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THE MAXIMAL SPECTRAL RADIUS OF A DIGRAPH WITH $(M+1)^2 - S$ EDGES*

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Abstract. It is known that the spectral radius of a digraph with k edges is at most $\sqrt{k}$, and that this inequality is strict except when k is a perfect square. For $k = m^2 + \ell$, ℓ fixed, m large, Friedland showed that the optimal digraph is obtained from the complete digraph on m vertices by adding one extra vertex, a corresponding loop, and then connecting it to the first $\lfloor \ell/2 \rfloor$ vertices by pairs of directed edges (for even ℓ an extra edge is added to the new vertex).

Using a combinatorial reciprocity theorem, and a classification by Backelin on the digraphs on s edges having a maximal number of walks of length two, the following result is obtained: for fixed $0 < s \neq 4$, $k = (m+1)^2 - s$, m large, the maximal spectral radius of a digraph with k edges is obtained by the digraph which is constructed from the complete digraph on $m+1$ vertices by removing the loop at the last vertex together with $\lfloor s/2 \rfloor$ pairs of directed edges that connect to the last vertex (if s is even, remove an extra edge connecting the last vertex).

Key words. Spectral radius, Digraphs, $(0,1)$-matrices, Perron-Frobenius theorem, Number of walks.

AMS subject classifications. 05C50, 05C20, 05C38.

Digraphs drawn by **dot** [8].

1. Introduction. By a *digraph* we understand a finite directed graph with no multiple edges, but possibly loops. Let $G(m,p,q)$ be the digraph on $\{1,\ldots,m+1\}$, where there is an edge from i to j if $i,j \leq m$ or if $i \leq p$ and $j = m+1$ or if $i = m+1$ and $j \leq q$. Let $M(G(m,p,q)) = M(m,p,q)$ denote the adjacency matrix of $G(m,p,q)$; it is a $(0,1)$-matrix with m^2+p+q ones. If $\mathbb{I}_{a,b}$ denotes the $a \times b$ matrix with all ones, and $\mathbb{O}_{a,b}$ the matrix with all zeroes, then

$$M(m,p,q) = \left[\begin{array}{c|c} \mathbb{I}_{m,m} & \begin{matrix}\mathbb{I}_{p,1}\\ \mathbb{O}_{m-p,1}\end{matrix} \\ \hline \mathbb{I}_{1,q}\ \mathbb{O}_{1,m-q} & 0 \end{array}\right]. \tag{1.1}$$

For $0 < \ell < 2m+1$, we put $M(m,\ell) = M(m, \lceil \ell/2 \rceil, \ell - \lceil \ell/2 \rceil)$, i.e. the $M(m,p,q)$-matrix with $p+q = \ell$,$p \geq q$, $p-q$ minimal. We denote the corresponding digraph with $G(m,\ell)$. As an example, $G(5,2,1) = G(5,3)$ is shown in Figure 1.1, and has adjacency matrix

$$M(5,2,1) = M(5,3) = \begin{bmatrix} 1&1&1&1&1&1\\ 1&1&1&1&1&0\\ 1&1&1&1&1&0\\ 1&1&1&1&1&0\\ 1&1&1&1&1&0\\ 1&1&0&0&0&0 \end{bmatrix}. \tag{1.2}$$

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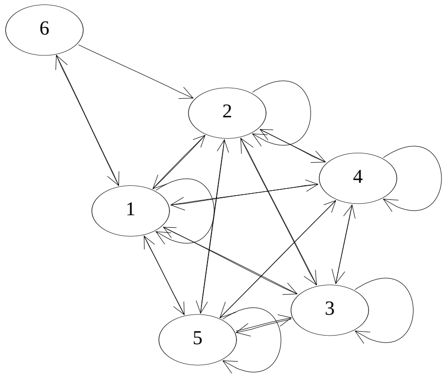

FIG. 1.1. $G(5,2,1) = G(5,3)$

Brualdi and Hoffman [2] studied the following question: if the number of ones in a $(0,1)$-matrix A is a specified integer d, give an upper bound for the spectral radius $\rho(A)$. They showed that when $d = m^2$ or $d = m^2 + 1$ then the maximal spectral radius is m, and is obtained by the matrix $M(m,0)$ and $M(m,1)$, respectively. They also gave an example, due to Don Coppersmith, which shows that $M(3,3)$ is not optimal. For the special case of symmetric $(0,1)$-matrices, they determined the maximal spectral radius when $d = \binom{m}{2}$ (the maximal spectral radius is optained by the complete graph on d vertices) and gave a conjecture for what happens when

$$\binom{m}{2} < d < \binom{m+1}{2}.$$

This conjecture was later proved by Rowlinson [12]. Friedland [6] showed that for a fixed ℓ there is an $L(\ell)$ such that for $m \geq L(\ell)$ the maximal spectral radius of a $(0,1)$-matrix with $m^2 + \ell$ ones is achieved by the matrix $M(m,\ell)$. He conjectured that for any m, ℓ, $0 < \ell < 2m+1$, the maximal spectral radius of a $(0,1)$-matrix with $m^2 + \ell$ ones is achieved by some $(m+1) \times (m+1)$ matrix. For $m^2 + \ell = (m+1)^2 - 4$, he showed that the optimal matrix is *not* $M(m,\ell)$ but rather

$$\left[\begin{array}{c|c} \mathbb{I}_{m-1,m-1} & \mathbb{I}_{m-1,2} \\ \hline \mathbb{I}_{2,m-1} & \mathbb{O}_{2,2} \end{array}\right]. \tag{1.3}$$

The counter-example due to Don Coppersmith, mentioned above, is of this type. It is reasonable to believe that these are the *only exceptions* and that for other ℓ, $M(m,\ell)$ is optimal.

We show a weaker result: for a fixed $s \neq 4$ there is an $S(s)$ such that for $m \geq S(s)$ the maximal spectral radius of a digraph with $(m+1)^2 - s$ edges is achieved by the digraph $G(m, 2m+1-s)$.

Our main tools are the following:

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1. A combinatorial reciprocity theorem by Fröberg [7], Carlitz-Scoville-Vaughan [5], and Gessel [9], which asserts that for a digraph A, the generating series

$$H_A(t) = \sum_{n=0}^{\infty} \chi_n(A) t^n, \tag{1.4}$$

of $\chi_n(A)$, the number of walks of length $n-1$ in A, is related to the series of the complementary digraph by $H_A(t)H_{\bar{A}}(-t) = 1$,
2. A classification by Backelin [1] of the digraphs of s edges with maximal number of walks of length 2.

The proof runs as follows: Backelin's classification shows that for $s > 6$, m sufficiently large, the digraph with the following adjacency matrix has the maximal number of walks of length 2 among digraphs with s edges and $m+1$ vertices:

$$\left[\begin{array}{c|c} \mathbb{O}_{m,m} & \begin{array}{c}\mathbb{O}_{m-s/2,1}\\ \mathbb{I}_{s/2,1}\end{array} \\ \hline \mathbb{O}_{1,m-s/2}\ \mathbb{I}_{1,m-s/2} & 1 \end{array}\right]. \tag{1.5}$$

The generating series for walks in that graph is

$$1 + (m+1)t + st^2 + ct^3 + O(t^4),$$

so the generating series for the complementary graph, which has adjacency matrix $M(m, 2m+1-s)$, is

$$\frac{1}{1-(m+1)t + st^2 - ct^3 + O(t^4)}.$$

For any other digraph with s edges and $m+1$ vertices, we have that the generating series is

$$\frac{1}{1-(m+1)t + st^2 - dt^3 + O(t^4)},$$

with $d < c$. A perturbation analysis yields that the pole of smallest modulus is located at $m^{-1} + sm^{-3} - cm^{-4}$ in the first case and at $m^{-1} + sm^{-3} - dm^{-4}$ in the second, so the first series has smaller radius of convergence, hence faster growth of the coefficients. Consequently, the first graph has the larger spectral radius.

2. The proof. For any digraph A, let

$$H_A(t) = \sum_{n=0}^{\infty} \chi_n(A) t^n, \tag{2.1}$$

where $\chi_n(A)$ denotes the number of walks of length $n-1$ in A, with the convention that $\chi_0(A) = 1$, $\chi_1(A) =$ the number of vertices in A, and $\chi_2(A) =$ the number of edges in A. Let

$$R(A) = \frac{1}{\lim_{n\to\infty} \sqrt[n]{\chi_n(A)}} \tag{2.2}$$

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be the radius of convergence of $H_A(t)$, and let $\rho(A) = 1/R(A)$. If the adjacency matrix $M(A)$ of A is irreducible then $\rho(A) > 0$ is the largest eigenvalue of $M(A)$.

For $(m+1)^2 - s > m^2 + 1$, let $\mathcal{DI}(m,s)$ denote the finite set of digraphs on $\{1,\ldots,m+1\}$ having precisely $(m+1)^2-s$ edges. Let $\mathcal{PDI}(m,s) \subset \mathcal{DI}(m,s)$ denote the subset consisting of those digraphs whose $(0,1)$-adjacency matrix can be regarded as the Young diagram of a numerical partition of $(m+1)^2-s$; in other words, the rows and columns of the adjacency matrix should be weakly decreasing. Then $\mathcal{PDI}(m,s)$ is finite, and the cardinality does not depend on m as long as m is sufficiently large. Furthermore every digraph in $\mathcal{PDI}(m,s)$ is connected in the directed sense, i.e. there is a directed walk between any two vertices. Hence the adjacency matrix of an element in $\mathcal{PDI}(m,s)$ is by definition *irreducible*. Furthermore, by a result of Schwarz [13],

$$\max\{\,\rho(A) | A \in \mathcal{DI}\,\} = \max\{\,\rho(A) | A \in \mathcal{PDI}(m,s)\,\}\,. \tag{2.3}$$

Let $\bar{A}$ denote the complementary graph of A, i.e. the digraph on $\{1,\ldots,m+1\}$ which has an edge $i \to j$ iff there isn't an edge $i \to j$ in A. Then the following relation holds (see the discussion at the end of this article)

$$H_A(t)H_{\bar{A}}(-t) = 1. \tag{2.4}$$

If $A \in \mathcal{PDI}(m,s)$, then $\bar{A}$ is a digraph on $m+1$ vertices with s edges. We have that

$$H_{\bar{A}}(t) = 1 + (m+1)t + st^2 + ct^3 + O(t^4) \tag{2.5}$$

where c is the number of walks in $\bar{A}$ of length 2.

2.1. The case $s > 6$. Suppose first that $s > 6$. Backelin [1] showed that among all digraphs with s edges, the so-called *saturated stars* have the maximal number of walks of length 2. By a saturated star with $s = 2k-1$ edges is meant the digraph with edges $(1,i)$ and $(i,1)$ for $1 \le i \le k$; for $s = 2k$ we add the edge $(1,k+1)$. So the saturated stars with 9 and 10 edges looks as in Figure 2.1

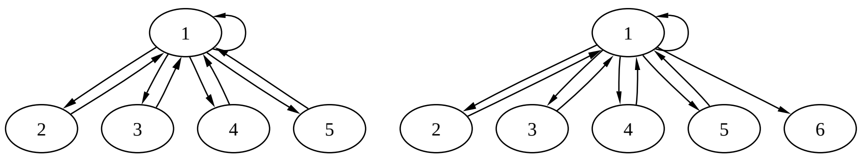

FIG. 2.1. *Saturated star digraphs with 9 and 10 edges*

Note that if R is a saturated star, then the graph on $m+1$ vertices, which has an edge $i \to j$ iff $(m+2-i) \to (m+2-j)$ is not an edge in R, is of the form $G(m,s)$. For instance, the digraphs above have adjacency matrices

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{2.6}$$

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and if we take $m = 7$ we get the following adjacency matrices for the relabeled complementary graphs:

$$\begin{bmatrix} 1&1&1&1&1&1&1&1\\ 1&1&1&1&1&1&1&1\\ 1&1&1&1&1&1&1&1\\ 1&1&1&1&1&1&1&0\\ 1&1&1&1&1&1&1&0\\ 1&1&1&1&1&1&1&0\\ 1&1&1&1&1&1&1&0\\ 1&1&1&0&0&0&0&0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1&1&1&1&1&1&1&1\\ 1&1&1&1&1&1&1&1\\ 1&1&1&1&1&1&1&1\\ 1&1&1&1&1&1&1&0\\ 1&1&1&1&1&1&1&0\\ 1&1&1&1&1&1&1&0\\ 1&1&1&1&1&1&1&0\\ 1&1&0&0&0&0&0&0 \end{bmatrix}. \tag{2.7}$$

Now suppose that R is a saturated star, and that A is the digraph on $m + 1$ vertices obtained from R as above. So $\bar{A}$ and R differ only in that $\bar{A}$ has some isolated vertices. Let R' be a a different digraph with s edges. Let $\bar{B}$ be the digraph obtained by adjoining isolated vertices, so that the total number of vertices becomes $m + 1$. Let $B = \bar{\bar{B}}$. Then

$$H_{\bar{B}}(t) = 1 + (m + 1)t + st^2 + dt^3 + O(t^4), \tag{2.8}$$

where d is the number of walks in R' of length 2. By Backelin's result, $c > d$. Hence we have that

$$\begin{aligned} H_A(t) - H_B(t) &= \frac{1}{1 - (m + 1)t + st^2 - ct^3 + O(t^4)} \\ &\quad - \frac{1}{1 - (m + 1)t + st^2 - dt^3 + O(t^4)} \\ &= (c - d)t^3 + O(t^4) \end{aligned} \tag{2.9}$$

so we have at once that A has strictly more walks of length 2 than B has. By induction, we can show that

LEMMA 2.1. *The exponent of t^i in* (2.9) *is a polynomial in m of degree $i - 3$, with leading coefficient $(i - 2)(c - d)$.*

Thus, for any j, by taking m sufficiently large, we can achieve that the coefficients of t^i, $i \le j$, in (2.9) are all positive. Recall that $\mathcal{PDI}(m, s)$ is finite. Hence, for any j, if we take m sufficiently large, then A has the maximal number of walks of length j among the $B \in \mathcal{PDI}(m, s)$.

In [14] it is shown that if $\mathfrak{G}$ is a digraph with adjacency matrix M, then $H_{\mathfrak{G}}(t) = P(t)/Q(t)$, where $Q(t) = \det(I - tM)$, and $P(t)$ is a polynomial of smaller degree then Q. Hence $H_{\bar{A}}(t), H_{\bar{B}}(t), H_A(t), H_B(t)$ are all rational functions. Let $r = r(m, c)$ be the pole of $H_A(t)$ that is closest to origin. Then $1/r$ is the eigenvalue of M of largest modulus, so from the Perron-Frobenius theorem it follows that if $\mathfrak{G}$ is connected in the directed sense then r is a positive real number. Writing

$$H_A(t) = \frac{1}{1 - (m + 1)t + st^2 - ct^3 + t^4 \frac{b_0 + b_1 t + \cdots + b_{N_1} t^{N_1}}{1 + a_1 t + a_2 t^2 + \cdots + a_{N_2} t^{N_2}}} \tag{2.10}$$

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we have that $r(m,c)$ is the smallest real root of

$$(2.11)\quad (1 + a_1 t + a_2 t^2 + \cdots + a_{N_2} t^{N_2})(1 - (m+1)t + st^2 - ct^3) + t^4(b_0 + b_1 t + \cdots + b_{N_1} t^{N_1}) = 0.$$

We are interested in the asymptotic behavior of r as $m \to \infty$. As we will show below, we can expand $r(m,c)$ as a Laurent series in m as

$$(2.12)\quad r(m,c) = m^{-1} + sm^{-3} - cm^{-4} + (2s^2 + b_0)m^{-5} + (b_1 - a_1 b_0 - 5cs)m^{-6} + O(m^{-7}).$$

Applying similar analysis for the pole $r^*(m,d)$ of $H_B(t)$ that has the smallest modulus, we get

$$r(m,c) - r^*(m,d) = -(c-d)m^{-4} + O(m^{-5}),$$

so

$$r(m,c) < r^*(m,d).$$

Thus, for large m, $H_A(t)$ has strictly smaller radius of convergence than $H_B(t)$. When we combine this with Lemma 2.1 we see that by taking m sufficiently large, we can achieve that $H_A(t) \gg H_B(t)$, i.e. that *all* coefficients of $H_A(t)$ are at least the corresponding coefficients of $H_B(t)$. In fact, the inequality is strict for exponents more than 2.

2.1.1. Perturbation analysis of the positive root. We now show how to derive the expansion (2.12). Replacing $m+1$ by $1/\varepsilon$ in (2.11), and clearing denominators, we get

$$(2.13)\quad (1 + a_1 t + a_2 t^2 + \cdots + a_{N_2} t^{N_2})(\varepsilon - t + \varepsilon st^2 - \varepsilon ct^3) + \varepsilon t^4(b_0 + b_1 t + \cdots + b_{N_1} t^{N_1}) = 0.$$

The unperturbed equation is

$$(2.14)\quad (1 + a_1 t + a_2 t^2 + \cdots + a_{N_2} t^{N_2})(-t) = 0,$$

which has a root at $t = 0$. We introduce the scaling $t = \varepsilon T$ and get

$$(2.15)\quad (1 + a_1 \varepsilon T + a_2 \varepsilon^2 T^2 + \cdots + a_{N_2} \varepsilon^{N_2} T^{N_2})(\varepsilon - \varepsilon T + \varepsilon^3 sT^2 - \varepsilon^4 cT^3) + \varepsilon^5 T^4(b_0 + b_1 \varepsilon T + \cdots + b_{N_1} \varepsilon^{N_1} T^{N_1}) = 0$$

hence

$$(2.16)\quad (1 + a_1 \varepsilon T + a_2 \varepsilon^2 T^2 + \cdots + a_{N_2} \varepsilon^{N_2} T^{N_2})(1 - T + \varepsilon^2 sT^2 - \varepsilon^3 cT^3) + \varepsilon^4 T^4(b_0 + b_1 \varepsilon T + \cdots + b_{N_1} \varepsilon^{N_1} T^{N_1}) = 0.$$

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The unperturbed equation is now $1 - T = 0$. Hence, $T = O(1)$ and $T^{-1} = O(1)$, so this is the correct scaling. We make the substitution $Y = T - 1$ and get

$$\begin{aligned}(2.17)\quad &\left[1 + a_1\varepsilon(Y+1) + a_2\varepsilon^2(Y+1)^2 + \cdots + a_{N_2}\varepsilon^{N_2}(Y+1)^{N_2}\right]\left[-Y + \varepsilon^2 s(Y+1)^2 - \right.\\ &\left.\varepsilon^3 c(Y+1)^3\right] + \varepsilon^4(Y+1)^4(b_0 + b_1\varepsilon(Y+1) + \cdots + b_{N_1}\varepsilon^{N_1}(Y+1)^{N_1}) = 0.\end{aligned}$$

It is now clear that Y can be expanded in powers of ε, so we make the *Ansatz*

$$Y = \sum_{i=1}^{\infty} w_i\varepsilon^i. \tag{2.18}$$

Collecting the coefficients of the powers of ε in (2.17) we have

$$\begin{aligned}
&(2.19) && 1: && 0\\
&(2.20) && \varepsilon: && -w_1\\
&(2.21) && \varepsilon^2: && s - a_1w_1 - w_2\\
&(2.22) && \varepsilon^3: && 2sw_1 + a_1s - c - a_1w_1^2 - a_1w_2 - a_2w_1 - w - 3\\
&(2.23) && \varepsilon^4: && 2sw_2 + sw_1^2 - 3cw_1 - a_1c + a_2s\\
&(2.24) && && + 3a_1sw_1 + b_0 - 2a_1w_1w_2 - a_1w_3 - 2a_2w_1^2 - a_2w_2 - a_3w_1 - w_4.
\end{aligned}$$

These should be zero, which allows us to solve for the w_i's, obtaining

$$w_1 = 0, \quad w_2 = s, \quad w_3 = -c, \quad w_4 = 2s^2 + b_0. \tag{2.25}$$

So

$$Y = s\varepsilon^2 - c\varepsilon^3 + (2s^2 + b_0)\varepsilon^4 + O(\varepsilon^5), \tag{2.26}$$

hence

$$T = 1 + s\varepsilon^2 - c\varepsilon^3 + (2s^2 + b_0)\varepsilon^4 + O(\varepsilon^5), \tag{2.27}$$

hence

$$t = \varepsilon + s\varepsilon^3 - c\varepsilon^4 + (2s^2 + b_0)\varepsilon^5 + O(\varepsilon^6). \tag{2.28}$$

This concludes the proof for the case $s > 6$.

2.2. The exceptional cases. It remains to take care of the case $s \leq 6$. Backelin's classification says that if $s \in \{1, 3, 5\}$, then the saturated stars are optimal. Hence, it remains to check the cases $s = 2$, $s = 4$, $s = 6$.

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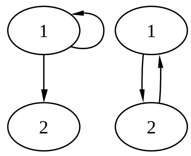
2.2.1. $s = 2$**.** For $s = 2$ there are two non-isomorphic graphs R, namely



corresponding to matrices $M_1 = \left[\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\right]$ and $M_2 = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$. Recall that when M_1 or M_2 is subtracted from the bottom right corner of the $(m+1) \times (m+1)$ matrix of all ones, the result should be weakly decreasing in rows and columns. We call the resulting matrices

$$A_1 = \begin{bmatrix} 1 & \cdots & 1 & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \cdots & 1 & 1 & 1 \\ 1 & \cdots & 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} 1 & \cdots & 1 & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \cdots & 1 & 1 & 0 \\ 1 & \cdots & 1 & 0 & 1 \end{bmatrix}. \tag{2.29}$$

The matrix A_2 is not weakly decreasing in the last row, so it is not really necessary to continue with the calculations, but we proceed anyway in order to demonstrate how this is done. We have that

$$H_{M_1}(t) = H_{M_2}(t) = \frac{1+t}{1-t}, \tag{2.30}$$

so that

$$H_{A_1}(t) = H_{A_2}(t) = \frac{1}{\frac{1-t}{1+t} - m + 1} = \frac{1+t}{1 - tm - (m-1)t^2}. \tag{2.31}$$

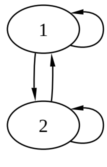
The smallest positive root of the denominator is

$$\frac{-m + \sqrt{m^2 + 4\,m - 4}}{2(m-1)}, \tag{2.32}$$

so the spectral radius is

$$\frac{2(m-1)}{-m + \sqrt{m^2 + 4\,m - 4}}. \tag{2.33}$$

2.2.2. $s = 4$**.** From Backelin's classification we have that for $s = 4$, the digraph



with adjacency matrix $\left[\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right]$ has the most walks of length 2. This is in accordance with [6] where it is shown that the matrix (1.3) has maximal spectral radius among $(0,1)$-matrices with $(m+1)^2 - 4$ ones.

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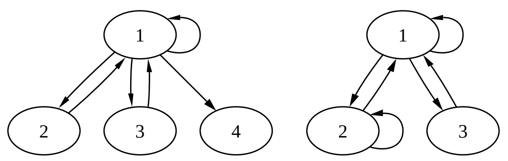

FIG. 2.2. *Digraphs with 6 edges having maximal number of walks of length 2*

2.2.3. $s = 6$**.** The remaining exceptional case in Backelin's classification is for $s = 6$. Then the digraphs in Figure 2.2, with adjacency matrices

$$M_1 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad M_2 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \tag{2.34}$$

both have 14 of walks of length 2. Note that the first digraph is a saturated star. The relabeled complemented matrices are

$$A_1 = \begin{bmatrix} 1 & \cdots & 1 & 1 & 1 & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \cdots & 1 & 1 & 1 & 1 & 1 \\ 1 & \cdots & 1 & 1 & 1 & 1 & 0 \\ 1 & \cdots & 1 & 1 & 1 & 1 & 0 \\ 1 & \cdots & 1 & 0 & 0 & 0 & 0 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 1 & \cdots & 1 & 1 & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \cdots & 1 & 1 & 1 & 1 \\ 1 & \cdots & 1 & 1 & 1 & 0 \\ 1 & \cdots & 1 & 1 & 0 & 0 \\ 1 & \cdots & 1 & 0 & 0 & 0 \end{bmatrix}. \tag{2.35}$$

The generating series for the complemented relabeled digraphs are

$$H_{A_1}(t) = \frac{1 + t - 2t^2}{1 - mt - mt^2 + 3t^2 + 2mt^3 - 6t^3} =: \frac{p_1(t)}{p_2(t)} \tag{2.36}$$

$$H_{A_2}(t) = \frac{1 + 2t - t^2 - t^3}{1 + t - mt + 3t^2 - 2mt^2 - 2t^3 + mt^3 - 2t^4 + mt^4} =: \frac{q_1(t)}{q_2(t)}. \tag{2.37}$$

Regarding the positive root of $p_2(t)$ as a function of m, and expanding that function as a power series about infinity, we get

$$r_1(m) := m^{-1} - m^{-2} + 7m^{-3} - 33m^{-4} + 191m^{-5} + O(m^{-6}),$$

whereas the expansion of the positive root of $q_2(t)$ is

$$r_2(m) := m^{-1} - m^{-2} + 7m^{-3} - 33m^{-4} + 196m^{-5} + O(m^{-6}).$$

The first root is therefore slightly smaller for large m; the difference is *miniscule*, but *vive la différence*! In fact, since $r_1(m) - r_2(m) < 0$ for large m, and $r_1(4) - r_2(4) \approx -0.003$, it will suffice to show that $p_2(t) = q_2(t) = 0$ has no solution to demonstrate that $r_1(m) - r_2(m) < 0$ for $m \geq 4$, as shown in Figure 2.3. Using Macaulay 2 [10] we can verify that $\{1\}$ is a Gröbner bases for the ideal generated by $\{p_2(t), q_2(t)\}$ in $\mathbb{Q}(m)[t]$. Hence $p_2(t), q_2(t)$ are co-prime in $\mathbb{Q}(m)[t]$, so they can not have a common zero.

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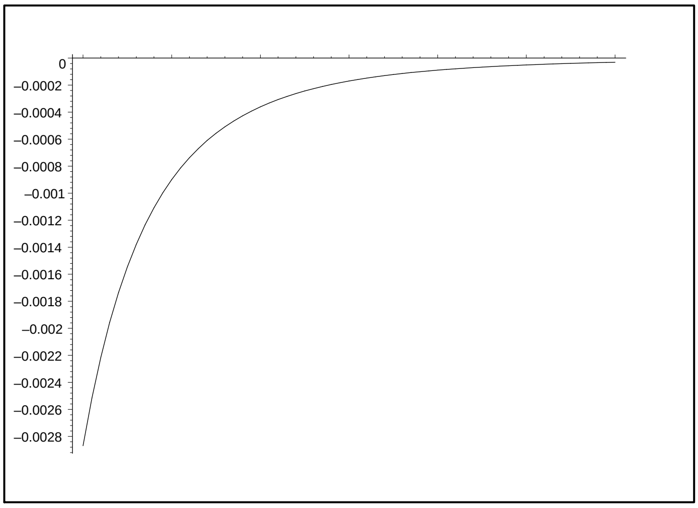
FIG. 2.3. *Difference* $r_1(m) - r_2(m)$ *for* $4 \leq m \leq 10$.



2.3. A note on the relation $H_A(t)H_{\bar{A}}(-t) = 1$. The relation (2.4) was proved by Carlitz, Scoville and Vaughan in 1976 [5]. Gessel, in his PhD thesis [9], proved a sharper version, counting the number of walks starting in a specified subset of the set of vertices. Nice expositions of this are [4] and [3].

However, already in 1975 Fröberg [7] had showed that if $R = \mathbb{C}\langle x_1, \ldots, x_n\rangle/I$, where $\mathbb{C}\langle x_1, \ldots, x_n\rangle$ is the free associate algebra on $x_1, \ldots, x_n$, and I is an ideal generated by monomials of degree 2, J the ideal generated by those quadratic monomials not in I, $R' = \mathbb{C}\langle x_1, \ldots, x_n\rangle/J$, $H_R(t)$ the Hilbert series of the graded algebra R and $H_{R'}(t)$ the Hilbert series of R', then

$$H_R(t)H_{R'}(-t) = 1. \tag{2.38}$$

If G is a digraph on $\{1, \ldots, n\}$, then taking I to be the ideal generated by $x_i x_j$ for all i, j such that $i \to j$ is *not* a directed edge in G, we recover (2.4).

More abstractly, since the ring R' is the *Koszul dual* of R, (2.38) follows. This point of view is explored by Reiner in [11].

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