Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 10, pp. 240-256, November 2003

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J-SPECTRAL FACTORIZATION FOR RATIONAL MATRIX FUNCTIONS WITH ALTERNATIVE REALIZATION*

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Abstract. In this paper, recent results by Petersen and Ran on the J-spectral factorization problem to rational matrix functions with constant signature that are not necessarily analytic at infinity are extended. In particular, a full parametrization of all $\widetilde{J}$-spectral factors that have the same pole pair as a given square J-spectral factor is given. In this case a special realization involving a quintet of matrices is used.

Key words. J-spectral factorization, Algebraic Riccati equations, Spectral decomposition of pencils.

AMS subject classifications. 47A68, 47A56, 47A62, 15A24.

1. Introduction. In the present paper, we consider factorizations of the rational matrix function Φ of the form

$$\Phi(\lambda) = W_1(\lambda) J W_1(-\overline{\lambda})^* = W(\lambda)\widetilde{J}W(-\overline{\lambda})^*, \tag{1.1}$$

where W_1 is a given minimal square factor and W is a nonsquare factor, and

$$\widetilde{J} = \begin{bmatrix} J & 0 \\ 0 & J_{22} \end{bmatrix}. \tag{1.2}$$

Here J satisfies $J = J^* = J^{-1}$ and J_{22} is invertible and Hermitian. In this regard, an $m \times p$ rational matrix function $W(\lambda)$ is called a **minimal $\widetilde{J}$-spectral factor** of $\Phi(\lambda)$ if (1.1) is a minimal factorization. Φ may be a regular rational matrix function taking Hermitian values on the imaginary axis; for such a minimal $\widetilde{J}$-spectral factorization to exist, the number of positive and negative eigenvalues of the matrix $\Phi(\lambda)$ must be the same (i.e., Φ has constant signature) for all imaginary λ, except for the poles and zeros of Φ. See [19] for this and other necessary conditions for the existence of a minimal square J-spectral factorization. For a necessary and sufficient condition for existence of J-spectral factorization with square factor W which has neither zeros nor poles in the open right half plane, see [18]. We shall assume throughout that $\Phi(\alpha) = J$ for a given pure imaginary number α.

For a history and relevant literature of the problem of finding symmetric factors of selfadjoint rational matrix functions see e.g. [9]. In particular, we are interested in the following two recent contributions. Firstly, in [9] necessary and sufficient conditions

*Received by the editors on 13 January 2003. Accepted for publication on 25 August 2003. Handling Editor: Harm Bart.

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Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 10, pp. 240-256, November 2003

are given for the existence of a complete set of minimal J-spectral factorizations of a selfadjoint rational matrix function with constant signature. Also, in [14] the problem of parameterizing the set of all nonsquare minimal spectral factors of a rational matrix function taking positive semi-definite values on the imaginary axis is considered.

In the present paper, we discuss the $\widetilde{J}$-spectral factorization of a rational matrix function with constant signature into $\widetilde{J}$- spectral factors where we have $J = J^* = J^{-1}$. Our analysis has a heavy reliance on the discussions in [16] and [14] (see also [15] and [17]). In particular, it was shown in the main result of [16] that if

$$\Phi(\lambda) = J + C(\lambda I - A)^{-1}B \tag{1.3}$$

is a realization of a rational matrix function Φ with constant signature and

$$W_1(\lambda) = I_m + C_1(\lambda I - A_1)^{-1}\widetilde{B}_1$$

is a minimal square J-spectral factor W_1 of Φ, then any other minimal $\widetilde{J}$-spectral factor with the same pole pair (C_1, A_1) is given by

$$W(\lambda) = \begin{bmatrix} I_m & 0 \end{bmatrix} + C_1(\lambda I - A_1)^{-1}\begin{bmatrix} XC_1^*J + \widetilde{B}_1 & X_2 \end{bmatrix}, \tag{1.4}$$

where $X = X^*$ and X_2 and $J_{22} = J_{22}^*$ satisfy

$$XZ^* + ZX - XC_1^*JC_1X = X_2J_{22}X_2^*, \tag{1.5}$$

with $Z = A_1 - \widetilde{B}_1C_1$. The converse of this claim was also shown to be true.

In this paper we study $\widetilde{J}$-spectral factorization using the concept of realization. Usually a realization for a rational matrix function W is a representation of the form

$$W(\lambda) = D + C(\lambda I - A)^{-1}B, \tag{1.6}$$

which holds whenever W is analytic and invertible at infinity. However, here we consider arbitrary regular rational matrix functions which are not necessarily analytic and invertible at infinity. Of course, the problem in this case could be handled by considering a change of variable, replacing λ by $\frac{1}{\lambda-\alpha}$, where $\alpha \in i\mathbb{R}$ is a point where Φ has neither a pole nor a zero. However, recently alternative realizations were proposed which allow one to study arbitrary regular rational matrix functions without constraints on the behaviour at infinity (see [6], [7] and Section 5.2 of [3]). One such representation of the function W is

$$W(\lambda) = D + (\alpha - \lambda)C(\lambda G - A)^{-1}B, \tag{1.7}$$

where A and G are $n \times n$ matrices with $\alpha G - A$ invertible, B is an $n \times m$ matrix and C is an $m \times n$ matrix and finally D is an invertible $m \times m$ matrix. (This realization is valid provided W is analytic at $\lambda = \alpha$.)

In this paper we show how this realization can be used to solve the following problem:

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Find a full parametrization of all $\widetilde{J}$-spectral factors that have the same pole pair as a given square J-spectral factor.

The paper consists of four sections, including the introduction. Section 2 is preliminary in character and describes some key elements of rational matrix functions with the alternative realization (1.7). In the third section we solve the J-spectral factorization problem for arbitrary rational matrix functions. Conclusive remarks are given and ongoing topics of research highlighted in the last section.

2. Preliminaries. Firstly, we give terminology and notation. By a **Cauchy contour** γ we mean the positively oriented boundary of a bounded Cauchy domain in $\mathbb{C}$. Such a contour consists of a finite number of non-intersecting closed rectifiable Jordan curves. The set of points inside γ is called the **inner domain** of γ and will be denoted by Δ_+. The **outer domain** of γ is the set $\Delta_- = \mathbb{C}_\infty \setminus \overline{\Delta_+}$. By convention $0 \in \Delta_+$ and by definition $\infty \in \Delta_-$.

Next, we consider operator pencils. Let $\mathcal{X}$ be a complex Banach space and let G and A be bounded linear operators on $\mathcal{X}$. For $\lambda \in \mathbb{C}$ the expression $\lambda G - A$ is called a **(linear) operator pencil** on $\mathcal{X}$. Given a non-empty subset Δ of the Riemann sphere $\mathbb{C}_\infty$, we say that $\lambda G - A$ is **Δ-regular** if $\lambda G - A$ (or just G if $\lambda = \infty$) is invertible for each $\lambda \in \Delta$. The spectrum of $\lambda G - A$ denoted by $\sigma(G, A)$ is the subset of $\mathbb{C}_\infty$ determined by the following properties:

1. $\infty \in \sigma(G, A)$ if and only if G is not invertible and
2. $\sigma(G, A) \cap \mathbb{C}$ consists of all those $\lambda \in \mathbb{C}$ for which $\lambda G - A$ is not invertible. Its complement (in $\mathbb{C}_\infty$) is the **resolvent set of** $\lambda G - A$, denoted by $\rho(G, A)$.

Next, we recall a spectral decomposition theorem which summarizes in a way suitable for our purposes the extension (see, for instance, [5], [6] and [20]) to operator pencils of the classical Riesz theory about separation of spectra.

THEOREM 2.1. *Let γ be a Cauchy contour with Δ_+ and Δ_- as inner and outer domains, respectively. Furthermore, suppose that $\lambda G - A$ is a γ-regular pencil of operators on the Banach space $\mathcal{X}$. Then there exists a projection P and an invertible operator E, both acting on $\mathcal{X}$, such that relative to the decomposition*

$$\mathcal{X} = \ker P \oplus \mathrm{im} P \tag{2.1}$$

the following partitioning holds

$$(\lambda G - A)E = \begin{bmatrix} \lambda\Omega_1 - I_1 & 0 \\ 0 & \lambda I_2 - \Omega_2 \end{bmatrix} : \ker P \oplus \mathrm{im} P \to \ker P \oplus \mathrm{im} P, \tag{2.2}$$

where I_1 (respectively, I_2) denotes the identity operator on $\ker P$ *(respectively,* $\mathrm{im} P$*), the pencil $\lambda\Omega_1 - I_1$ is $(\Delta_+ \cup \gamma)$-regular and $\lambda I_2 - \Omega_2$ is $(\Delta_- \cup \gamma)$-regular. Furthermore, P and E (and hence also the operators Ω_1 and Ω_2) are uniquely determined. In fact, we have that*

$$P = \frac{1}{2\pi i}\int_\gamma G(\lambda G - A)^{-1} d\lambda; \tag{2.3}$$

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$$E = \frac{1}{2\pi i}\int_\gamma (1-\lambda^{-1})(\lambda G - A)^{-1} d\lambda; \tag{2.4}$$

$$\Omega = \begin{bmatrix} \Omega_1 & 0 \\ 0 & \Omega_2 \end{bmatrix} = \frac{1}{2\pi i}\int_\gamma (\lambda-\lambda^{-1}) G(\lambda G - A)^{-1} d\lambda; \tag{2.5}$$

We call the 2×2 operator matrix in (2.2) the **γ-spectral decomposition** of the pencil $\lambda G - A$ and the operator Ω in (2.5) will be referred to as the **associated operator** corresponding to $\lambda G - A$ and γ. For the projection P and the operator E, we shall use the terms **separating projection** and **right equivalence operator**, respectively, in Theorem 2.1.

For the proof of Theorem 2.1 we refer to [6] (also, Chapter 4 of [5]). Here we mention a few crucial steps in the proof, which we shall also use later. For

$$Q = \frac{1}{2\pi i}\int_\gamma (\lambda G - A)^{-1} G d\lambda \tag{2.6}$$

it can be shown that

$$PG = GQ, \quad PA = AQ, \tag{2.7}$$

and hence the pencil $\lambda G - A$ admits the following partitioning

$$\lambda G - A = \begin{bmatrix} \lambda G_1 - A_1 & 0 \\ 0 & \lambda G_2 - A_2 \end{bmatrix} : \ker Q \oplus \mathrm{i}mQ \to \ker P \oplus \mathrm{i}mP. \tag{2.8}$$

The next step is to show that the pencil $\lambda\Omega_1 - I_1$ is $(\Delta_+ \cup \gamma)$-regular and $\lambda I_2 - \Omega_2$ is $(\Delta_- \cup \gamma)$-regular. Since $0 \in \Delta_+$ and $\infty \in \Delta_-$ it follows that A_1 and G_2 are invertible. Thus we may set

$$E = \begin{bmatrix} A_1^{-1} & 0 \\ 0 & G_2^{-1} \end{bmatrix} : \ker P \oplus \mathrm{i}mP \to \ker Q \oplus \mathrm{i}mQ \tag{2.9}$$

and $\Omega_1 = G_1 A_1^{-1}$ and $\Omega_2 = A_2 G_2^{-1}$. Then (2.2) holds and it follows that the pencil $\lambda\Omega_1 - I_1$ is $(\Delta_+ \cup \gamma)$-regular and $\lambda I_2 - \Omega_2$ is $(\Delta_- \cup \gamma)$-regular. Next, we can prove that E is also given by (2.4) and Ω by (2.5).

Similarly, (see Theorem 2.1) the associate pencil $\lambda G^\times - A^\times$ may be decomposed as follows:

$$\begin{aligned} &(\lambda G^\times - A^\times)E^\times \\ &= \begin{bmatrix} \lambda\Omega_1^\times - I_1^\times & 0 \\ 0 & \lambda I_2^\times - \Omega_2^\times \end{bmatrix} : \ker P^\times \oplus \mathrm{i}mP^\times \to \ker P^\times \oplus \mathrm{i}mP^\times, \end{aligned} \tag{2.10}$$

where $I_1^\times$ (respectively, $I_2^\times$) denotes the identity operator on $\ker P^\times$ (respectively, $\mathrm{i}mP^\times$) and $\Omega_1^\times = G_1^\times (A_1^\times)^{-1}$ and $\Omega_2^\times = A_2^\times (G_2^\times)^{-1}$. Again by Theorem 2.1 the pencil $\lambda\Omega_1^\times - I_1^\times$ is $(\Delta_+ \cup \gamma)$-regular and $\lambda I_2^\times - \Omega_2^\times$ is $(\Delta_- \cup \gamma)$-regular. Furthermore,

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A publication of the International Linear Algebra Society
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we have that

$$P^\times = \frac{1}{2\pi i}\int_\gamma G^\times(\lambda G^\times - A^\times)^{-1}d\lambda; \tag{2.11}$$

$$E^\times = \frac{1}{2\pi i}\int_\gamma (1-\lambda^{-1})(\lambda G^\times - A^\times)^{-1}d\lambda; \tag{2.12}$$

$$\Omega^\times = \begin{bmatrix} \Omega_1^\times & 0 \\ 0 & \Omega_2^\times \end{bmatrix} = \frac{1}{2\pi i}\int_\gamma (\lambda-\lambda^{-1})G^\times(\lambda G^\times - A^\times)^{-1}d\lambda. \tag{2.13}$$

Indeed, we may put

$$Q^\times = \frac{1}{2\pi i}\int_\gamma (\lambda G^\times - A^\times)^{-1}G^\times d\lambda. \tag{2.14}$$

It can be shown that

$$P^\times G^\times = G^\times Q^\times, \quad P^\times A^\times = A^\times Q^\times, \tag{2.15}$$

and thus the pencil $\lambda G^\times - A^\times$ admits the following partitioning:

$$\begin{aligned} &\lambda G^\times - A^\times \\ &= \begin{bmatrix} \lambda G_1^\times - A_1^\times & 0 \\ 0 & \lambda G_2^\times - A_2^\times \end{bmatrix} : \ker Q^\times \oplus \mathrm{im} Q^\times \to \ker P^\times \oplus \mathrm{im} P^\times. \end{aligned} \tag{2.16}$$

The next step is to show that the pencil $\lambda\Omega_1^\times - I_1^\times$ is $(\Delta_+ \cup \gamma)$- regular and $\lambda I_2^\times - \Omega_2^\times$ is $(\Delta_- \cup \gamma)$-regular. Since $0 \in \Delta_+$ and $\infty \in \Delta_-$ it follows that $A_1^\times$ and $G_2^\times$ are invertible. Thus we may set

$$E^\times = \begin{bmatrix} A_1^{\times -1} & 0 \\ 0 & G_2^{\times -1} \end{bmatrix} : \ker P^\times \oplus \mathrm{im} P^\times \to \ker Q^\times \oplus \mathrm{im} Q^\times \tag{2.17}$$

and $\Omega_1^\times = G_1^\times A_1^{\times -1}$ and $\Omega_2^\times = A_2^\times G_2^{\times -1}$. Then (2.10) holds and it follows that the pencil $\lambda\Omega_1^\times - I_1^\times$ is $(\Delta_+ \cup \gamma)$-regular and $\lambda I_2^\times - \Omega_2^\times$ is $(\Delta_- \cup \gamma)$-regular. Next, we can prove that $E^\times$ is also given by (2.12) and $\Omega^\times$ by (2.13).

Let W be a regular $m \times m$ rational matrix function which has an invertible value at the point $\alpha \in \mathbb{C}$. Then W admits a realization of the form (1.7) (see [6]) given by

$$W(\lambda) = D + (\alpha - \lambda)C(\lambda G - A)^{-1}B,$$

where we assume $\alpha G - A$ is invertible. The realization (1.7) of $W(\lambda)$ is said to be **minimal** if the size of the matrices G and A is as small as possible among all realizations of W. In this case, if G and A are $n \times n$, say, then the number n is called the **McMillan degree of** W, with this number being denoted by $\delta(W)$. The realization is minimal if and only if it is controllable and observable, more precisely, if and only if the maps $C(\lambda G - A)^{-1} : \mathbb{C}^n \to \mathcal{R}(\sigma)$ and $B^*(\lambda G^* - A^*)^{-1} : \mathbb{C}^n \to \mathcal{R}(\sigma)$ are one-to-one. Here $\mathcal{R}(\sigma)$ denotes the set of $n \times 1$ rational vector functions with

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A publication of the International Linear Algebra Society
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poles off σ, where σ is the set of zeros of $\det(\lambda G - A)$ including infinity. This is easily seen by using an appropriate Möbius transformation. Indeed, in this case, put

$$\phi(\lambda) = \frac{\alpha\lambda + 1}{\lambda},$$

and define

$$V(\lambda) = W(\phi(\lambda)).$$

One checks that

$$V(\lambda) = D - C(\alpha G - A)^{-1}(\lambda I + G(\alpha G - A)^{-1})^{-1}B.$$

This realization for V is minimal if and only if the realization (1.7) is minimal. But for this standard type of realization it is well-known that minimality is equivalent to observability and controllability. A straightforward computation for this particular realization for V shows that the standard definition of observability and controllability are equivalent to

$$C(\alpha G - A)^{-1}(\lambda I + G(\alpha G - A)^{-1})^{-1} : \mathbb{C}^n \to \mathcal{R}(\widehat{\sigma})$$

being one-to-one as well as

$$B^*(\lambda I + (\overline{\alpha}G^* - A^*)^{-1}G^*)^{-1} : \mathbb{C}^n \to \mathcal{R}(\widehat{\sigma})$$

being one-to-one. Here $\widehat{\sigma} = \phi^{-1}(\sigma)$. Next, we observe that

$$C(\alpha G - A)^{-1}(\lambda I + G(\alpha G - A)^{-1})^{-1} = \frac{1}{\lambda}C(\phi(\lambda)G - A)^{-1}.$$

If we set $\nu = \phi(\lambda)$, then $\dfrac{1}{\lambda} = \nu - \alpha$. So

$$(\nu - \alpha)C(\nu G - A)^{-1} : \mathbb{C}^n \to \mathcal{R}(\sigma)$$

is one-to-one. Likewise, we have

$$B^*(\lambda I + (\overline{\alpha}G^* - A^*)^{-1}G^*)^{-1} : \mathbb{C}^n \to \mathcal{R}(\widehat{\sigma})$$

is one-to-one if and only if

$$B^*(\overline{\alpha}G^* - A^*)^{-1}(\lambda I + G^*(\overline{\alpha}G^* - A^*)^{-1})^{-1} : \mathbb{C}^n \to \mathcal{R}(\sigma)$$

is one-to-one. A similar argument as before shows that this is equivalent to

$$B^*(\nu G^* - A^*)^{-1} : \mathbb{C}^n \to \mathcal{R}(\sigma)$$

being one-to-one. Note that the realization (1.7) for $W(\lambda)$ also has an inverse given by

$$W(\lambda)^{-1} = D^{-1} - (\alpha - \lambda)D^{-1}C(\lambda G^\times - A^\times)^{-1}BD^{-1}, \tag{2.18}$$

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where $G^\times = G - BD^{-1}C$ and $A^\times = A - \alpha BD^{-1}C$. A minimal realization is essentially unique, more precisely, let

$$W(\lambda) = D_i + (\alpha - \lambda)C_i(\lambda G_i - A_i)^{-1}B_i, \quad i = 1, 2 \tag{2.19}$$

be two minimal realizations for the same rational matrix function $W(\lambda)$. Then $D_1 = D_2$ and there exists unique invertible matrices E and F such that

$$E(\lambda G_1 - A_1)F = \lambda G_2 - A_2; \quad C_1F = C_2; \quad EB_1 = B_2. \tag{2.20}$$

We shall say that the two realizations are **strictly equivalent**, by abuse of expression, sometimes also that they are **similar**.

Next, we describe the form in which pole and zero data will be given. Let $W(\lambda)$ be an arbitrary regular (i.e., analytic and invertible at $\lambda = \alpha$) rational matrix function. Then we know from the discussion above that W and its inverse W^{-1} may be represented as (1.7) and (2.18), respectively. The pair of matrices $(C_p, \lambda G_p - A_p)$ is called a **pole pair** for W if there exists a $\widetilde{B}$ such that

$$W(\lambda) = D + (\alpha - \lambda)C_p(\lambda G_p - A_p)^{-1}\widetilde{B} \tag{2.21}$$

is a minimal realization. For the zero structure we use $W(\cdot)^{-1}$. So, a pair of matrices $(\lambda G_z - A_z, B_z)$ is called a **null pair** for W if there exists a $\widetilde{C}$ such that

$$W(\lambda)^{-1} = D^{-1} - (\alpha - \lambda)D^{-1}\widetilde{C}(\lambda G_z - A_z)^{-1}B_zD^{-1} \tag{2.22}$$

is a minimal realization. Suppose that $(C_p, \lambda G_p - A_p)$ and $(\lambda G_z - A_z, B_z)$ are pole and null pairs for an arbitrary rational matrix function W. Then we also have the following minimal realization for W^{-1} :

$$W(\lambda)^{-1} = D^{-1} - (\alpha - \lambda)D^{-1}C_p(\lambda G_p^\times - A_p^\times)^{-1}\widetilde{B}D^{-1} \tag{2.23}$$

where $G_p^\times = G_p - \widetilde{B}D^{-1}C_p$ and $A_p^\times = A_p - \alpha\widetilde{B}D^{-1}C_p$. Since two minimal realizations of the same function $W(\cdot)^{-1}$ are strictly equivalent (or similar), there exists unique invertible matrices E and F such that

$$E(\lambda G_z - A_z)F = \lambda G_p^\times - A_p^\times; \quad \widetilde{C}F = C_p; \quad EB_z = \widetilde{B}; \tag{2.24}$$

which implies that

$$E^{-1}(\lambda G_p - A_p) - (\lambda G_z - A_z)F = (\lambda - \alpha)B_zC_p. \tag{2.25}$$

In this case, we have that

$$W(\lambda) = D + (\alpha - \lambda)C_p(\lambda G_p - A_p)^{-1}EB_z \tag{2.26}$$

and

$$W(\lambda)^{-1} = D^{-1} - (\alpha - \lambda)D^{-1}C_pF^{-1}(\lambda G_z - A_z)^{-1}B_zD^{-1}. \tag{2.27}$$

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Finally, the following identities will be useful in the sequel. Let $W(\lambda)$ be as in (1.7) where $\lambda G - A$ is γ-regular. Assume that $\det W(\lambda) \neq 0$ for each $\lambda \in \gamma$ and set $G^{\times} = G - BD^{-1}C$ and $A^{\times} = A - \alpha BD^{-1}C$. Then for $\lambda \in \gamma$ we have

$$W(\lambda)^{-1}C(\lambda G - A)^{-1} = D^{-1}C(\lambda G^{\times} - A^{\times})^{-1}; \tag{2.28}$$

$$(\lambda G - A)^{-1}BW(\lambda)^{-1} = (\lambda G^{\times} - A^{\times})^{-1}BD^{-1}; \tag{2.29}$$

$$(\lambda G^{\times} - A^{\times})^{-1} = (\lambda G - A)^{-1} - (\alpha - \lambda)(\lambda G - A)^{-1}BW(\lambda)^{-1}C(\lambda G - A)^{-1}. \tag{2.30}$$

3. Minimal J-Spectral Factorization for Arbitrary Rational Matrix Functions. Let Φ be a rational matrix function with constant signature, for which we assume the existence of a square minimal J-spectral factorization

$$\Phi(\lambda) = W_1(\lambda)JW_1(-\bar{\lambda})^*.$$

Here we describe explicitly all minimal nonsquare $\widetilde{J}$-spectral factors of the rational matrix function Φ with constant signature matrix $\widetilde{J}$, and with the same pole pair as a given minimal square J-spectral factor. The formulas for these $\widetilde{J}$- spectral factors are given in terms of the components of an algebraic Riccati equation and a given minimal square J-spectral factor.

Let Φ be a rational matrix function with constant signature, for which we assume the existence of a square minimal J-spectral factorization $\Phi(\lambda) = W_1(\lambda)JW_1(-\bar{\lambda})^*$. The main problem that we wish to consider may be stated as follows. Given Φ in realized form we wish to find all minimal spectral factorizations $\Phi = W\widetilde{J}W^*$, where W is possibly nonsquare. Our aim is to obtain a minimal realization for W's of this type. The approach that we will adopt in solving this problem is comparable to the one in [16]. In particular, we explicitly describe all minimal nonsquare $\widetilde{J}$-spectral factors W of Φ, for which $W(\alpha) = \begin{bmatrix} I_m & 0 \end{bmatrix}$ and with the same pole pair as W_1. Throughout, we assume that $\widetilde{J}$ is given by (1.2).

THEOREM 3.1. *Suppose that the rational matrix function Φ with constant signature matrix has a realization*

$$\Phi(\lambda) = J + (\alpha - \lambda)C(\lambda G - A)^{-1}B$$

and a minimal square J-spectral factor W_1 given by the minimal realization

$$W_1(\lambda) = I_m + (\alpha - \lambda)C_1(\lambda G_1 - A_1)^{-1}\widetilde{B}_1 \tag{3.1}$$

whenever $\alpha = -\overline{\alpha}$. Set $Y = G_1 - \widetilde{B}_1C_1$ and $Z = A_1 - \alpha\widetilde{B}_1C_1$. For any $X = X^*$ form

$$YX + XY^* - XC_1^*JC_1X$$

and let X_2 and $J_{22} = J_{22}^*$ invertible be such that

$$YX + XY^* - XC_1^*JC_1X = X_2J_{22}X_2^*. \tag{3.2}$$

Then for any such X, X_2 and J_{22} the function

$$W(\lambda) = \begin{bmatrix} I_m & 0 \end{bmatrix} + (\alpha - \lambda)C_1(\lambda G_1 - A_1)^{-1}\begin{bmatrix} XC_1^*J + \widetilde{B}_1 & X_2 \end{bmatrix} \tag{3.3}$$

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is a $\widetilde{J}$-spectral factor of Φ, where $\widetilde{J}$ is given by (1.2)*. Moreover, for any such X, the matrix $(Z-\alpha Y)X$ is selfadjoint as well.*

Conversely, given $\widetilde{J}$ as in (1.2) *all $\widetilde{J}$-spectral factors of Φ are given by* (3.3) *where X and X_{22} satisfy* (3.2).

Proof. ($\Leftarrow$) We consider a nonsquare rational matrix of the form

$$W(\lambda) = \begin{bmatrix} I_m & 0 \end{bmatrix} + (\alpha-\lambda)C_1(\lambda G_1 - A_1)^{-1}\begin{bmatrix} X_1 + \widetilde{B}_1 & X_2 \end{bmatrix}. \tag{3.4}$$

We can rewrite (3.4) in terms of the square J-spectral factor (3.1) as

$$W(\lambda) = \begin{bmatrix} W_1(\lambda) + R_1(\lambda) & R_2(\lambda) \end{bmatrix}, \tag{3.5}$$

where $R_1(\lambda) = (\alpha-\lambda)C_1(\lambda G_1 - A_1)^{-1}X_1$ and $R_2(\lambda) = (\alpha-\lambda)C_1(\lambda G_1 - A_1)^{-1}X_2$. If we form a $\widetilde{J}$-spectral product with $W(\lambda)$ in the form given by (3.5) we obtain

$$\begin{aligned}
W(\lambda)\widetilde{J}W(-\overline{\lambda})^* &= \begin{bmatrix} W_1(\lambda) + R_1(\lambda) & R_2(\lambda) \end{bmatrix} \widetilde{J} \begin{bmatrix} W_1(-\overline{\lambda})^* + R_1(-\overline{\lambda})^* \\ R_2(-\overline{\lambda})^* \end{bmatrix} \\
&= \begin{bmatrix} W_1(\lambda) + R_1(\lambda) & R_2(\lambda) \end{bmatrix} \begin{bmatrix} J & 0 \\ 0 & J_{22} \end{bmatrix} \begin{bmatrix} W_1(-\overline{\lambda})^* + R_1(-\overline{\lambda})^* \\ R_2(-\overline{\lambda})^* \end{bmatrix} \\
&= (W_1(\lambda) + R_1(\lambda))J(W_1(-\overline{\lambda})^* + R_1(-\overline{\lambda})^*) + R_2(\lambda)J_{22}R_2(-\overline{\lambda})^* \\
&= W_1(\lambda)JW_1(-\overline{\lambda})^* + R_1(\lambda)JW_1(-\overline{\lambda})^* + W_1(\lambda)JR_1(-\overline{\lambda})^* \\
&\quad + R_1(\lambda)JR_1(-\overline{\lambda})^* + R_2(\lambda)J_{22}R_2(-\overline{\lambda})^* \\
&= \Phi(\lambda) + R_1(\lambda)JW_1(-\overline{\lambda})^* + W_1(\lambda)JR_1(-\overline{\lambda})^* \\
&\quad + R_1(\lambda)JR_1(-\overline{\lambda})^* + R_2(\lambda)J_{22}R_2(-\overline{\lambda})^*.
\end{aligned}$$

Thus we have that $\Phi(\lambda) = W(\lambda)\widetilde{J}W(-\overline{\lambda})^*$ if and only if

$$\begin{aligned} R_1(\lambda)JW_1(-\overline{\lambda})^* + W_1(\lambda)JR_1(-\overline{\lambda})^* + R_1(\lambda)JR_1(-\overline{\lambda})^* \\ + R_2(\lambda)J_{22}R_2(-\overline{\lambda})^* = 0. \end{aligned} \tag{3.6}$$

Next, we multiply (3.6) on the left by $W_1(\lambda)^{-1}$ and on the right by $W_1(-\overline{\lambda})^{-*}$ and use the fact that

$$W_1(\lambda)^{-1}C_1(\lambda G_1 - A_1)^{-1} = C_1(\lambda Y - Z)^{-1},$$

where $Y = G_1 - \widetilde{B}_1C_1$ and $Z = A_1 - \alpha\widetilde{B}_1C_1$. By a straightforward calculation we find that (3.6) is equivalent to

$$\begin{aligned} (\alpha-\lambda)C_1(\lambda Y - Z)^{-1}X_1J - (\overline{\alpha}+\lambda)JX_1^*(\lambda Y^* + Z^*)^{-1}C_1^* \\ = (\alpha-\lambda)(\overline{\alpha}+\lambda)C_1(\lambda Y - Z)^{-1}(X_1JX_1^* + X_2J_{22}X_2^*)(\lambda Y^* + Z^*)^{-1}C_1^*. \end{aligned} \tag{3.7}$$

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A publication of the International Linear Algebra Society
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From the spectral decomposition of the associate pencil $\lambda Y - Z$ of the pencil $\lambda G_1 - A_1$ we obtain

$$(\lambda Y - Z)^{-1} = E^{\times} \begin{bmatrix} (\lambda \Omega_1^{\times} - I_1^{\times})^{-1} & 0 \\ 0 & (\lambda I_2^{\times} - \Omega_2^{\times})^{-1} \end{bmatrix}. \tag{3.8}$$

Hence, it follows that

$$(\alpha - \lambda)(\lambda Y - Z)^{-1} = E^{\times} \begin{bmatrix} (\alpha - \lambda)(\lambda \Omega_1^{\times} - I_1^{\times})^{-1} & 0 \\ 0 & (\alpha - \lambda)(\lambda I_2^{\times} - \Omega_2^{\times})^{-1} \end{bmatrix}, \tag{3.9}$$

where

$$(\alpha - \lambda)(\lambda \Omega_1^{\times} - I_1^{\times})^{-1} = -\sum_{\nu=0}^{\infty} \alpha \lambda^{\nu} (\Omega_1^{\times})^{\nu} + \sum_{\nu=0}^{\infty} \lambda^{\nu+1} (\Omega_1^{\times})^{\nu}$$

and

$$(\alpha - \lambda)(\lambda I_2^{\times} - \Omega_2^{\times})^{-1} = \sum_{\nu=0}^{\infty} \alpha \lambda^{-\nu-1} (\Omega_2^{\times})^{\nu} - \sum_{\nu=0}^{\infty} \lambda^{-\nu} (\Omega_2^{\times})^{\nu}.$$

Similarly, we have that

$$\begin{aligned} &-(\overline{\alpha} + \lambda)(\lambda Y^* + Z^*)^{-1} = \\ &\begin{bmatrix} -(\overline{\alpha} + \lambda)(\lambda \Omega_1^{\times *} + I_1^{\times})^{-1} & 0 \\ 0 & -(\overline{\alpha} + \lambda)(\lambda I_2^{\times} + \Omega_2^{\times *})^{-1} \end{bmatrix} E^{\times *}, \end{aligned} \tag{3.10}$$

where

$$-(\overline{\alpha} + \lambda)(\lambda \Omega_1^{\times *} + I_1^{\times})^{-1} = -\sum_{\nu=0}^{\infty} (-1)^{\nu} \overline{\alpha} \lambda^{\nu} (\Omega_1^{\times *})^{\nu} - \sum_{\nu=0}^{\infty} (-1)^{\nu} \lambda^{\nu+1} (\Omega_1^{\times *})^{\nu}$$

and

$$-(\overline{\alpha} + \lambda)(\lambda I_2^{\times} + \Omega_2^{\times *})^{-1} = -\sum_{\nu=0}^{\infty} (-1)^{\nu} \overline{\alpha} \lambda^{-\nu-1} (\Omega_2^{\times *})^{\nu} - \sum_{\nu=0}^{\infty} (-1)^{\nu} \lambda^{\nu} (\Omega_2^{\times *})^{\nu}.$$

Substituting (3.9) and (3.10) into formula (3.7) and comparing the coefficients of λ^{-1} yields

$$C_1 E^{\times} \begin{bmatrix} 0 & 0 \\ 0 & \alpha I_2^{\times} - \Omega_2^{\times} \end{bmatrix} X_1 J - J X_1^* \begin{bmatrix} 0 & 0 \\ 0 & \overline{\alpha} I_2^{\times} - \Omega_2^{\times *} \end{bmatrix} E^{\times *} C_1^* = 0. \tag{3.11}$$

Notice that (3.7) also implies that

$$(\alpha - \lambda) C_1 (\lambda Y - Z)^{-1} X_1 J u = 0 \quad (u \in \ker\ C_1^*). \tag{3.12}$$

The pair $(C_1, \lambda Y - Z)$ is a null-kernel pair and hence (3.12) can be rewritten as

$$X_1 J u = 0 \quad (u \in \ker\ C_1^*). \tag{3.13}$$

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A publication of the International Linear Algebra Society
Volume 10, pp. 240-256, November 2003

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From (3.11) and (3.12) it follows that there exists a selfadjoint matrix X such that

$$XC_1^* = X_1 J. \tag{3.14}$$

Indeed, we define X on im C_1^* by setting

$$XC_1^* u = X_1 J u \quad (u \in \mathbf{C}^m).$$

From (3.13) it follows that X is well-defined and uniquely defined on im C_1^*. Consider the orthogonal decomposition $\mathbb{C}^n = \text{i}m\ C_1^* \oplus \ker\ C_1$, and the following partitioning of X

$$X = \begin{bmatrix} X_{11} \\ X_{21} \end{bmatrix} :\ \text{i}m\ C_1^* \to \begin{bmatrix} \text{i}m\ C_1^* \\ \ker\ C_1 \end{bmatrix}.$$

Note that

$$\begin{aligned} C_1 E^\times \begin{bmatrix} 0 & 0 \\ 0 & \alpha I_2^\times - \Omega_2^\times \end{bmatrix} X_1 J &= C_1 E^\times \begin{bmatrix} 0 & 0 \\ 0 & \alpha I_2^\times - \Omega_2^\times \end{bmatrix} X C_1^* \\ &= C_1 E^\times \begin{bmatrix} 0 & 0 \\ 0 & \alpha I_2^\times - \Omega_2^\times \end{bmatrix} X_{11} C_1^*. \end{aligned}$$

Since $C_1 E^\times \begin{bmatrix} 0 & 0 \\ 0 & \alpha I_2^\times - \Omega_2^\times \end{bmatrix} X_1 J$ is selfadjoint by (3.11), we conclude that $C_1 E^\times \begin{bmatrix} 0 & 0 \\ 0 & \alpha I_2^\times - \Omega_2^\times \end{bmatrix} X_{11} C_1^*$ is selfadjoint. This implies that

$$E^\times \begin{bmatrix} 0 & 0 \\ 0 & \alpha I_2^\times - \Omega_2^\times \end{bmatrix} X_{11}$$

is selfadjoint.

Now define X on $\mathbb{C}^n$ by

$$X = \begin{bmatrix} X_{11} & X_{21}^* \\ X_{21} & X_{22} \end{bmatrix} \text{ on } \begin{bmatrix} \text{i}m\ C_1^* \\ \ker\ C_1 \end{bmatrix},$$

where X_{22} is an arbitrary selfadjoint linear transformation on $\ker\ C_1$. We note that this suggests that there is freedom in the choice for X.

Using (3.11) and (3.12) and the fact that there exists an X such that (3.14) holds we can rewrite (3.7) in the following equivalent form

$$\begin{aligned} C_1(\lambda Y - Z)^{-1} \{ & (\alpha - \lambda) X(\lambda Y^* + Z^*) - (\overline{\alpha} + \lambda)(\lambda Y - Z) X \\ & - (\alpha - \lambda)(\overline{\alpha} + \lambda) X C_1^* J C_1 X - (\alpha - \lambda)(\overline{\alpha} + \lambda) X_2 J_{22} X_2^* \} \\ & \times (\lambda Y^* + Z^*)^{-1} C_1^* = 0. \end{aligned} \tag{3.15}$$

By using the fact that $(C_1, \lambda Y - Z)$ is a null-kernel pair, we see that (3.15) is equivalent to

$$\begin{aligned} \{ & (\alpha - \lambda) X(\lambda Y^* + Z^*) - (\overline{\alpha} + \lambda)(\lambda Y - Z) X - (\alpha - \lambda)(\overline{\alpha} + \lambda) X C_1^* J C_1 X \\ & - (\alpha - \lambda)(\overline{\alpha} + \lambda) X_2 J_{22} X_2^* \} (\lambda Y^* + Z^*)^{-1} C_1^* = 0. \end{aligned}$$

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Taking adjoints we see that the above formula is equivalent to

$$C_1(\lambda Y-Z)^{-1}\{(\alpha-\lambda)X(\lambda Y^*+Z^*)-(\overline{\alpha}+\lambda)(\lambda Y-Z)X \\ -(\alpha-\lambda)(\overline{\alpha}+\lambda)XC_1^*JC_1X-(\alpha-\lambda)(\overline{\alpha}+\lambda)X_2J_{22}X_2^*\}=0.$$

Finally, by again using the fact that $(C_1, \lambda Y-Z)$ is a null-kernel pair, we notice that (3.15) is equivalent to

$$(\overline{\alpha}+\lambda)(\lambda Y-Z)X-(\alpha-\lambda)X(\lambda Y^*+Z^*)+(\alpha-\lambda)(\overline{\alpha}+\lambda)XC_1^*JC_1X \\ +(\alpha-\lambda)(\overline{\alpha}+\lambda)X_2J_{22}X_2^*=0. \tag{3.16}$$

Next, note that (3.16) is an equality between two matrix polynomials of degree two. Since like coefficients are equal we obtain the following equations:

$$YX+XY^*-XC_1^*JC_1X=X_2J_{22}X_2^*; \tag{3.17}$$

$$\frac{1}{\overline{\alpha}-\alpha}\{(XZ^*-ZX)+(\overline{\alpha}YX-\alpha XY^*)\}-XC_1^*JC_1X=X_2J_{22}X_2^*; \tag{3.18}$$

$$\frac{1}{\alpha}ZX+\frac{1}{\overline{\alpha}}XZ^*-XC_1^*JC_1X=X_2J_{22}X_2^*. \tag{3.19}$$

Now we show that $(Z-\alpha Y)X$ is selfadjoint whenever $\alpha=-\overline{\alpha}$. Indeed, since $\alpha=-\overline{\alpha}$, (3.16) becomes

$$(-\lambda Y+Z)X-X(\lambda Y^*+Z^*)+(\lambda-\alpha)XC_1^*JC_1X=-(\lambda-\alpha)X_2J_{22}X_2^*. \tag{3.20}$$

Setting $\lambda=\alpha$ in (3.20) leads to

$$(-\alpha Y+Z)X-X(\alpha Y^*+Z^*)=0. \tag{3.21}$$

Again, since $\alpha=-\overline{\alpha}$ and by using (3.21) we get that $(Z-\alpha Y)X$ must be selfadjoint.

Finally, by using the selfadjointness of $(Z-\alpha Y)X$ and because $\alpha=-\overline{\alpha}$ we observe that both (3.18) and (3.19) are equal to (3.17). So we can choose X_2 and J_{22} such that (3.2) holds whenever $\alpha=-\overline{\alpha}$ and $(Z-\alpha Y)X$ is selfadjoint.

We observe that the above proof suggests that there is some freedom in the choice of X. However, X is in fact unique (also compare with [14]). Suppose that

$$C_1(\lambda G_1-A_1)^{-1}XC_1^*J=C_1(\lambda G_1-A_1)^{-1}X^{'}C_1^*J$$

for some selfadjoint matrix $X^{'}$, and also that

$$(\overline{\alpha}+\lambda)(\lambda Y-Z)X-(\alpha-\lambda)X(\lambda Y^*+Z^*)+(\alpha-\lambda)(\overline{\alpha}+\lambda)XC_1^*JC_1X \\ =(\overline{\alpha}+\lambda)(\lambda Y-Z)X^{'}-(\alpha-\lambda)X^{'}(\lambda Y^*+Z^*)+(\alpha-\lambda)(\overline{\alpha}+\lambda)X^{'}C_1^*JC_1X^{'}.$$

Then the observability of $(C_1, \lambda G_1-A_1)$ implies that $XC_1^*=X^{'}C_1^*$. Thus

$$im\,(X-X^{'})\subset\ker C_1.$$

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But then

$$XC_1^* JC_1 X = X^{'} C_1^* JC_1 X,$$

and thus

$$(\alpha - \lambda)(X - X^{'})(\lambda Y^* + Z^*) - (\overline{\alpha} + \lambda)(\lambda Y - Z)(X - X^{'}) = 0.$$

From this we deduce that $im\,(X - X^{'})$ is $(\lambda Y - Z)$- invariant. As it is also contained in $\ker C_1$, and as $(C_1, \lambda Y - Z)$ is observable, we obtain that $(\lambda Y - Z) im\,(X - X^{'}) = (0)$. It then follows that $im\,(X - X^{'}) = (0)$, i.e., $X = X^{'}$.

($\Rightarrow$) We still have to establish the proof of the direct statement. Given the formula for $W(\lambda)$ one derives that

$$\begin{aligned} &W(\lambda)\widetilde{J}W(-\bar{\lambda})^* = J + (\alpha - \lambda)C_1(\lambda G_1 - A_1)^{-1}(X_1 J + \widetilde{B}_1 J) \\ &- (\overline{\alpha} + \lambda)(JX_1^* + J\widetilde{B}_1^*)(\lambda G_1^* + A_1^*)^{-1}C_1^* \\ &- (\alpha - \lambda)(\overline{\alpha} + \lambda)C_1(\lambda G_1 - A_1)^{-1}\left\{(XC_1^* + \widetilde{B}_1 J)(\widetilde{B}_1^* + JC_1 X) + X_2 J_{22} X_2^*\right\} \\ &\times (\lambda G_1^* + A_1^*)^{-1}C_1^*. \end{aligned}$$

Employing (3.2) we see that

$$\begin{aligned} &(\alpha - \lambda)(\overline{\alpha} + \lambda)\left\{(XC_1^* + \widetilde{B}_1 J)(\widetilde{B}_1^* + JC_1 X) + X_2 J_{22} X_2^*\right\} \\ &= (\alpha - \lambda)X\left\{(\lambda Y^* + Z^*) + (\overline{\alpha} + \lambda)C_1^*\widetilde{B}_1^*\right\} \\ &+ \quad (\overline{\alpha} + \lambda)\left\{(\alpha - \lambda)\widetilde{B}_1 C_1 - (\lambda Y - Z)\right\}X + (\alpha - \lambda)(\overline{\alpha} + \lambda)\widetilde{B}_1 J\widetilde{B}_1^* \\ &= (\alpha - \lambda)X(\lambda G_1^* + A_1^*) - (\overline{\alpha} + \lambda)(\lambda G_1 - A_1)X + (\alpha - \lambda)(\overline{\alpha} + \lambda)\widetilde{B}_1 J\widetilde{B}_1^*. \end{aligned}$$

Inserting this in the formula above easily leads to

$$W(\lambda)\widetilde{J}W(-\bar{\lambda})^* = W_1(\lambda)JW_1(-\bar{\lambda})^* = \Phi(\lambda). \quad \square$$

In the following corollary we consider the relationship between special choices of $\widetilde{J}$ and $\widetilde{J}$-spectral factors of Φ.

Corollary 3.2. *Let $\widetilde{J}$ be given by* (1.2). *Under the assumptions of Theorem 3.1 the following hold.*

(a) *Let $\Pi_+(J) = \Pi_+(\widetilde{J})$, where $\Pi_+(J)$ $(resp., \Pi_+(\widetilde{J}))$ denotes the number of positive eigenvalues of J $(resp., \widetilde{J})$. There is a one-to-one correspondence between $\widetilde{J}$-spectral factors of Φ with pole pair $(C_1, \lambda G_1 - A_1)$ and with value $\begin{bmatrix} I_m & 0 \end{bmatrix}$ at α, and pairs of matrices (X, X_2) satisfying*

$$YX + XY^* - XC_1^* JC_1 X \leq 0$$

and

$$YX + XY^* - XC_1^* JC_1 X = X_2 J_{22} X_2^*.$$

This one-to-one correspondence is given by (3.3).

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A publication of the International Linear Algebra Society
Volume 10, pp. 240-256, November 2003

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(b) *Let $\Pi_-(J) = \Pi_-(\widetilde{J})$, where $\Pi_-(J)$ $(resp., \Pi_-(\widetilde{J}))$ denotes the number of negative eigenvalues of J $(resp., \widetilde{J})$. There is a one-to-one correspondence between $\widetilde{J}$-spectral factors of Φ with pole pair $(C_1, \lambda G_1 - A_1)$ and with value $\begin{bmatrix} I_m & 0 \end{bmatrix}$ at α, and pairs of matrices (X, X_2) satisfying*

$$YX + XY^* - XC_1^*JC_1X \geq 0$$

and

$$YX + XY^* - XC_1^*JC_1X = X_2J_{22}X_2^*.$$

This one-to-one correspondence is given by (3.3).

(c) *Let $\Pi_+(J) = \Pi_+(\widetilde{J})$ and $\Pi_-(J) = \Pi_-(\widetilde{J})$. There is a one-to-one correspondence between $\widetilde{J}$-spectral factors of Φ with pole pair $(C_1, \lambda G_1 - A_1)$ and with value $\begin{bmatrix} I_m & 0 \end{bmatrix}$ at α, and matrices X satisfying*

$$YX + XY^* - XC_1^*JC_1X = 0.$$

This one-to-one correspondence is given by (3.3).

Part (c) of the above corollary corresponds to an analogue of the square case which, for instance, is discussed in [9].

We note from the $\widetilde{J}$-spectral factorization in (1.1) and the proof of Theorem 3.1 that

$$\begin{aligned} &W(\lambda)\widetilde{J}W(-\overline{\lambda})^* \\ &= (W_1(\lambda) + R_1(\lambda))J(W_1(-\overline{\lambda})^* + R_1(-\overline{\lambda})^*) + R_2(\lambda)J_{22}R_2(-\overline{\lambda})^*, \end{aligned} \tag{3.22}$$

where we have that $W_1 + R_1$ has a fixed pole pair $(C_1, \lambda G_1 - A_1)$. If we make the assumption that J_{22} is positive semidefinite, i.e., $J_{22} \geq 0$, then it follows that $(W_1 + R_1)J(W_1 + R_1)^* \geq \Phi$. On the other hand, if we suppose that J_{22} is negative semidefinite, i.e., $J_{22} \leq 0$, then it follows that $(W_1 + R_1)J(W_1 + R_1)^* \leq \Phi$.

Remark 3.3. Our next remark represents a much weaker analogue of Corollary 2.3 of [16]. It may be formulated and proved as follows. Assume Φ has a minimal square J-spectral factor

$$W_1(\lambda) = I_m + (\alpha - \lambda)C_1(\lambda G_1 - A_1)^{-1}\widetilde{B}_1.$$

All square rational matrix functions V such that $VJV^* \leq \Phi$, V has a pole pair of the form $(C_1, \lambda G_1 - A_1)$ and $V(\alpha) = I_m$ are given by

$$V(\lambda) = I_m + (\alpha - \lambda)C_1(\lambda G_1 - A_1)^{-1}(XC_1^*J + \widetilde{B}_1), \tag{3.23}$$

where X solves

$$YX + XY^* - XC_1^*JC_1X = X_2X_2^*. \tag{3.24}$$

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A publication of the International Linear Algebra Society
Volume 10, pp. 240-256, November 2003

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Here $Y = G_1 - \widetilde{B}_1C_1$ and $Z = A_1 - \alpha\widetilde{B}_1C_1$. Indeed, we write $V(\lambda) = I + (\alpha-\lambda)C_1(\lambda G_1 - A_1)^{-1}(X_1 + \widetilde{B}_1)$ and $R_2(\lambda) = (\alpha-\lambda)C_1(\lambda G_1 - A_1)^{-1}X_2$ and consider

$$\Phi(\lambda) = \begin{bmatrix} V(\lambda) & R_2(\lambda) \end{bmatrix} \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} V(-\overline{\lambda})^* \\ R_2(-\overline{\lambda})^* \end{bmatrix}$$

where $\begin{bmatrix} V(\lambda) & R_2(\lambda) \end{bmatrix} = \begin{bmatrix} I_m & 0 \end{bmatrix} + (\alpha-\lambda)C_1(\lambda G_1 - A_1)^{-1} \begin{bmatrix} X_1^*J + \widetilde{B}_1 & X_2 \end{bmatrix}$. It follows from the proof of Theorem 3.1 that $X_1 = XC_1^*J$ with $X = X^*$ satisfying

$$YX + XY^* - XC_1^*JC_1X = X_2X_2^* \geq 0.$$

The converse is proved by taking a V as in (3.23) and then forming $\Phi(\lambda)$ as above with $J_{22} = I$ and $R_2(\lambda) = (\alpha-\lambda)C_1(\lambda G_1 - A_1)^{-1}X_2$. Then it follows that $VJV^* \leq \Phi$, V has a pole pair of the form $(C_1, \lambda G_1 - A_1)$, $V(\alpha) = I$ and $R_2(\alpha) = 0$.

The following result explains the square case.

Corollary 3.4. *Suppose that the rational matrix function Φ with constant signature matrix J has a realization*

$$\Phi(\lambda) = J + (\alpha-\lambda)C(\lambda G - A)^{-1}B$$

and a minimal square J-spectral factor W_1 given by the minimal realization

$$W_1(\lambda) = I_m + (\alpha-\lambda)C_1(\lambda G_1 - A_1)^{-1}\widetilde{B}_1 \tag{3.25}$$

whenever $\alpha = -\overline{\alpha}$. Set $Y = G_1 - \widetilde{B}_1C_1$ and $Z = A_1 - \alpha\widetilde{B}_1C_1$. Let $X = X^*$ be such that

$$YX + XY^* - XC_1^*JC_1X = 0. \tag{3.26}$$

Then for any such X, the function

$$W(\lambda) = I_m + (\alpha-\lambda)C_1(\lambda G_1 - A_1)^{-1}(XC_1^*J + \widetilde{B}_1) \tag{3.27}$$

is a J-spectral factor of Φ. Moreover, for any such X, the matrix $(Z - \alpha Y)X$ is selfadjoint.

Conversely, all J-spectral factors of Φ are given by (3.27) *where X satisfies* (3.26).

The proof of this corollary is almost exactly the same as for the nonsquare case in Theorem 3.1.

4. Conclusions and Future Directions. The problem that we solve in this paper is to give a full parametrization of all $\widetilde{J}$-spectral factors that have the same pole pair as a given square J-spectral factor of a rational matrix function with constant signature and with special realization of the type given by (1.7), i.e.,

$$W(\lambda) = D + (\alpha-\lambda)C(\lambda G - A)^{-1}B.$$

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A publication of the International Linear Algebra Society
Volume 10, pp. 240-256, November 2003

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Explicit formulas for these $\widetilde{J}$-spectral factors are given in terms of a solution of a particular algebraic Riccati equation. Also, it is possible to recover most of the formulas in [16] by applying an inverse Möbius transformation and using other heuristic tools.

Some work can still be done in the case of analytic rational matrix functions given as in (1.6), in other words

$$W(\lambda) = D + C(\lambda I - A)^{-1}B.$$

For instance, in [10] it was proved that although minimal J-spectral square factors may not always exist, there always is a possibly nonsquare minimal $\widetilde{J}$-spectral factor. The following open question may be posed; what is the smallest possible size of such a nonsquare factor?

Another fertile area of research is the further characterization of the null-pole structure of rational matrix functions ([1, 2, 3, 4, 11]) with alternative realization. In this regard, it would be an interesting exercise to follow the lead taken in [16] and discuss the common null structure of rational matrix functions (see also [11]) that arises from the analysis of J-spectral factors with more general realization. Another question would be whether this zero structure can be obtained in terms of the kernel of a generalized Bezoutian. A prerequisite for this is of course that an appropriate notion of a Bezoutian ([8, 12, 13, 16]) should be unearthed for spectral factors with alternative realization.

Acknowledgment. The authors are indebted to André Ran whose suggestions led to a significant improvement of the presentation of this paper.

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