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ABSOLUTELY FLAT IDEMPOTENTS*

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Abstract. A real n-by-n idempotent matrix A with all entries having the same absolute value is called *absolutely flat*. The possible ranks of such matrices are considered along with a characterization of the triples: size, constant, and rank for which such a matrix exists. Possible inequivalent examples of such matrices are also discussed.

Key words. Idempotent matrix, Absolutely flat, Projection.

AMS subject classifications. 15A21, 15A24, 05A15, 46B20.

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1. Introduction. We consider a problem, suggested, in part by [4] and specifically mentioned to the other authors by Harel. The problem considered in [4] (see also [2, 3, 5]) is about the isomorphic classification of the ranges of nicely bounded projections in some classical Banach spaces. It has been solved in [4] in the special case of projections of small norms and another special case is that of absolutely flat idempotents. We also found this question of independent interest.

For which positive integers n, does there exist an n-by-n real, idempotent matrix A of rank r, all of whose entries are a positive constant c in absolute value (*absolutely flat*)? From the equation $A^2 = A$, it readily follows that c must be $1/k$ for some positive integer $k \leq n$. Thus, the key parameters of our problem are n, k, r: for which triples of positive integers is there a matrix A of desired type? Since $kA = B$ is a ± 1 matrix, an equivalent formulation concerns the existence of a ± 1 matrix B such that $B^2 = kB$, and we reserve the letter B for such a ± 1 matrix that comes from a given A in sections 2 and 3 below.

Since the minimal polynomial of A must divide (and, in fact, equal in our case) $x^2 - x$, A is diagonalizable ([1, p. 145]) and all of its eigenvalues must be 0 or 1.

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Importantly, $\mathrm{Tr}[A] = \mathrm{rank}\, A$, as each is simply the count of the number of eigenvalues equal to 1. We first derive two number theoretic necessary conditions that constrain feasible triples n, k, r. Then, we show that for odd n, only $r = 1$ is possible and that all triples $n, k, 1$ meeting the necessary conditions do occur. Finally, for even n, all triples meeting the necessary conditions occur, completing a characterization of feasible triples. We also discuss the existence of multiple matrices, distinct modulo obvious symmetries of the problem, which are absolutely flat idempotents for the same parameters n, k, r.

2. The Elementary Necessary Conditions. A signature matrix is a diagonal matrix S with diagonal entries ± 1. It is clear that similarity does not change the property of idempotence. Further, permutation and signature similarity do not change the set of absolute values of the entries of a matrix. Thus, permutation and signature similarity do not change whether A is an absolutely flat idempotent, nor do they change the parameters n, k, r if A is.

From the equality of rank and trace, an absolutely flat idempotent must have at least one positive diagonal entry, and, therefore, after a permutation similarity, we may assume a positive number as the (1,1) entry of A. Then, any absolutely flat idempotent may be normalized, by signature similarity, so that all the entries in its first column are positive. We generally assume this normalization. From $A^2 = A$, it then follows that the number of negative entries in each row is constant. Call this number $u \geq 0$. Similarly, let $m \geq 0$ denote the number of negative entries on the main diagonal of A (an absolutely flat idempotent of parameters n, k, r). The trace of A is $[(n-m)/k] - [m/k] = (n-2m)/k$, but since rank equals trace, we have $n - 2m = kr$ or

$$n = rk + 2m, \tag{2.1}$$

the first of our necessary conditions. The second follows from $B^2 = kB$, with B in normalized form. The inner product of the first (any) row of B with the first (normalized) column has $n - u$ positive summands and u negative summands, with the net sum being k. Thus, $n - 2u = k$ or

$$n - k = 2u. \tag{2.2}$$

Since $m \geq 0$, it follows from (2.1) that

$$rk \leq n, \tag{2.3}$$

and it follows from (2.2) that

$$n \text{ and } k \text{ have the same parity.} \tag{2.4}$$

It also follows from (2.1) that n is odd if and only if r and k are odd.

3. The Odd Case. Many triples with n odd and $r > 1$ (and necessarily odd) satisfy the requirements (2.1) and (2.2). However, interestingly, absolutely flat idempotents never exist in such cases.

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THEOREM 3.1. *For an odd integer n, there is an absolutely flat idempotent A with parameters n, k, r if and only if $r = 1$ and $k \leq n$ is odd. In this event, the matrix A is unique up to signature/permutation similarity;*

$$A = \frac{1}{k}\begin{bmatrix} 1 & \cdots & 1 & -1 & \cdots & -1 \\ 1 & \cdots & 1 & -1 & \cdots & -1 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 1 & \cdots & 1 & -1 & \cdots & -1 \\ 1 & \cdots & 1 & -1 & \cdots & -1 \end{bmatrix},$$

in which there are $m = u = (n - rk)/2$ columns of -1's.

Proof. If $r = 1$, and $k \leq n$ is odd, it is easily checked that the displayed matrix A shows existence. Furthermore, in this event, any absolutely flat idempotent that is normalized via (permutation and) signature similarity to have positive first column and then by permutation similarity to have all positive entries in the first $n - u$ columns, will have all rows equal and appear as the displayed A. It follows that $m = u = (n - rk)/2$ and that this is the number of negative columns.

If n is odd, we already know that $k \leq n$ is odd and r is odd. We show that $r = 1$ in two cases: $k = 1; k > 1$. Let $(n, k, r) = (2l + 1, 2t + 1, 2s + 1)$. Consider B in normalized form, so that $B^2 = kB$, with B a ± 1 matrix, and partition B as

$$B = \begin{bmatrix} 1 & f^T \\ e & C \end{bmatrix} \tag{3.1}$$

in which e is the $2l$-by-1 vector of 1's and f is a $2l$-by-1 vector consisting of $(l + t)$ 1's followed by $(l - t)$ -1's. From $B^2 = kB$, it follows that

$$\begin{aligned} f^T e &= 2t \\ f^T C &= 2tf^T \\ Ce &= 2te \\ ef^T + C^2 &= (2t + 1)C. \end{aligned} \tag{3.2}$$

Multiplication of both sides of the last equation on the left by C and use of $Ce = 2te$ yields

$$\begin{aligned} 0 &= C^3 - (2t + 1)C^2 + 2tef^T \\ &= C^3 - (4t + 1)C^2 + 2t(2t + 1)C. \end{aligned} \tag{3.3}$$

Since C is a ± 1 matrix and is of even dimension, it follows that $2 \mid C^2$ (entry-wise) and then, by a simple induction, that $2^q \mid C^{2^q}$ for each positive integer q. Thus, $2^q \mid \mathrm{Tr}\big[C^{2^q}\big]$ for all positive integers q.

Now distinguish two possibilities: $k = 1$ ($t = 0$); and $k > 1$ ($t > 0$). In the former case, (3.3) gives $C^3 = C^2$ and, thus, by induction, $C^{2^q} = C^2$. Therefore, $2^q \mid C^2$ for all positive integers q, which gives $C^2 = 0$. But then C is nilpotent; $\mathrm{Tr}[C] = 0$, and $\mathrm{Tr}[B] = \mathrm{Tr}[A] = \text{rank } A = r = 1$, as was to be shown.

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Now, suppose $t > 0$ ($k > 1$). First, rank C = rank B, as the first column of B is $1/2t$ times the sum of the last $2l$ columns of B (by the first and third equations of (3.2)), and the equation, $f^T C = 2tf^T$, implies that f^T can be written as a linear combination of rows of C. From (3.3), C is diagonalizable with distinct eigenvalues from $\{0, 2t, 2t+1\}$. Let $a \geq 0$ be the number of eigenvalues of C equal to $2t$, $b \geq 0$ be the number equal to $(2t+1)$; then there are $2l - a - b$ of them equal to 0. Since rank C = rank B, we have

$$a + b = 2s + 1.$$

Also, $r = \mathrm{Tr}[B]/(2t+1)$, so that $2ta+(2t+1)b = \mathrm{Tr}[C] = \mathrm{Tr}[B]-1 = (2s+1)(2t+1)-1$, or $2ta + (2t+1)b = (2s+1)2t + 2s$. These two equations have the unique solution $(a, b) = (1, 2s)$. We may now calculate $\mathrm{Tr}\left[C^{2^q}\right]$ as

$$(2t)^{2^q} + 2s(2t+1)^{2^q}.$$

Since $2^q \mid C^{2^q}$, still, and thus $2^q \mid \mathrm{Tr}\left[C^{2^q}\right]$ for all positive integers q, we have $s = 0$, or $r = 1$, as was to be shown. This concludes the proof. □

4. The Even Case. When n is even, conditions (2.1) and (2.2) still govern existence, but the overall situation is remarkably different from the odd case. Now, there is existence whenever the conditions are met. Here we exhibit an absolutely flat idempotent for each triple n, k, r meeting the conditions (2.1) and (2.2).

For a given positive integer k, define

$$P = \frac{1}{k}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad M = \frac{1}{k}\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}. \tag{4.1}$$

Because of (2.2), $k = 2t$ must be even, and we have

$$P^2 = \frac{1}{t}P,\ PM = \frac{1}{t}M,\ M^2 = 0,\ MP = 0. \tag{4.2}$$

Solutions may now be constructed using the P's and M's as blocks. For example, a solution for (n, k, r) = (8,2,3) is

$$\begin{bmatrix} P & M & M & M \\ M & P & M & M \\ M & M & P & M \\ P & M & M & M \end{bmatrix}.$$

Since n is even and, therefore, k is even, we assume our parameters are of the form $(n, k, r) = (2l, 2t, r)$; r need not be even. From (2.3), it follows that $tr \leq l$. As proof of the following theorem, we give a general strategy for constructing absolutely flat idempotents with parameters $2l, 2t$ and r, $tr \leq l$.

Theorem 4.1. *Let $n = 2l$, $k = 2t$ and r be positive integers such that $tr \leq l$. Then, there is an absolutely flat idempotent with parameters n, k, r. In particular, whenever n is even, there is an absolutely flat idempotent whenever conditions (2.1) and (2.2) are met.*

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Proof. Let $(n,k,r) = (2l, 2t, r)$. By the elementary necessary conditions, express $2l$ as $2tr + 2m$ for some $m \in \mathbb{N}$. Let P and M be the matrices as in (4.1). Examine now the block matrix,

$$
A = \begin{bmatrix}
P & \cdots & P & & & & & & & \\
\vdots & \ddots & \vdots & & & & & & & \\
P & \cdots & P & & & & & & & \\
 & & & \ddots & & & & & & \\
 & & & & P & \cdots & P & & & \\
 & & & & \vdots & \ddots & \vdots & & & \\
 & & & & P & \cdots & P & & & \\
P & \cdots & P & & & & & M & \cdots & M \\
\vdots & \ddots & \vdots & & & & & \vdots & \ddots & \vdots \\
P & \cdots & P & & & & & M & \cdots & M
\end{bmatrix}. \tag{4.3}
$$

The matrix A consists of r t-by-t blocks of P's along the main diagonal and an m-by-t block of P's in the lower left-hand corner. All other blocks in A are M's. It is then an elementary exercise in block matrix multiplication (using (4.2)) that $A^2 = A$. As the trace of A is r, it follows that the rank of A is r. This completes the proof. ◻

Theorems 3.1 and 4.1 provide a complete characterization of the triples n, k, r for which absolutely flat idempotents exist. We note that any positive integer k (n) may occur, but for odd k (n), only rank 1 matrices exist. On the other hand, any rank may occur. In either case, n need be sufficiently large.

5. Multiple Solutions. By appealing to the Jordan canonical form, any two n-by-n idempotents of the same rank are similar. However, for our problem, restriction to permutation and signature similarity is more natural; of course, permutation and signature similarities send one solution for n, k, r to another for the same n, k, r. Although it has not been important for our earlier results, transposition is another natural operation sending one solution to another. It is natural to ask how many solutions, distinct up to permutation, signature similarity, and transposition can occur. When $r = 1$, it is easily worked out that there is only one (when there is one). The form mentioned in Theorem 3.1 is canonical (even when n is even).

However, already for the parameters (8,2,2), there can be distinct solutions. For example,

$$
A_1 = \frac{1}{2}\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 \\
1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 \\
1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 \\
1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 \\
1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 \\
1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 \\
1 & 1 & -1 & -1 & -1 & 1 & 1 & 1
\end{bmatrix}
$$

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and

$$A_2 = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & -1 & 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & -1 & -1 & -1 & 1 & 1 \\ 1 & 1 & 1 & -1 & -1 & -1 & 1 & 1 \\ 1 & 1 & 1 & -1 & -1 & -1 & 1 & 1 \\ 1 & 1 & 1 & -1 & 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & -1 & -1 & -1 & 1 & 1 \\ 1 & 1 & 1 & -1 & -1 & -1 & 1 & 1 \\ 1 & 1 & 1 & -1 & -1 & -1 & 1 & 1 \end{bmatrix}$$

are both absolutely flat (8,2,2) idempotents. To see that A_1 is not permutationally or signature similar to A_2 (or its transpose), we mention an idea that we used to discover some of the construction herein, but was not needed in the proofs thus far. We say that two rows (columns) of an n-by-n $\pm 1/k$ matrix are *of the same type* if they are either identical or negatives of each other. It is an easy exercise that the number of distinct row types (number of distinct column types) is unchanged by either signature similarity or permutation similarity.

Additionally, we define the *row* (*column*) *multiplicity* of an absolutely flat idempotent matrix, A, to be the multiset consisting of the number of rows (columns) for each row (column) type. It is again an easy exercise that permutation/signature similarity does not change the row (column) multiplicity of an absolutely flat idempotent matrix. In the matrix A_1 the row (column) multiplicity is $\{6,2\}$ ($\{6,2\}$), while in A_2 the row (column) multiplicity is $\{6,2\}$ ($\{4,4\}$). Thus, A_1 cannot be transformed to A_2 by any combination of permutation/signature similarities and/or transposition (though they are similar).

Of course, the number of row types in a matrix is at least the rank. We note that the construction technique of Theorem 4.1 always produces a solution with the same number of row types as rank. The (8,2,3) example, A_3, below demonstrates that larger numbers of row types are possible. However, it may be shown that for rank 2 absolutely flat idempotents, only 2 row and column types are possible.

$$A_3 = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 \\ 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 \\ 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 \\ 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 & 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 \end{bmatrix}$$

LEMMA 5.1. *A rank 2 absolutely flat matrix has precisely 2 row types and 2 column types.*

Proof. We prove the result for row types as the case of columns is similar. Let A be a rank 2 absolutely flat matrix. Performing permutation and signature similarity we may assume that the first column of A consists only of positive entries, as this

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doesn't change the number of row types. Since A has rank 2, there are at least 2 distinct row types. Let x and y be the two rows corresponding to these row types, and let w be an arbitrary other row in A. Then,

$$w = ax + by$$

for some $a, b \in \mathbb{Q}$. Clearly, we must have $a + b = 1$ because the initial entries of w, x, and y are all the same. Since x and y are different rows, it follows from the absolutely flat property that $a - b = 1$ or $a - b = -1$. In the first case, we have $a = 1$ and $b = 0$, and in the second, it follows that $a = 0$ and $b = 1$. This completes the proof. □

We now consider the problem of counting all different rank 2 absolutely flat idempotent matrices. As we are interested in distinct solutions up to permutation and signature similarity, we first put our matrix in a normalized form. Let A be a rank 2 absolutely flat idempotent matrix with parameters $(n, k, 2)$. As before, we can perform a permutation and signature similarity to make the first column of A positive. Let a (b) be the number of all positive (negative) columns of A. Through another permutation similarity, we may assume that the first a columns of A are positive and that the next b columns of A are negative. From Lemma 5.1, the remaining $n - a - b$ columns of A are of one type. Let v be one of these columns (necessarily containing both a positive and a negative entry) and let c be the number of them in A. Notice that the other $d = n - a - b - c$ columns must be $-v$. Since $c + b = u$ and $d + b = u$ (the number of negative entries in each row must be u), it follows that $d = c$. These normalizations partition our matrix as

$$F = \begin{bmatrix} P_{a,a} & M_{a,b} & W_{a,c} & -W_{a,c} \\ P_{b,a} & M_{b,b} & X_{b,c} & -X_{b,c} \\ P_{c,a} & M_{c,b} & Y_{c,c} & -Y_{c,c} \\ P_{c,a} & M_{c,b} & Z_{c,c} & -Z_{c,c} \end{bmatrix}. \tag{5.1}$$

Here, the $P_{i,j}$ are positive matrices of sizes i-by-j; $M_{i,j}$ are negative matrices of sizes i-by-j; and $W_{i,j}$, $X_{i,j}$, $Y_{i,j}$, and $Z_{i,j}$ are matrices of sizes i-by-j with exactly 1 row type. Through further permutation, it is clear that the columns of $W_{a,c}$, $X_{b,c}$, $Y_{c,c}$, and $Z_{c,c}$ can be assumed to begin with all positive entries and end with all negative ones:

$$\begin{bmatrix} + & \cdots & + \\ \vdots & \ddots & \vdots \\ + & \cdots & + \\ - & \cdots & - \\ \vdots & \ddots & \vdots \\ - & \cdots & - \end{bmatrix}. \tag{5.2}$$

Let a_p (a_m) be the number of positive (negative) rows in $W_{a,c}$; b_p (b_m) be the number of positive (negative) rows in $X_{b,c}$; c_{1p} (c_{1m}) be the number of positive (negative) rows in $Y_{c,c}$; and c_{2p} (c_{2m}) be the number of positive (negative) rows in $Z_{c,c}$. The

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final matrix produced after this sequence of operations is called the *standard* form of A.

We now derive necessary conditions on the parameters defined above for the matrix as in (5.1) to be idempotent. Clearly, we must have $a_p + a_m = a$, $b_p + b_m = b$, $c_{1p} + c_{1m} = c$, $c_{2p} + c_{2m} = c$, $a + b + 2c = n$, and $b + c = u$. Examining the inner product of the first row and the first column, we see that $a - b = k$, and looking at the inner products of each row type with the second column type produces the equations,

$$a_p - a_m - b_p + b_m + c_{1p} - c_{1m} - c_{2p} + c_{2m} = k \tag{5.3}$$

and

$$a_p - a_m - b_p + b_m - c_{1p} + c_{1m} + c_{2p} - c_{2m} = -k. \tag{5.4}$$

Adding equations (5.3) and (5.4) gives us

$$\begin{aligned} 0 &= a_p/2 - a_m/2 + b_m/2 - b_p/2 \\ &= a_p - b_p + (b - a)/2 \\ &= a_p - b_p - k/2, \end{aligned}$$

and a similar computation with the subtraction of (5.3) and (5.4) produces the equation, $c_{1p} = c_{2p} + k/2$. Many of these necessary conditions are actually redundant, and so we will only consider the system,

$$\begin{aligned} a - b &= k \\ b + c &= u \\ a_p &= b_p + k/2 \\ c_{1p} &= c_{2p} + k/2. \end{aligned} \tag{5.5}$$

In fact, we have the following

THEOREM 5.2. *A matrix in standard form in which $a, b, c, a_p, b_p, c_{1p}, c_{2p}$ are all nonnegative and satisfy (5.5) is an (n,k,2) idempotent.*

Proof. Assume that A is in standard form with $a, b, c, a_p, b_p, c_{1p}, c_{2p} \geq 0$ and (5.5) satisfied. To prove idempotence, we need to check three inner products. The inner product of the first row type and first column type is just $a - b = k$, and the inner product of the first row type and the second column type is

$$\begin{aligned} a_p - (a - a_p) - b_p + (b - b_p) + (c_{2p} + k/2) - (c - c_{2p} - k/2) - c_{2p} + (c - c_{2p}) \\ = 2a_p - 2b_p - a + b + k \\ = k \end{aligned}$$

as desired. A similar computation involving the second row type and the second column type gives us

$$\begin{aligned} a_p - (a - a_p) - b_p + (b - b_p) - (c_{2p} + k/2) + (c - c_{2p} - k/2) + c_{2p} - (c - c_{2p}) \\ = 2a_p - 2b_p - a + b - k \\ = -k. \end{aligned}$$

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Finally, adding the equations $2b+2c = 2u$ and $a-b = k$ gives us that $a+b+2c = n$, completing the proof. □

In what follows, the multiplicities of an absolutely flat idempotent matrix will be important. Let $x_A = a_p + b_p + c_{1p} + c_{2p} = 2b_p + 2c_{2p} + k$ and set $y_A = a + b$. Then, the row and column multiplicities of A in standard form are $\{x_A, n - x_A\}$ and $\{y_A, n - y_A\}$, respectively. The following lemma is a natural consequence of the symmetries of the problem.

Lemma 5.3. *Let A be a rank 2 absolutely flat idempotent in standard form with row and column multiplicities of $\{x_A, n-x_A\}$ and $\{y_A, n-y_A\}$ as above. Then, A is permutation/signature equivalent to a matrix B in standard form with $x_B = n - x_A$ and $y_B = y_A$. Similarly, A is permutation/signature equivalent to a matrix B in standard form with $y_B = n - y_A$ and $x_B = x_A$.*

Proof. Let A be as in (5.1). After permuting the last $2c$ columns and the corresponding last $2c$ rows, A becomes

$$\begin{bmatrix} P_{a,a} & M_{a,b} & -W_{a,c} & W_{a,c} \\ P_{b,a} & M_{b,b} & -X_{b,c} & X_{b,c} \\ P_{c,a} & M_{c,b} & -Z_{c,c} & Z_{c,c} \\ P_{c,a} & M_{c,b} & -Y_{c,c} & Y_{c,c} \end{bmatrix}.$$

Through further permutation, the columns of $-W_{a,c}$, $-X_{b,c}$, $-Z_{c,c}$, and $-Y_{c,c}$ can be made to look like those in (5.2). Now, this final matrix, B, is in normal form with $x_B = n - x_A$, and $y_B = y_A$ as desired.

As for the second statement in the lemma, first perform a signature similarity on A that makes each column of,

$$\begin{bmatrix} W_{a,c} \\ X_{b,c} \\ Y_{c,c} \\ Z_{c,c} \end{bmatrix},$$

either all positive or all negative, and then perform a permutation similarity to bring our matrix back into standard form. It is clear that this new matrix, B, has $y_B = n - y_A$. If $x_B = x_A$, then we are done. Otherwise, $x_B = n - x_A$, and we can proceed as above to form an equivalent matrix, B', with $x_{B'} = n - x_B = x_A$ and $y_{B'} = y_B = n - y_A$. This completes the proof of the lemma. □

We are now in a position to give bounds for the number of rank 2 absolutely flat idempotent matrices up to permutation/signature similarity and transposition. A straightforward verification (using (2.1) and (2.2)) shows that

$$\begin{aligned} &a_p = k/2,\ a_m = \lceil n/4 \rceil \\ &b_p = 0,\ b_m = \lceil m/2 \rceil \\ &c_{1p} = k/2,\ c_{1m} = \lfloor m/2 \rfloor \\ &c_{2p} = 0,\ c_{2m} = \lfloor n/4 \rfloor \end{aligned}$$

satisfy (5.5) and, therefore, produce an $(n, k, 2)$ absolutely flat idempotent matrix by Theorem 5.2 (this is, in fact, the solution found in Theorem 4.1). In this case, the row multiplicity is $\{k, n-k\}$ and the column multiplicity is $\{2\lceil n/4 \rceil, 2\lfloor n/4 \rfloor\}$.

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Let $a, b, c, a_p, b_p, c_{1p}, c_{2p}$ be an arbitrary solution to (5.5). Set $t = a_p - k/2 = b_p$, $q = c_{1p} - k/2 = c_{2p}$, and let $l = \lceil n/4 \rceil - a_m$. Since $a_p + a_m - b_p - b_m = k$, it follows that $l = \lceil m/2 \rceil - b_m$. If we set $p = \lfloor m/2 \rfloor - c_{1m}$ and $y = \lfloor n/4 \rfloor - c_{2m}$, then from $c_{1p} + c_{1m} = c_{2p} + c_{2m}$ we must have $p = y$. Finally, the equation $b + c = u$ implies that $p = q + t - l$. It is easily seen that these conditions are also sufficient, and so we have the following.

THEOREM 5.4. *All solutions to (5.5) in nonnegative integers are given by*

$$\begin{aligned}
&a_p = k/2 + t, \ a_m = \lceil n/4 \rceil - l \\
&b_p = t, \ b_m = \lceil m/2 \rceil - l \\
&c_{1p} = k/2 + q, \ c_{1m} = \lfloor m/2 \rfloor - q - t + l \\
&c_{2p} = q, \ c_{2m} = \lfloor n/4 \rfloor - q - t + l
\end{aligned}$$

in which $t, q \in \mathbb{N}$, $l \in \mathbb{Z}$, *and*

$$q + t - \lfloor m/2 \rfloor \ \leq \ l \ \leq \ \lceil m/2 \rceil .$$

In particular, when $m = 0$, we must have $q = t = l = 0$, giving us the immediate

COROLLARY 5.5. *Up to permutation/signature similarity and transposition, there is only one rank 2 absolutely flat idempotent matrix with* $n = 2k$.

With a careful consideration of Theorem 5.4, we can produce bounds for the number of inequivalent $(n, k, 2)$ absolutely flat idempotents. Notice that for the parameterized solutions in Theorem 5.4, we have $x_A = 2t+2q+k$ and $y_A = k+2\lceil m/2 \rceil+2t-2l$. In particular, the conditions in Theorem 5.4 imply that $x_A = k + 2i$ and $y_A = k + 2j$ for some $i, j \in \{0, \ldots, m\}$.

In fact, the converse is true. Namely, let $i, j \in \{0, \ldots, m\}$; then, we claim that (5.5) has a solution, A, in which $x_A = k + 2i$ and $y_A = k + 2j$. To see this, fix $i \in \{0, \ldots, m\}$, and let l and t be such that $0 \leq t \leq i$ and $i - \lfloor m/2 \rfloor \leq l \leq \lceil m/2 \rceil$. Next, set $q = i - t$. Then, q, t, l gives rise to a solution of (5.5) by Theorem 5.4, and we have $x_A = k + 2i$. Moreover, it is clear that the value of $t - l$ may be taken to be any number from $\{-\lceil m/2 \rceil, \ldots, \lfloor m/2 \rfloor\}$. This proves the claim.

Since we are looking for inequivalent solutions, we will only consider (by Lemma 5.3) $i, j \in \{0, \ldots, \lfloor m/2 \rfloor\}$. As transposition (which switches the row and column multiplicities) could make two solutions permutation/signature equivalent, it follows from the discussion above that we have at least

$$\sum_{j=0}^{\lfloor m/2 \rfloor} \sum_{i=j}^{\lfloor m/2 \rfloor} 1 = \binom{\lfloor m/2 \rfloor + 2}{2} \tag{5.6}$$

inequivalent $(n, k, 2)$ absolutely flat idempotents.

We now discuss bounding the number of solutions from above. Given $i, j \in \{0, \ldots, \lfloor m/2 \rfloor\}$ with $i \geq j$ (recall that transposition may be used to swap row and column multiplicities), we will count the number of triples, (q,t,l), that give rise to a rank 2 absolutely flat idempotent, A, with $x_A = k + 2i$ and $y_A = k + 2j$. From Theorem 5.4, it follows that $t + q = i$ and $\lceil m/2 \rceil + t - l = j$ in which $t, q \in \mathbb{N}$ and

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$i - \lfloor m/2 \rfloor \leq l \leq \lceil m/2 \rceil$. When $l = \lceil m/2 \rceil$, we must have $t = j$ and $q = i - j$, and when $l = \lceil m/2 \rceil - j$, it follows that $t = 0$ and $q = i$. It is easy to see, therefore, that there are $j + 1$ solutions to such a system given $i \geq j$. Hence, the total number of inequivalent solutions is bounded above by,

$$\sum\nolimits_{j=0}^{\lfloor m/2 \rfloor} \sum\nolimits_{i=j}^{\lfloor m/2 \rfloor} (j+1) = \frac{\lfloor m/2 \rfloor \left(\lfloor m/2 \rfloor + 1\right) \left(\lfloor m/2 \rfloor + 2\right)}{6} + \binom{\lfloor m/2 \rfloor + 2}{2}.$$

Combining this computation with (5.6) gives us the following.

THEOREM 5.6. *Let N be the number of inequivalent $(n, k, 2)$ absolutely flat idempotent matrices. Then,*

$$\binom{\lfloor m/2 \rfloor + 2}{2} \leq N \leq \frac{\lfloor m/2 \rfloor \left(\lfloor m/2 \rfloor + 1\right) \left(\lfloor m/2 \rfloor + 2\right)}{6} + \binom{\lfloor m/2 \rfloor + 2}{2}.$$

When $m = 1$, it is clear that $N = 1$, and thus we have

COROLLARY 5.7. *Up to permutation/signature similarity and transposition, there is only one rank 2 absolutely flat idempotent matrix with $n = 2k + 2$.*

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