_1)$ and add the result to the i_0th row.

The result is a matrix of the form

$$\begin{pmatrix} I - B & 0 \\ -Z' & I \end{pmatrix},$$

where $B(k, l) = X(k, l)$ for all (k, l) except for $(k, l) = (i, j)$ and where

$$B(i, j) \;=\; X(i, j) + A(i, i_1) \left[\prod_{k=1}^{m-1} A(i_k, i_{k+1}) \right] A(i_m, j).$$

The matrix B is as described and since $I - B$ was produced by elementary operations on $I - A$, we have that $\det(I - A) = \det(I - B)$. ◻

THEOREM 3.7. *Let $G = (\mathcal{V}, \mathcal{E})$ be an irreducible directed graph with a weight function $w : \mathcal{E} \to \mathbb{R}^+$. Suppose $\alpha : \mathcal{E} \to \mathbb{Z}^+$ and that (G_α, w_α) is the graph expansion on G corresponding to α as defined in* [6, 7]. *Then $\rho(G, w, \alpha) = \rho(G_\alpha, w_\alpha)$.*

Proof. Let (G, w) be a weighted graph, and let α be a map from the edges to the positive integers. In [6, 7] the expansion graph (G_α, w_α) replaces each edge e from G with a path $\mathcal{P}(e)$ of length $\alpha(e)$ beginning at the initial vertex of e, passing through $\alpha(e) - 1$

new vertices, and ending at the terminal vertex of e. The first edge in $\mathcal{P}(e)$ is assigned weight $w(e)$, all other edges in $\mathcal{P}(e)$ are assigned weight 1.

Let i denote the initial vertex of $\mathcal{P}(e)$, let $i_1, i_2, \ldots, i_{\alpha(e)-1}$ denote the $\alpha(e) - 1$ new vertices of the graph and let j denote the terminal vertex of $\mathcal{P}(e)$. Let t be any real number. Then the matrix $W_\alpha t$ along with the indices $i, i_1, i_2, \ldots, i_{\alpha(e)-1}, j$ satisfy the hypotheses of Lemma 3.6. Therefore, we may delete the rows and columns of $W_\alpha t$ with indices $i_1, i_2, \ldots, i_{\alpha(e)-1}$, and add $w(e)t^{\alpha(e)}$ to the (i, j) entry and the result is a matrix $A(t)$ where $\det(I - A(t)) = \det(I - W_\alpha t)$.

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Repeating this for all edges e in G, we see that if $A(t)$ is the adjacency matrix of the measured graph (G, w, α), then $\det(I - A(t)) = \det(I - W_\alpha t)$. Since the reciprocal of the smallest real root of the equation $\det(I - W_\alpha t) = 0$ is equal to $\rho(G_\alpha, w_\alpha)$, we have that $\rho(G, w, \alpha) = \rho(G_\alpha, w_\alpha)$. ☐

In order to obtain some of our bounds we need to be able to define a length function which reverses the process of graph expansion. We establish the existence of this length function below. The proof also follows from Lemma 3.6.

THEOREM 3.8. *Let $G = (\mathcal{V}, \mathcal{E})$ be an irreducible directed graph with a weight function $w : \mathcal{E} \to \mathbb{R}^+$. Suppose $\alpha : \mathcal{E} \to \mathbb{Z}^+$ and that (G_α, w_α) is the graph expansion on G corresponding to α as defined in* [6, 7]. *Then there is a length function β such that $\rho(G_\alpha, w_\alpha, \beta) = \rho(G, w)$.*

Proof. Every edge $e \in \mathcal{E}$ gives rise to a path of edges $\mathcal{P}(e)$ in $G_\alpha = (\mathcal{V}_\alpha, \mathcal{E}_\alpha)$, where the length of $\mathcal{P}(e)$ is $\alpha(e)$. Further, each edge $f \in \mathcal{E}_\alpha$ belongs to a unique path $\mathcal{P}(e)$ where $e \in \mathcal{E}$. We define a length function β on G_α by $\beta(f) = 1/\alpha(e)$ for $f \in \mathcal{P}(e)$.

Let $B(t)$ be the adjacency matrix for $(G_\alpha, w_\alpha, \beta)$. Let e be an edge in G and let $f_1, f_2, \ldots, f_{\alpha(e)}$ be the edges forming the path $\mathcal{P}(e)$ in G_α. Let $i_0, i_1, \ldots, i_{\alpha(e)}$ denote the vertices which the path $\mathcal{P}(e)$ passes through. Then the matrix $B(t)$ with the indices $i_0, i_1, \ldots, i_{\alpha(e)}$ satisfy the hypotheses of Lemma 3.6. Therefore we may delete the columns and rows corresponding to $i_1, i_2, \ldots, i_{\alpha(e)-1}$, and add the product from $k = 0$ to $\alpha(e) - 1$ of the (i_k, i_{k+1}) entries of $B(t)$ to the $(i_0, i_{\alpha(e)})$ entry of $B(t)$. The result will be a matrix $C(t)$ with $\det(I - B(t)) = \det(I - C(t))$. Note that the product

$$\prod_{k=0}^{\alpha(e)-1} B(t)(i_k, i_{k+1}) \;=\; \prod_{k=1}^{\alpha(e)} w_\alpha(f_k) t^{1/\alpha(e)} = w(e)t.$$

If we repeat this process for all edges e in G, the result will be a matrix which has in the (i, j) entry the sum of the weights of edges e from i to j multiplied by t. That is, we will have the (i, j) entry of Wt. Since $\det(I - Wt) = \det(I - B(t))$, we have $\rho(G, w) = \rho(G_\alpha, w_\alpha, \beta)$. ☐

Our estimates of the spectral radius of a measured graph will involve the elasticities of the matrix W. To be more specific, it will involve the elasticity of the edges in a graph, a notion which is defined below.

DEFINITION 3.9. Let $G = (\mathcal{V}, \mathcal{E})$ be a directed graph, and let $w : \mathcal{E} \to \mathbb{R}^+$ be a weight function. For $e \in \mathcal{E}$, define $\mu(e)$, the *elasticity* of e, by

$$\mu(e) = e_{i,j}(W) w(e)/w(i, j),$$

where e is an edge from vertex i to vertex j.

REMARK 3.10. The elasticity $e_{i,j}(W)$ is $w(i, j)/\rho(W)$ times the derivative of $\rho(W)$ with respect to $w(i, j)$. Similarly, for $e \in \mathcal{E}$,

$$\mu(e) \;=\; \frac{w(e)}{\rho(W)} \frac{\partial \rho(W)}{\partial w(e)}.$$

Furthermore, we retain the identity, $\sum_{e \in \mathcal{E}} \mu(e) = 1$.

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ELA
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THEOREM 3.11. *Let $G = (\mathcal{V}, \mathcal{E})$ be an irreducible directed graph and let $\mathcal{G} = (G, w, \alpha)$ be a measured graph. Then*

$$\log(\rho(\mathcal{G})) \;\geq\; \frac{\log(\rho(G,w))}{\sum_{e\in\mathcal{E}} \alpha(e)\mu(e)}. \tag{3.1}$$

Proof. Let $A(t)$ be the adjacency matrix for the measured graph $\mathcal{G}$. Let $W = (w(i,j))$. Applying Theorem 2.2 to the matrices $A(t)$ and Wt we see that for any $t > 0$,

$$\log(\rho(A(t))) \;\geq\; \log(\rho(Wt)) + \sum_{w(i,j)>0} \log\left(\frac{A(t)(i,j)}{w(i,j)t}\right) e_{i,j}(W). \tag{3.2}$$

Next we note that

$$\frac{A(t)(i,j)}{w(i,j)t} \;=\; \sum_{e\in[i,j]} \frac{w(e)}{w(i,j)} t^{\alpha(e)-1}$$

and, since $\sum_{e\in[i,j]} w(e) = w(i,j)$, we have that

$$\log\left(\frac{A(t)(i,j)}{w(i,j)t}\right) \;\geq\; \sum_{e\in[i,j]} \frac{w(e)}{w(i,j)}(\alpha(e)-1)\log(t). \tag{3.3}$$

Substituting this expression into inequality (3.2) we obtain

$$\begin{aligned}\log(\rho(A(t))) \;&\geq\; \log(\rho(Wt)) + \sum_{e\in\mathcal{E}} \mu(e)(\alpha(e)-1)\log(t)\\ &=\; \log(\rho(W)) + \sum_{e\in\mathcal{E}} \mu(e)\alpha(e)\log(t).\end{aligned} \tag{3.4}$$

Setting $t = 1/\rho(\mathcal{G})$, we now have

$$0 \;\geq\; \log(\rho(W)) - \sum_{e\in\mathcal{E}} \mu(e)\alpha(e)\log(\rho(\mathcal{G})). \tag{3.5}$$

Upon rearranging, we get

$$\log(\rho(\mathcal{G})) \;\geq\; \frac{\log(\rho(G,w))}{\sum_{e\in\mathcal{E}} \alpha(e)\mu(e)},$$

which is the desired inequality. ◻

The case of equality in the previous theorem is discussed below.

THEOREM 3.12. *Let $\mathcal{G} = (G, w, \alpha)$ be an irreducible measured graph. We have equality in equation* (3.1) *if and only if $\rho(G,w) = \rho(\mathcal{G}) = 1$ or there exist constants $k, \delta_1, \delta_2, \ldots, \delta_n > 0$ such that for all $e \in [i,j]$, $\alpha(e) = k + \delta_i - \delta_j$. In this case, $k = \sum_{e\in\mathcal{E}} \alpha(e)\mu(e)$.*

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ELA
http://math.technion.ac.il/iic/ela

Proof. If $\rho(\mathcal{G}) = 1$, it is clear that we have equality in equation (3.1) if and only if $\rho(G, w) = 1$. Assume for the remainder of the proof that $\rho(\mathcal{G}) \neq 1$.

First note that equality in (3.1) is derived from the case of equality in (3.2) (at least when $t = 1/\rho(\mathcal{G})$) and (3.3). Inequality (3.3) is a property of logarithms, equality here occurs if and only if $e, e' \in [i, j]$ implies that $\alpha(e) = \alpha(e')$.

Assume that $\alpha(e) = k + \delta_i - \delta_j$, for all $e \in [i, j]$, as in the statement of the theorem. Then we have equality in (3.3). Let $\alpha(i, j) = \alpha(e)$, for $e \in [i, j]$. Then for any $t > 0$,

$$\alpha(i, j) \log(t) \;=\; (k + \delta_i - \delta_j) \log(t)$$

or

$$t^{\alpha(i,j)} \;=\; t^k t^{\delta_i} / t^{\delta_j}.$$

Let $D(t)$ be the diagonal matrix with $D(t)(i, i) = t^{\delta_i}$. It follows that

$$A(t) \;=\; t^{k-1} D(t)(Wt)(D(t))^{-1}.$$

By Theorem 2.2, this implies that there is equality in (3.2) for any $t > 0$. Coupled with equality in (3.3), we obtain equality in (3.1).

Now assume there is equality in equation (3.1). Then we have equality in (3.3) which means that $e, e' \in [i, j]$ implies that $\alpha(e) = \alpha(e')$. Thus we may set $\alpha(i, j) = \alpha(e)$, for $e \in [i, j]$.

Theorem 2.2 implies that $A(1/\rho(\mathcal{G})) = cD(W\rho(\mathcal{G})^{-1})D^{-1}$, for some scalar $c > 0$ and a positive diagonal matrix D. The equation above means that for all $1 \le i, j \le n$,

$$w(i, j)(\rho(\mathcal{G}))^{-\alpha(i,j)} \;=\; \frac{cD(i, i)w(i, j)}{\rho(\mathcal{G})D(j, j)}.$$

Taking logarithms and rearranging, we obtain

$$(1 - \alpha(i, j)) \log(\rho(\mathcal{G})) = \log c + \log(D(i, i)) - \log(D(j, j)). \tag{3.6}$$

Let $k = 1 - \log c / \log(\rho(\mathcal{G}))$ and let $\delta_i = \log(D(i, i)) / \log(\rho(\mathcal{G}))$, for $1 \le i \le n$. The equality $\alpha(i, j) = k + \delta_i - \delta_j$, for all $1 \le i, j \le n$, now follows.

We may assume that all of the δ_i are positive since adding a single constant to all of the δ_i's does not affect the value of $\delta_i - \delta_j$. It follows from the definition of c that $c = \rho(\mathcal{G})/\rho(W)$. Therefore, we see that $k = \log(\rho(W))/\log(\rho(\mathcal{G}))$. Since we are assuming equality in Theorem 3.11, we have that $k = \sum_{w(i,j)>0} \alpha(i, j)e_{i,j}(W) = \sum_{e\in\mathcal{E}} \mu(e)\alpha(e)$. □

The formalism we have developed in this section can now be directly applied to the situation of graph expansions as studied in [6, 7].

3.1. Weighted graph expansions. First we consider an irreducible measured graph $\mathcal{G} = (G, w, \alpha)$, where $G = (\mathcal{V}, \mathcal{E})$ and $\alpha : \mathcal{E} \to \mathbb{Z}^+$. Then there is a graph $G_\alpha = (\mathcal{V}_\alpha, \mathcal{E}_\alpha)$ and a weight function w_α associated with $\mathcal{G}$ as defined in [6, 7]. In this situation we will say that α is an *expansion of the weighted graph* (G, w) and

Electronic Journal of Linear Algebra ISSN 1081-3810
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Volume 10, pp. 163-178, July 2003

ELA
http://math.technion.ac.il/iic/ela

(G_α, w_α) is the *weighted expansion graph.* Recall that we showed in Theorem 3.7 that $\rho(G, W, \alpha) = \rho(G_\alpha, w_\alpha)$.

Theorem 3.11 allows us to obtain a finer estimate on $\rho(G_\alpha)$ than in [6] as follows:

COROLLARY 3.13. *Let (G, w) be an irreducible weighted graph and (G_α, w_α) be a weighted expansion graph. Then*

$$\log(\rho(G_\alpha, w_\alpha)) \;\geq\; \frac{\log(\rho(G, w))}{\sum_{e\in\mathcal{E}} \mu(e)\alpha(e)}. \tag{3.7}$$

To obtain an upper bound we will reverse the roles of the graph and the expansion graph. Suppose that α is an expansion of a weighted graph (G, w) and that (G_α, w_α) is the resulting weighted expansion graph. Then as noted in Theorem 3.8, there is a length function β on G_α such that $\rho(G_\alpha, w_\alpha, \beta) = \rho(G, w)$. Recall that β assigned the length of $1/\alpha(e)$ to any edge f in G_α which belongs to the path $\mathcal{P}(e)$ corresponding to the expansion of the edge e in G.

By applying Theorem 3.11 to $\mathcal{G} = (G_\alpha, w_\alpha, \beta)$ and (G_α, w_α) we obtain the following:

COROLLARY 3.14. *Let (G, w) be an irreducible weighted graph and (G_α, w_α) be a weighted expansion graph. Then*

$$\log(\rho(G, w)) \sum_{e\in\mathcal{E}_\alpha} \mu_\alpha(e)\frac{1}{\alpha(e)} \;\geq\; \log\left(\rho(G_\alpha, w_\alpha)\right). \tag{3.8}$$

REMARK 3.15. Let α be an expansion on a weighted graph (G, w). Suppose that $1 \le m \le \alpha(e) \le M$, for all edges e in the graph G. The following inequalities from [6] follow from the previous two corollaries:

If $\rho(G, w) \geq 1$, then

$$\rho(G, w)^{1/M} \;\leq\; \rho(G_\alpha, w_\alpha) \;\leq\; \rho(G, w)^{1/m} \;\leq\; \rho(G, w).$$

If $\rho(G, w) \leq 1$, then

$$\rho(G, w) \;\leq\; \rho(G, w)^{1/m} \;\leq\; \rho(G_\alpha, w_\alpha) \;\leq\; \rho(G, w)^{1/M}.$$

Furthermore if $\rho(G, w) \neq 1$, then $\rho(G, w) \neq \rho(G_\alpha, w_\alpha)$.

3.2. The expansion of a single edge. What is new about Corollaries 3.13 and 3.14 is that they give finer inequalities than those in [6] when α is a nonconstant expansion. An interesting case of this is the case of the expansion of a single edge.

COROLLARY 3.16. *Let (G, w) be an irreducible weighted graph, and let $a \geq 0$ be an integer. Fix an edge $e_0 \in \mathcal{E}$ and define a length function α such that $\alpha(e) = 1$ if $e \neq e_0$ and $\alpha(e_0) = a + 1$. Let f_0 be an edge in G_α which is part of the path $\mathcal{P}(e_0)$. Let μ and μ_α denote the elasticity functions on edges in the graphs G and G_α, respectively. Then*

$$\log\left(\rho(G_\alpha, w_\alpha)\right) \;\geq\; \frac{\log(\rho(G, w))}{1 + a\mu(e_0)} \tag{3.9}$$

Electronic Journal of Linear Algebra ISSN 1081-3810
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and

$$\log(\rho(G,w))(1 - a\mu_\alpha(f_0)) \;\geq\; \log\left(\rho(G_\alpha, w_\alpha)\right). \tag{3.10}$$

Furthermore, let i_0 and j_0 be the initial and terminal vertices of e_0, respectively. Equality holds in either of the above inequalities if and only if $[i_0, j_0] = \{e_0\}$ and the vertices in G can be partitioned into p sets $V_0 = \{j_0\}$, V_1, V_2, $\ldots$, $V_{p-2}, V_{p-1} = \{i_0\}$ such that $w(i,j) > 0$ and $i \in V_m$ implies $j \in V_{m+1}$ or $(i,j) = (i_0, j_0)$.

Proof. The inequality (3.9) follows immediately from Corollary 3.13, noting $\sum_{e\in\mathcal{E}} \mu(e)\alpha(e) = a\mu(e_0) + \sum_{e\in\mathcal{E}} \mu(e)$ and $\sum_{e\in\mathcal{E}} \mu(e) = 1$.

Inequality (3.10) follows almost as easily from Corollary 3.14. In our case

$$\sum_{e\in\mathcal{E}_\alpha} \frac{1}{\alpha(e)}\mu_\alpha(e) \;=\; 1 - \left(\frac{a}{a+1}\right) \sum_{f\in\mathcal{P}(e_0)} \mu_\alpha(f). \tag{3.11}$$

If follows from Proposition 2.3 that for all $f \in \mathcal{P}(e_0)$, $\mu_\alpha(f) = \mu_\alpha(f_0)$. There are $a+1$ such edges, therefore

$$\sum_{f\in\mathcal{E}_\alpha} \frac{1}{\alpha(f)}\mu_\alpha(f) \;=\; 1 - a\mu_\alpha(f_0). \tag{3.12}$$

For the case of equality, we consider the situation in light of Theorem 3.12. We have equality if and only if for all $e \in [i,j]$,

$$\alpha(e) \;=\; k + \delta_i - \delta_j \tag{3.13}$$

for some constants $k, \delta_1, \delta_2, \ldots, \delta_n$. Let $\mathcal{C}$ be any cycle in the graph G. Let τ be the number of times that the edge e_0 occurs in $\mathcal{C}$. Consider the sum $\sum_{e\in\mathcal{C}}(\alpha(e) - 1)$. By the definition of α, this is τa, but from equation (3.13), this is $k\|\mathcal{C}\|$. We therefore get the relationship $\|\mathcal{C}\| = \tau a/k$.

Let $p = a/k$. It is clear that p is an integer since we can take $\mathcal{C}$ to be a cycle which passes through e_0 exactly once. Now, for each integer $m \in \{1, 2, \ldots, p-1\}$, let V_m be the collection of vertices v in G for which there is a path of length m from j_0 to v.

We have established that the length of any cycle in the graph G is equivalent to 0 mod p. Suppose that $1 \leq m_1, m_2 < p$ and that $v \in V_{m_1} \cap V_{m_2}$. Let l be the length of any path from v to i_0. If $m_1 \neq m_2$, then we can form two different cycles starting at the vertex j_0, passing through v, and ending with the edge e_0. We can construct one with length $m_1 + l + 1$ and the other with length $m_2 + l + 1$. The difference of the lengths $m_1 - m_2$ must be equal to 0 mod p, but this is impossible. Therefore, the sets $\{V_m\}$ are pairwise disjoint.

We also have established that the length of any cycle is p times the number of times it contains the edge e_0. Let $i \in V_{p-1}$ and consider the shortest cycle from j_0 to i back to j_0. This cycle must be of length p, otherwise it would have to contain e_0 more than once and would not be the shortest such path. The shortest path from j_0 to i is of length $p-1$ and therefore there is an edge connecting i to j_0. But this

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ELA
http://math.technion.ac.il/iic/ela

edge must be e_0, otherwise we have a cycle of length p which does not pass through e_0. Hence, $V_{p-1} = \{i_0\}$ and $[i_0, j_0] = \{e_0\}$.

Finally, we note that the set V_m can alternatively be described as the set of vertices v for which $W^m(j_0, v) > 0$. Therefore $w(i,j) > 0$ and $i \in W_m$ implies $j \in W_{m+1}$ or $(i,j) = (i_0, j_0)$. □

Given an expansion that lengthens a single edge, we can conclude an inequality on its elasticity before and after the expansion. It follows directly from the bounds in Corollary 3.16.

Corollary 3.17. *With the same hypotheses as Corollary 3.16, for $\rho(G,w) > 1$, if $f_0 \in \mathcal{P}(e_0)$, then*

$$\mu(e_0) \;\geq\; \frac{\mu_\alpha(f_0)}{1 - a\mu_\alpha(f_0)}.$$

The reverse inequality holds in the case $\rho(G,w) < 1$.

3.3. Limiting cases. We now consider the situation where we have a sequence of measured graphs $\mathcal{G}_n = (G, w, \alpha_n)$, where $\alpha_n(e) \to 0$ for some collection of edges $\mathcal{E}' \subset \mathcal{E}$. We note that the case where $\alpha_n(e) \to \infty$ on a set of edges was previously considered in the context of graph expansions [6, 7]. Here we consider the inverse situation, the case where α is nonincreasing and $\alpha_n(e) \to 0$ for some collection of edges $\mathcal{E}' \subset \mathcal{E}$. To keep the setting more natural, we consider the case where α_n satisfies the following property: for all $e \in \mathcal{E}$, either $\alpha_n(e) = 1$, for all $n \geq 0$, or $\lim_{n\to\infty} \alpha_n(e) = 0$.

There will be several cases, depending on $G' = (\mathcal{V}', \mathcal{E}')$, the subgraph of G induced by the edges in $\mathcal{E}'$.

Theorem 3.18. *Let α_n be a sequence of length functions on an irreducible weighted graph (G,w). Let $\mathcal{G}_n = (G, w, \alpha_n)$. Suppose that $G' = (\mathcal{V}', \mathcal{E}')$ is an induced subgraph of G such that*

$\lim_{n\to\infty} \alpha_n(e) = 0$, for all $e \in \mathcal{E}'$, and $\alpha_n(e) = 1$, for all $e \in \mathcal{E} - \mathcal{E}'$. If $\rho(G', w) > 1$, then $\lim_{n\to\infty} \rho(\mathcal{G}_n) = \infty$.

Proof. Let $A_n(t)$ denote the adjacency matrix of the measured graph $\mathcal{G}_n$. Let $B_n(t)$ be the adjacency matrix of the measured subgraph (G', w, α_n) arranged so that the rows and columns of $B_n(t)$ are indexed in the same way as the rows and columns of $A_n(t)$ (if a vertex i appears in G but not G', then the ith row and column of $B_n(t)$ contain all zeros). Clearly, for any $t > 0$, $\rho(A_n(t)) \geq \rho(B_n(t))$.

Let $W' = B_n(1)$ be the weight matrix for the weighted graph (G', w). Then for any $t > 0$, $\lim_{n\to\infty} B_n(t) = W'$. Let $\epsilon > 0$ be given. We know that $\rho(W') > 1$ so that there is an N such that $n > N$ implies that the minimum value of t for which $\rho(B_n(t)) = 1$ is less than ϵ. Since $\rho(A_n(t)) \geq \rho(B_n(t))$, for sufficiently large n, the minimum value of t for which $\rho(A_n(t)) = 1$ is also less than ϵ. Therefore, as $n \to \infty$, the reciprocals of the smallest real numbers for which $\rho(A_n(t)) = 1$ tend to ∞. That is, $\lim_{n\to\infty} \rho(\mathcal{G}_n) = \infty$. □

The case where $\rho(G', w) < 1$ is more complicated.

Theorem 3.19. *Let α_n be a sequence of length functions on an irreducible weighted graph (G,w). Let $\mathcal{G}_n = (G, w, \alpha_n)$. Suppose that $G' = (\mathcal{V}', \mathcal{E}')$ is a subgraph*

Electronic Journal of Linear Algebra ISSN 1081-3810
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ELA
http://math.technion.ac.il/iic/ela

of G such that $\lim_{n\to\infty}\alpha_n(e)=0$, for all $e\in\mathcal{E}'$, and $\alpha_n(e)=1$ for all $e\in\mathcal{E}-\mathcal{E}'$. Let W,W' be the adjacency matrices for (G,w) and (G',w) respectively.

Suppose that $W\neq W'$ and that $\rho(G',w)<1$. Then

$$\lim_{n\to\infty}\rho(\mathcal{G}_n) \;=\; \frac{1}{\inf\{t \;:\; \rho((W-W')t+W')=1\}}.$$

In the case where $W=W'$ and $\rho(G',w)<1$,

$$\lim_{n\to\infty}\rho(\mathcal{G}_n)=0.$$

Proof. Let $A_n(t)$ denote the adjacency matrix of the measured graph $\mathcal{G}_n$. Let $B_n(t)$ be the adjacency matrix of the measured graph (G',w,α_n). Then since $\alpha_n\equiv 1$ on $\mathcal{E}-\mathcal{E}'$, $A_n(t)-B_n(t)=(W-W')t$. Further, we have for any $t>0$, $\lim_{n\to\infty}B_n(t)=B_n(1)=W'$. Let $t_n=\inf\{t\mid\rho(A_n(t))=1\}$ and let $t_*=\inf\{t\mid\rho((W-W')t+W')=1\}$. Notice that in the case where $W\neq W'$, t_* is finite and it suffices to show that $\lim_{n\to\infty}t_n=t_*$.

Assume first that $t_1<1$. Then for all n, $t_n<1$ (see Remark 3.15) and $B_n(t_n)\leq B_n(1)=W'$. Therefore, we have that

$$1 \;=\; \rho(A(t_n)) \;\leq\; \rho((W-W')t+W')$$

which implies $t_*\leq t_n$.

Let $\epsilon>0$. If $W\neq W'$, then $\rho((W-W')(t_*+\epsilon)+W')>1$ since G is irreducible. Since $B_n(t_*+\epsilon)\to B_n(1)$ as $n\to\infty$, for sufficiently large n, we have $\rho((W-W')(t_*+\epsilon)+B_n(t_*+\epsilon))>1$. This implies that $t_n<t_*+\epsilon$.

Similarly, if $t_1\geq 1$ and $\epsilon>0$ is given, then $t_*-\epsilon<t_n\leq t_*$ for sufficiently large n. Therefore $\lim_{n\to\infty}t_n=t_*$.

Finally, assume $W'=W$. That is, $\alpha_n\to 0$, for each edge of G and $\rho(G,w)<1$. Then $\rho(\mathcal{G}_n)<1$ for all n. Let $t>0$. Then since $A_n(t)\to A_n(1)=W$, for sufficiently large n, $\rho(A_n(t))<1$. Therefore, $\lim_{n\to\infty}\rho(\mathcal{G}_n)<1/t$. Since this is true for any $t>0$, we have that $\lim_{n\to\infty}\rho(\mathcal{G}_n)=0$. ◻

The case where G' does not contain a cycle is an interesting subcase of the previous theorem where the limit can be calculated more explicitly. If G' does not contain a cycle, then there is a partial ordering $\prec$ on $\mathcal{E}'$ defined as follows. For $e,f\in\mathcal{E}'$, with $e\neq f$, we say $e\prec f$ if there is a path of edges, completely contained in G', which begins with e and ends with f. In this case we say that the edge e precedes f in G'. Number the edges $e_1,e_2,\ldots,e_M\in\mathcal{E}'$ so that $e_k\prec e_l$ implies that $k>l$.

Let $A^{(0)}$ be the adjacency matrix for the weighted graph (G,w), i.e., $A^{(0)}(i,j)=\sum_{e\in[i,j]}w(e)$. We define the matrices $A^{(1)},\ldots,A^{(M)}$ as follows. For $k>0$, let i_k and j_k be the initial and terminal vertices of e_k, respectively. Subtract $w(e_k)$ from the i_k,j_k entry of $A^{(k-1)}$, then multiply the j_kth row of the resulting matrix by $w(e_k)$ and add the result to the i_kth row. Call the new matrix $A^{(k)}$. Let $\widetilde{A}=A^{(M)}$.

THEOREM 3.20. *Let α_n be a sequence of length functions on an irreducible weighted graph (G,w). Let $\mathcal{G}_n=(G,w,\alpha_n)$. Suppose that $G'=(\mathcal{V}',\mathcal{E}')$ is a subgraph of G such that $\lim_{n\to\infty}\alpha_n(e)=0$, for all $e\in\mathcal{E}'$, and $\alpha_n(e)=1$, for all*

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ELA
http://math.technion.ac.il/iic/ela

$e \in \mathcal{E} - \mathcal{E}'$. If G' does not contain a cycle, then $\lim_{n\to\infty} \rho(\mathcal{G}_n) = \rho(\widetilde{A})$*, where $\widetilde{A}$ is defined in the previous paragraph.*

Proof. Let $A_n^{(0)}(t)$ be the adjacency matrix for $\mathcal{G}_n$. That is, $A_n^{(0)}(t)(i,j) = \sum_{e\in[i,j]} w(e)t^{\alpha_n(e)}$. We note that $\lim_{n\to\infty} \rho(\mathcal{G}_n)$ is the limit of the reciprocals of the smallest real roots of $\det(I - A_n^{(0)}(t))$.

For $k > 0$, let $A_n^{(k)}(t)$ be the matrix obtained by subtracting $w(e_k)t^{\alpha_n(e_k)}$ from the entry $A^{(k-1)}(t)(i_k, j_k)$, then adding the j_kth row of the resulting matrix to the i_kth row. It suffices to show that for any $t > 0$, $\lim_{n\to\infty} \det(I - A_n^{(k)}(t)) = \lim_{n\to\infty} \det(I - A_n^{(k-1)}(t))$ by the following argument. At the kth stage of the construction, we subtract the term corresponding to e_k from the matrix $A_n^{(k-1)}(t)$. Because of the ordering of the edges, we can prove by induction that the j_kth row of $A_n^{(k-1)}(t)$ does not contain any terms corresponding to an $e_l \in \mathcal{E}'$, where $l > k$. Therefore, the matrix $A_n^{(k)}(t)$ will only contain terms corresponding to edges $e_l \in \mathcal{E}'$, where $l > k$. Therefore, the last matrix $A_n^{(M)}(t)$ contains no terms that depend upon n, or $A_n^{(M)}(t) = \widetilde{A}t$. If we can establish that $\lim_{n\to\infty} \det(I - A_n^{(k)}(t)) = \lim_{n\to\infty} \det(I - A_n^{(k+1)}(t))$, then we shall have that $\lim_{n\to\infty} \det(I - A_n^{(0)}(t)) = \det(I - \widetilde{A}t)$, and $\lim_{n\to\infty} \rho(\mathcal{G}_n) = \rho(\widetilde{A})$ follows.

For simplicity, let $k = 1$ and fix $n > 0$. We will prove that $\lim_{n\to\infty} \det(I - A_n^{(0)}(t)) = \lim_{n\to\infty} \det(I - A_n^{(1)}(t))$. Suppose $A_n^{(0)}(t)$ is $m \times m$, we will introduce an $m \times (m+1)$ matrix R and an $(m+1) \times m$ matrix S such that $A_n^{(0)}(t) = RS$. Define

$$R(i,j) \;=\; \begin{cases} A_n^{(0)}(t)(i,j) & \text{if } 1 \le i,j \le m,\ (i,j) \ne (i_1,j_1), \\[2ex] A_n^{(0)}(t)(i_1,j_1) - w(e_1)t^{\alpha_n(e_1)} & \text{if } (i,j) = (i_1,j_1), \\[2ex] w(e_1)t^{\alpha_n(e_1)} & \text{if } (i,j) = (i_1, m+1), \\[2ex] 0 & \text{if } i \ne i_1 \text{ and } j = m+1 \end{cases}$$

and

$$S(i,j) \;=\; \begin{cases} 1 & \text{if } i = j \text{ or } (i,j) = (m+1, j_1)\ , \\[2ex] 0 & \text{otherwise.} \end{cases}$$

Then $A_n^{(0)}(t) = RS$. We let $B_n(t) = SR$. Since the matrices $A_n^{(0)}(t)$ and $B_n(t)$ are strongly shift equivalent (see [10]), we have that $\det(I - A_n^{(0)}(t)) = \det(I - B_n(t))$. The matrix $B_n(t)$ can obtained from $A_n^{(0)}(t)$ by subtracting $w(e_1)t^{\alpha_n(e_1)}$ from the (i_1, j_1)–entry, augmenting a column which has entry 0 except in the i_1–position which contains the entry $w(e_1)t^{\alpha_n(e_1)}$, and augmenting a row which has entries equal to the j_1–th row of $A_n^{(0)}(t)$.

Next we note that if we multiply $w(e_1)t^{\alpha_n(e_1)}$ by the $(m+1)$–st row of $I - B_n(t)$ and add the result to the i_1–st row of $I - B_n(t)$, the resulting matrix $C_n(t)$ is $(m+$

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$1) \times (m+1)$ with $\det(C_n(t)) = \det(I - B_n(t))$. The $(m+1)$–st column of $C_n(t)$ has 1 in the $(m+1, m+1)$–position, $w(e_1)(1 - t^{\alpha_n(e_1)})$ in the $(m+1, j_1)$–position, and 0's elsewhere. Thus, by expanding $\det(C_n(t))$ along the $(m+1)$–st column, we see that $\det(C_n(t)) = \det(C_{m+1,m+1}(t)) \pm w(e_1)(1 - t^{\alpha_n(e_1)}) \det(C_{m+1,j_0}(t))$, where $C_{i,j}(t)$ denotes the $m \times m$ matrix that results from deleting the i–th row and j–th column of $C_n(t)$.

Now we simply note that by construction, $C_{m+1,m+1}(t) = A_n^{(1)}(t)$. Further, $\lim_{n\to\infty} w(e_1)(1 - t^{\alpha_n(e_1)}) \det(C_{m+1,j_0}(t)) = 0$ since all entries of $C_n(t)$ converge to a finite value as n approaches infinity and $(1 - t^{\alpha_n(e_1)}) \to 0$. □

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REFERENCES

[1] M. Boyle. Positive K–theory and symbolic dynamics. To appear in *Dynamics and Randomness*, A. Maass, S. Martinez and J. San Martin, editors, Kluwer, pp. 31–52, 2002.

[2] M. Boyle. Symbolic dynamics and matrices. *Combinatorial and graph-theoretical problems in linear algebra*. IMA Vol. Math. Appl., Vol. 50, Springer, 1993.

[3] M. Boyle and D. Lind. Small polynomial matrix presentations of nonnegative matrices. *Linear Algebra Appl.*, 355: 49–70, 2002.

[4] H. Caswell. *Matrix Population Models (2nd Edition): Construction, Analysis, and Interpretation*. Sinauer, Sunderland, Massachusetts, 2001.

[5] H. De Kroon, A. Plaisier, J. van Groenendael, and H. Caswell. Elasticity: the relative contribution of demographic parameters to population growth rate. *Ecology*, 65:1427–1431, 1986.

[6] K.-H. Förster and B. Nagy On spectra of expansion graphs and matrix polynomials, II. *Electron. J. Linear Algebra*, 9:158–170, 2002.

[7] S. Friedland and H. Schneider. Spectra of expansion graphs. *Electron. J. Linear Algebra*, 6:2–10, 1999.

[8] H. Kim, N. Ormes, and F. Roush. The spectra of nonnegative integer matrices via formal power series. *J. Amer. Math. Soc.*, 13:773–806, 2000.

[9] S.J. Kirkland, M. Neumann, N. Ormes, and J. Xu. On the elasticity of the Perron root of a nonnegative matrix. *SIAM J. Matrix Anal. Appl.*, 24:454–464, 2002.

[10] D. Lind and B. Marcus. *An Introduction to Symbolic Dynamics and Coding*. Cambridge University Press, New York, 1995.

[11] W. Parry and S. Tuncel. *Classification Problems in Ergodic Theory*. London Mathematical Society Lecture Note Series, Vol. 97, Cambridge University Press, 1982.

[12] G.W. Stewart. *Introduction to Matrix Computations*. Academic Press, New York, 1973.

[13] P. Walters. *An Introduction to Ergodic Theory*. Springer Verlag, New York, 1982.