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A CONTRIBUTION TO COLLATZ'S EIGENVALUE INCLUSION THEOREM FOR NONNEGATIVE IRREDUCIBLE MATRICES*

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Abstract. The matrix calculus is widely applied in various branches of mathematics and control system engineering. In this paper properties of real matrices with nonnegative elements are studied. The classical Collatz theorem is unique and immediately applicable to estimating the spectral radius of nonnegative irreducible matrices. The coherence property is identified. Then the Perron–Frobenius theorem and Collatz's theorem are used to formulate the coherence property more precisely. It is shown how dual variation principles can be used for the iterative calculation of $x = X[A]$ and the spectral radius of A, where x is any positive n-vector, $X[A]$ is the corresponding positive eigenvector, and A is an $n \times n$ nonnegative irreducible real matrix.

Key words. Collatz's theorem, Coherent property, Eigenvalues.

AMS subject classifications. 15A48, 15A18, 15A42.

1. Introduction. Let $N = \{1, 2, \ldots, n\}$, $A = (a_{ij})$ be an $n \times n$ irreducible nonnegative matrix, $x = (x_1, x_2, \ldots, x_n)^t$ be any positive n-vector, $f_i(x) = (Ax)_i/x_i \equiv \sum_{j=1}^{n} a_{ij}x_j/X_i$,$(i \in N)$, $m(x) = \min_{i \in N} f_i(x)$, and $M(x) = \max_{i \in N} f_i(x)$. Collatz's eigenvalue inclusion theorem [6], refined by Wielandt [21], together with the Perron–Frobenius theorems, is a classic part of the theory of nonnegative matrices.

THEOREM 1.1. **The Theorem of Collatz and Wielandt.** *The spectral radius* $\Lambda[A]$ *of the nonnegative irreducible matrix A satisfies either*

$$m(x) < \Lambda[A] < M(x) \tag{1}$$

or

$$m(x) = M(x) = \Lambda[A] \tag{2}$$

for any $x > 0$. *(If* (2) *holds, then x is of course a positive eigenvector of A corresponding to the eigenvalue* $\Lambda[A]$*.)*

We will refer to any interval (κ, μ), where $\kappa = m(x)$, $\mu = M(x)$ for some positive x, as an "inclusion interval."

Collatz's theorem is immediately applicable to estimating the spectral radius $\Lambda[A]$. A sharper estimate or specialization of (1) has been developed by many mathematicians ([3], [12], [15], [17], [18], [19]).

Another important application of Collatz's theorem comes from the following observation. Let P denote the set of positive vectors with unit l_1-norm. By the

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Perron–Frobenius theorem and Collatz's theorem, the positive eigenvector $X[A] \in P$ and $\Lambda[A]$ are solutions of the following minimax and maximin principles:

$$\Lambda[A] = \max_{x \in P} m(x) = \min_{x \in P} M(x),$$

where the extreme values are achieved only if $x = X[A]$; see [20]. It will be shown in a later paper how these dual variation principles can be used for the iterative calculation of $x = X[A]$ and $\Lambda[A]$. Such an application is in part based on the fact that $m(x)$ and $M(x)$ are not independent if $x \in P$ is required. In fact, Collatz's theorem suggests that, if one of these two quantities is close to $\Lambda[A]$, so is the other one. We may refer to this as the "coherence property" of $m(x), M(x)$ (or of the $f_i(x)$). The purpose of this paper is to formulate the "coherence property" more precisely.

The main result will define the precise set of all possible $\{m(x), M(x)\}$ pairs for $x \in P$. As a corollary it will be established that, if any of the quantities $M(x) - \Lambda[A], \Lambda[A] - m(x)$ converges to 0 ($x \in P$), then $||x - X[A]||$ converges to 0 with the same speed. ($||\cdot||$ will denote in this paper the l_1-norm.)

2. Estimates of the Inclusion Intervals. The following lemma is fundamental.

LEMMA 2.1. *Let $S = \{x^j\}$ denote any sequence in P. If S has no accumulation point in P, then*

$$\lim_{j \to \infty} M(x^j) = +\infty.$$

Proof. Since the closure P is compact, S has an accumulation point y on the boundary of P. Thus, $y_i \geq 0$ $(i \in N)$ and the index set $Z = \{i \in N | y_i = 0\}$ is not empty.

Since $||y|| = 1$, $N \backslash Z$ is not empty either. Because A is assumed nonnegative irreducible, there is an $r \in Z$, $S \in N \backslash Z$ such that $a_{rs} > 0$.

Let $S'' = \{x^{\sigma(j)}\}$ denote a subsequence of S converging to y. Then

$$[x^{\sigma(j)}]_r \to y_r = 0, [x^{\sigma(j)}]_s \to y_s > 0 \tag{3}$$

as $j \to \infty$. For nonnegative $A = (a_{jj})$, $f_r(x) \geq a_{rr} + a_{rs}\frac{x_s}{x_r}$. Therefore, by (3), $f_r(x^{\sigma(j)}) \to +\infty$ as $j \to \infty$ and thus $M(x^{\sigma(j)}) \to \infty$. Hence, $+\infty$ is an accumulation point of any subsequence of $\{M(x^j)\}$ and therefore $M(x^j) \to \infty$. ∎

The next theorem will specify an upper bound for a chosen ratio $f_k(x)$ if a common lower bound of all other $f_i(x)$ is given and vice versa.

Birkhoff and Varga [2] observed that the results of the Perron–Frobenius theory (and consequently also Collatz's theorem) could be slightly generalized by allowing the matrices considered to have negative diagonal elements. They introduced the terms "essentially nonnegative matrix" for matrices whose off-diagonal elements are nonnegative and "essentially positive matrix" for essentially nonnegative, irreducible matrices. The only important change is that, whenever the Perron–Frobenius theory or Collatz's theorem refers to the spectral radius of a nonnegative matrix A, the

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corresponding quantity for an essentially nonnegative matrix $\tilde{A}$ is the (real) eigenvalue of the maximal real part in the spectrum of $\tilde{A}$, also to be denoted by $\Lambda[\tilde{A}]$. Of course $\Lambda[\tilde{A}]$ need not be positive and it is not necessarily dominant among the eigenvalues in the absolute-value sense. The term $X[\tilde{A}]$ will always denote the positive eigenvector of the unit l_1-norm of an essentially positive matrix $\tilde{A}$.

$A = (a_{ij})$ will henceforth denote an essentially positive matrix and $A_{(k)}$ its $(n-1)$st-order submatrix associated with a_{kk}, and $\nu_k = \Lambda[A_{(k)}]$. By a theorem of Frobenius [10],

$$\lambda \equiv \Lambda[A] > \Lambda[A_{(k)}] \equiv \nu_k$$

(see also [20], p. 30; [11], p. 69; and [9]).

THEOREM 2.2. *Let $k \in N$ and $\mu > \nu_k$.*

(i) *The system of $n-1$ equations*

$$f_i(x) = \mu \text{ for all } i \neq k \tag{4}$$

has a unique solution $x = g^k(\mu)$ in P and

$$f_k[g^k(\mu)] = a_{kk} + p_k^t(\mu I - A_k)^{-1} q_k \stackrel{\text{def}}{=} \theta_k(\mu).^1$$

Note: p_k is the matrix constructed from the kth row of matrix A after deleting a_{kk}, q_k is the matrix constructed from the kth column of matrix A after deleting a_{kk}, and $A_{(k)}$ is the matrix constructed from A by deleting row and column k.

(ii) *If $x \in P$ and $f_i(x) \geq \mu$ ($\leq \mu$) for each $i \neq k$, $i \in N$, then*

$$f_k(x) \leq \theta_k(\mu) \ (\geq \theta_k(\mu))$$

and $f_k(x) = \theta_k(\mu)$ holds only for $x = g^k(\mu)$.

(iii) *The function $\theta_k(\mu)$ is analytic and decreasing in (ν_k, ∞) and*

$$\theta_k(\lambda) = \lambda, \tag{5}$$

$$\theta_k(\mu) =\to a_{kk} \text{ as } \mu \to \infty, \tag{6}$$

$$\theta_k(\mu) =\to +\infty \text{ as } \mu \downarrow \nu_k. \tag{7}$$

Proof. (i) It may be assumed without loss of generality that $k = n$. Then A has the partitioned form

$$A = \begin{bmatrix} B & q \\ p^t & a_{nn} \end{bmatrix},$$

[1] Since it is clear from the context, the orders of identity matrices and of zero matrices will not be indicated. Thus, identity matrices of all orders will be denoted by I and rectangular zero matrices of all orders will be denoted by 0.

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where $B \equiv A_{(n)}$, $p \equiv p_n \geq 0$, and $q \equiv q_n \geq 0$.

Moreover, $p_n \neq 0, q_n \neq 0$, for, otherwise A would become reducible. Suppose that $\mu > \nu_n$.

Let $x = [x_1, x_2, \ldots, x_n]^t$ satisfy (4) with $k = n$. We note that in this case x_n cannot vanish. The vector y formed from the first $n-1$ components of x then satisfies

$$By + x_n q = \mu y. \tag{8}$$

By a result of Frobenius (see [10]), the matrix $\mu I - B$ has a nonnegative inverse if $B \geq 0$ and $\mu > \Lambda[B]$. It has a positive inverse if in addition B is irreducible (see also [11], p. 69, and [9]). Consequently, (8) has the nonnegative solution

$$y = (\mu I - B)^{-1} q, x_n \neq 0, \text{ so it can be set that } x_n = 1. \tag{9}$$

We will show that y is actually a positive vector.

If $A_{(n)}$ is reducible, then there is a permutation matrix P to bring it into the lower block triangular form

$$PBP^t = \begin{bmatrix} G_{11} & 0 & 0 & \ldots & 0 \\ G_{21} & G_{22} & 0 & \ldots & 0 \\ . & . & . & \ldots & . \\ G_{k1} & G_{k2} & G_{k3} & \ldots & G_{kk} \end{bmatrix}, \tag{10}$$

where the G_{ii} are irreducible $m_i \times m_i$ matrices.[2]

It can be seen that there is no loss of generality to assume that B is reducible, for, if it is irreducible, then the representation (10) consists of a single block $B = G_{11}$. This situation can be treated as a special case. Equation (8) can be rewritten as

$$(PBP^t - \mu I)Py + Pq = 0. \tag{11}$$

Suppose that Pq is partitioned into submatrices $r_1, r_2, r_3, \ldots, r_k$ of sizes $m_1 \times 1, m_2 \times 1, \ldots, m_k \times 1$, respectively, and Py into the submatrices $z_1, z_2, \ldots, z_k$ of corresponding sizes.

Then (11) takes the form

$$(\mu I - G_{jj})z_j = r_j + \sum_{m=1}^{j-1} G_{jm} z_m \tag{12}$$

for $1 \leq j \leq k$. (The sum is assumed to vanish if $j = 1$.) The spectrum of B is the union of the spectra of the matrices G_{jj}. Therefore, $\mu > \Lambda[G_{jj}]$ for $1 \leq j \leq k$. Since G_{jj} is irreducible by the quoted result of Frobenius, $(\mu I - G_{jj})^{-1}$ is a *positive* matrix for $1 \leq j \leq k$.

The positivity of the vectors $z_1, z_2, \ldots, z_k$ will be proved now by induction. The inductive hypothesis S_j, stating that $z_m > 0$ for $m < j$, is empty for $j = 1$ and thus S_1

[2] Any (1×1) matrix is considered irreducible here.

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can be considered true. Assuming now S_j correct for some j, such that $1 \leq j \leq k-1$, it follows that the right-hand side of (12) can be the 0-vector only if $r_j = 0$ and $G_{jm} = 0$ for $m < j$. However, then all rows of A intersecting G_{jj} would only have 0 elements outside the block G_{jj}, which would imply that A is reducible. Therefore, the right-hand side of (12) representing a nonnegative vector must have a positive component. Because of the positivity of $(\mu I - G_{jj})^{-1}$, z_j is also a positive vector. Hence, S_j implies S_{j+1} for $j < k$ and thus, by induction, $z_j > 0$ for $1 \leq j \leq k$. As a result,

$$y = (\mu I - B)^{-1} q > 0 \tag{13}$$

is proved. We note that, by substituting B^t for B and p for q in (13), we obtain the dual inequality

$$(\mu I - B^t)^{-1} p > 0. \tag{14}$$

Let $a = (||y|| + 1)^{-1}$. Then

$$x^* = g^n(\mu) \stackrel{\text{def}}{=} \alpha \begin{bmatrix} y \\ 1 \end{bmatrix}$$

is the unique solution of the problem (4) in P (in the case $k = n$). g^n represents an analytic mapping of (ν_n, ∞) into P. The value of $f_n(x)^*$ is obtained easily:

$$f_n(x^*) = (AX^*)/x_n^* = p^t y + a_{nn} = a_{nn} + p^t(\mu I - B)^{-1} q = \theta(\mu).$$

(ii) Let x denote any vector in P. We introduce the matrices

$$C(x) = A - diag[f_1(x), f_2(x), \ldots, f_n(x)]. \tag{15}$$

$C(x)$ is essentially positive for any $x \in P$. Moreover, its positive eigenvector is x itself and the corresponding eigenvalue is

$$\Lambda[C(X)] = 0.$$

We note the special case $x^* = g^n(\mu)$:

$$C(x^*) = A - diag[\mu, \ldots, \mu, \theta_n(\mu)]. \tag{16}$$

Suppose now that, for some $x \in P, x \neq x^*$,

$$f_i(x) \geq \mu (i = 1, 2, 3, \ldots, n-1) \tag{17}$$

and at the same time

$$f_n(x) \geq \theta_n(\mu).$$

Then, by (15) and (16), $C(x) \geq C(x^*)$. The matrices $C(x), C(x^*)$ are different, for, otherwise, $x = x^*$ would follow from the uniqueness of the positive eigenvector. If

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$\Lambda[C(x)] > \Lambda[C(x^*)]$, then, because of the Frobenius theorem, $\Lambda[C]$ is a strictly increasing function of all entries of the essentially positive C. Thus, (17) implies

$$f_n(x) < \theta_n(\mu). \tag{18}$$

It can be shown similarly that (17) and (18) remain valid with the inequalities reversed simultaneously.

(iii) From the definition of $\theta_n(\mu)$ follows

$$(d/d\mu)\theta_n(\mu) = -p^t(\mu I - B)^{-2}q = -[(\mu I - B^t)p]^t[(\mu I - B)q].$$

Since both vectors in brackets were shown to be positive [(13), (14)], $(d/d\mu)\theta_n(\mu) < 0$; hence $\theta_n(\mu)$ is decreasing.

By the theorem of Collatz, the statement

$$f_1(x) = f_2(x) = \cdots = f_{n-1}(x) = \lambda$$

implies that $f_n(x) = \lambda$. Hence, by the result obtained in (i), the statement (5) is proved.

The statement (6) is obvious by the definition of $\theta_k(\mu)$. The statement (7) will be proved by contradiction. If it is false, then $\theta_n(\mu)$, a decreasing function, has a finite limit as $\mu \downarrow \nu_n \equiv \nu$. But then, by Lemma 2.1, a sequence $\mu_j \downarrow \nu$ exists such that, if $g^n(\mu_j) = x^j$, then x^j converges to a vector $u = (u_1, u_2, \ldots, u_n)^t \in P$. Let v denote the vector formed of the first $n-1$ components of u. Then, applying (9) to x^j and letting $j \to \infty$, we get

$$(B - \nu I)v + u_n q = 0.$$

Since the spectrum of B is the union of spectra of the submatrices G_{ii} obtained by the partitioning (11), there is at least one submatrix G_{ss} such that $\Lambda[G_{ss}] = \Lambda[B] = \nu$. If Pq is partitioned as before and Pv is partitioned into submatrices $w_1, w_2, \ldots, w_k$ of corresponding sizes, then equations analogous to (13) are obtained, replacing only z_j with w_j and μ with ν. In particular,

$$(\nu I - G_{ss})W_s = v_n r_s + \sum_{m=1}^{s-1} G_{sm} w_m, \tag{19}$$

since all right-hand terms of (19) are known to be nonnegative. So it comes to such terms as

$$(G_{ss} - \nu I)w_s \leq 0. \tag{20}$$

Here $w_s > 0$ and G_{ss} is an essentially positive matrix such that $\Lambda(G_{ss} - \nu I) = \nu - \nu = 0$. If w_s is not an eigenvector of G_{ss}, then, by the theorem of Collatz, $(G_{ss} - \nu I)w_s$ must have a positive component. In view of (20), therefore, w_s is an eigenvector of G_{ss} and $(G_{ss} - \nu I)w_s = 0$.

Therefore, (19) implies by the positivity of the vectors w_i and of v_n that $G_{sm} = 0$ $(m = 1, 2, 3, \ldots, s-1)$ and $r_s = 0$.

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As a result, the rows of A containing G_{ss} are otherwise empty. However, then A would be reducible, contrary to assumption. Thus (7) has to be correct. □

COROLLARY 2.3. *Let*

$$\Phi(\mu) = \max\{\theta_i(\mu)|i \in N\},$$

$$\Psi(\mu) = \min\{\theta_i(\mu)|i \in N\},$$

and Φ^{-1} *denote the inverse function of* Φ. *Then, for any* $x \in P$,

$$\Psi(M(x)) \leq m(x) \leq \Phi^{-1}(M(x)). \tag{21}$$

Proof. Equation (21) is a consequence of (ii) in Theorem 2.2.

The functions Φ^{-1}, Ψ are continuous, piecewise analytic, and strictly decreasing in $[\lambda, \infty]$. Further,

$$\Phi^{-1}(\lambda) = \Psi(\lambda) = \lambda,$$

$$\lim_{\mu\to\infty} \Phi^{-1}(\mu) = \max_{i\in N}(\nu_i) \equiv X, \text{ and}$$

$$\lim_{i\in N} \Psi(\mu) = \min_{i\in N}(a_{ii}) \equiv \alpha.$$

Hence, every inclusion interval (κ, μ) (defined by $\kappa = m(x), \mu = M(x)$, valid for some $x \in P$) can be represented by a point in the closed, curvilinear, wedge-shaped region L^* enclosed by the curves $\Gamma_1 : \kappa = \Psi(\mu)$ and $\Gamma_2 : \kappa = \Phi^{-1}(\mu)$ (see Fig. 1). □

The following theorem is an application of this corollary to the design of algorithms using the Perron–Frobenius–Collatz minimax principle for the calculation of $X[A]$.

THEOREM 2.4. *Suppose that, for a sequence* $\{x^i\}$ *of positive unit vectors,* $m(x) \to \Lambda[A] \equiv \lambda$. *Then* $x^i \to X[A] \equiv \xi$. *Moreover, the sequences* $\{m(x^i)\}$, $\{x^i\}$ *are equiconvergent in the sense that an index* ν *and a constant* $K > 0$ *exist such that*

$$||x^i - \xi|| < K[\lambda - m(x^i)] \text{ if } i \geq \nu.$$

Similar statements can be made if $M(x^i) \to \lambda$ *is known.*

Proof. By the corollary of Theorem 2.2 and the theorem of Collatz,

$$\lambda \leq M(x^i) \leq \phi(m(x^i)). \tag{22}$$

Since $\phi(\lambda) = \lambda$, (22) implies $M(x^i) \to \lambda$. All accumulation points of the sequence $\{x^i\}$ are in P, for, otherwise, $M(x^i)$ would be unbounded, by Lemma 2.1. Let $\{x^{\rho(i)}\}$ denote a convergent subsequence, say, $x^{\rho(i)} \to y \in P$. Then

$$M(y) = m(y) = \lambda$$

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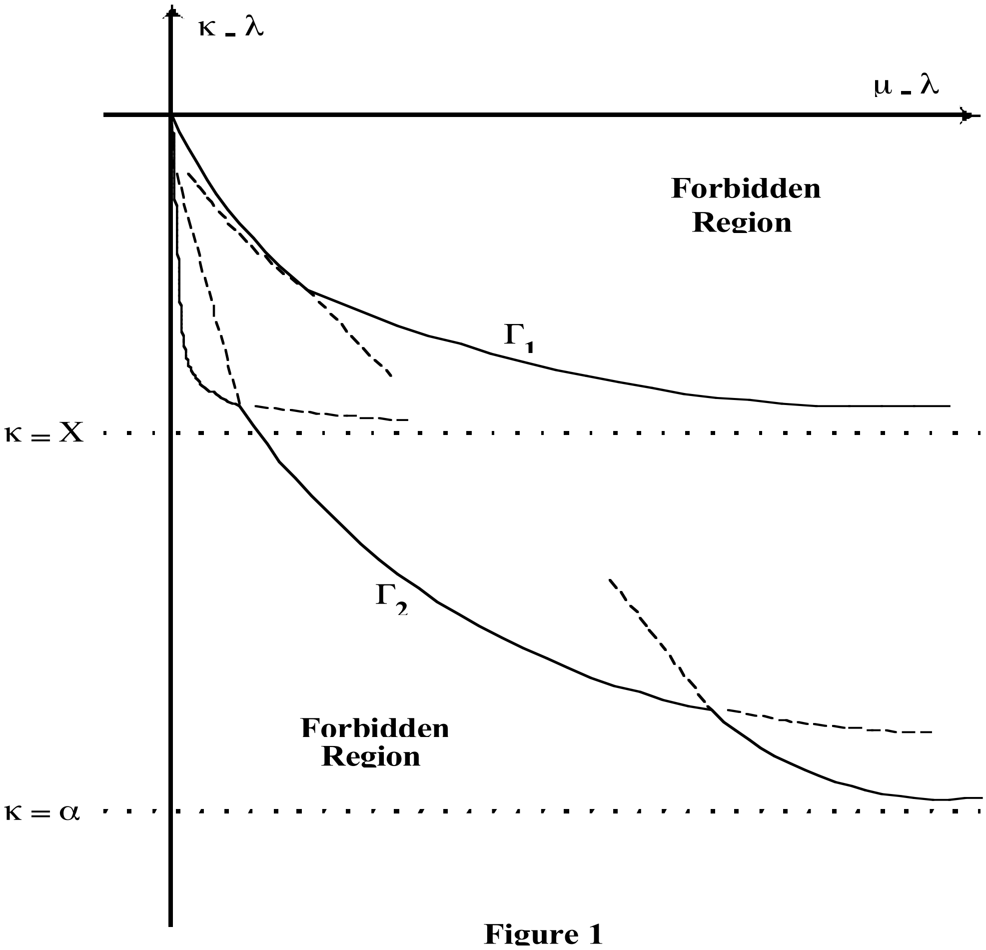

Figure 1
Locus of the Inclusion Intervals (κ , μ)
κ = m(x) μ = M(x)
Γ_1: κ = Φ^{-1}(μ) Γ_2: κ = ψ(μ)

and, therefore, $y = X[A] = \xi$, by the theorem of Collatz. This is true for *any* convergent subsequence $\{x^{\rho(i)}\}$. Hence $x^i \to \xi$.

The function $\Phi(\mu)$ is continuous and piecewise analytic in (χ, ∞). Therefore, it has a left-hand derivative $\dot{\Phi}_l(\lambda) \equiv k_1$. If $\lambda - m(x) = \delta \downarrow 0$, then $M(x) - \lambda \leq k_1\delta + 0(\delta^2)$. Thus, for each $i \in N$ and $k = 2\max(k_1, 1)$, and also for sufficiently large values of j,

$$|f_i(x^j) - \lambda| \leq k\delta,$$

from which

$$||Ax^j - \lambda x^j|| \leq k\delta \tag{23}$$

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follows easily.

Remembering now that λ is, by the Perron–Frobenius theorem, a simple eigenvalue of A, an elementary consideration shows that a constant β (dependent on A alone) exists such that, for any unit n-vector y,

$$||y-\xi|| \leq \beta||Ay-\lambda y||$$

by (23) therefore for sufficiently large j and $K=\beta k$. Thus,

$$||x^j-\xi|| \leq 0[\lambda - m(x^j)].$$

If $M(x^j)\to\lambda$ is assumed, the corresponding results can be obtained similarly. □

It should be noted that the result as such has been studied thoroughly in the literature and the work reported here is a step forward in achieving such a method from a different point of view. The main difference in this paper is the exact description of all possible pairs of maximal and minimal quotients $m(x)$ and $M(x)$ of a positive vector built from a nonnegative irreducible matrix A. A large number of authors have been dealing with methods for calculating the Perron vector by constructing a sequence $x^{(i)}$ with decreasing $M(x^{(i)})$. Their convergence is shown to be linear ([13], [14]). On the other hand, as described in section 1 of [7], a linear convergence was also derived. In the context of [8] and [5], a linear convergence was proved and numerical results were presented. It would be impossible to list by name the authors who studied thoroughly the above-mentioned method. We would, however, like to mention the contribution of a handful of mathematicians ([4], [1], [16]), as cited in the last paper.

Remark 2.5. It is interesting to compare this result to the behavior of the minimum sequence in the Rayleigh–Ritz method applicable to Hermitian matrices. Typically, in that case,

$$||x^j-\xi|| = 0(|\lambda - R^j|^{\frac{1}{2}})$$

only where R^j is the Rayleigh quotient belonging to the sequence element x^j, ξ is the eigenvector, and λ is the corresponding eigenvalue.

3. The Set of All Inclusive Intervals. The corollary to Theorem 2.2 states that, if all possible $\{m(x), M(x)\}$ pairs are represented in a (κ,μ) Cartesian coordinate system, their locus L is a (proper or improper) subset of the closed wedge-shaped region L^* bounded by the curves $k=\phi^{-1}(\mu)$ and $\kappa=\psi(\mu)$ $(\mu\geq\lambda)$. By Theorem 2.2, for any $\mu\geq\lambda$, an $x\in P$ can be found such that $M(x)=\mu$. Therefore, (21) implies that $\phi^{-1}(\mu)\geq\psi(\mu)$ for any $\mu>\lambda$.

The following questions remain to be resolved:

(1) Is $\phi^{-1}(\mu)\geq\psi(\mu)$ true for $\mu>\lambda$?

(2) Is every $(\kappa,\mu)\in L^*$ an inclusion interval (i.e., is $\kappa=m(x), \mu=M(x)$ for some $x\in P$)?

The answer given in Theorem 3.2 below is affirmative to both questions if $n\geq 3$. We will show first that neither of the answers is necessarily yes for $n=2$. Let $A=(a_{ij})$

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be a 2×2 essentially positive matrix. Then, with the notation of Theorem 2.2,

$$\theta_i(\mu) = a_{ii} + \frac{a_{ij}a_{ji}}{\mu - a_{jj}},$$

where $i = 1, 2$ and $j = 3 - i$.

It can be verified by direct calculation that, for $\mu > \lambda$,

$$\phi^{-1}(\mu) = \max\{\theta_1(\mu), \theta_2(\mu)\} \text{ and}$$

$$\psi(\mu) = \min\{\theta_1(\mu), \theta_2(\mu)\}.$$

Furthermore,

$$\phi^{-1}(\mu) > \psi(\mu) \text{ if } \mu > \lambda \text{ and } a_{11} \neq a_{22} \text{ and}$$

$$\phi^{-1}(\mu) = \psi(\mu) \text{ for all } \mu \geq \lambda \text{ if } a_{11} = a_{22}.$$

The set L is then the union of the curves described by $\kappa = \theta_1(\mu)$ and $\kappa = \theta_2(\mu)$ for $(\mu \geq \lambda)$ (thus $L^* \neq L$ if $a_{11} \neq a_{22}$).

In the proof of Theorem 3.2, we will need the answer to a question that is interesting in itself. Is it possible to modify a vector $x \in P$ continuously so that two of the ratios $f_i(x)$ should change? How are the changes of the two selected ratios related?

LEMMA 3.1. *Suppose that $x^o \in P$ and $f_i(x^o) = \phi_i$. Then positive numbers τ_i, τ_{ij} exist $(i, j \in n, i \neq j)$ such that, for any $r, s \in N$, $r \neq s$, the system*

$$\begin{aligned} f_i(x) &= \phi_i \text{ if } i \neq r, s, \\ f_r(x) &= \phi_r + \rho, \\ f_s(x) &= \phi_s + \sigma \end{aligned} \tag{24}$$

has a uniquely determined solution $x \in P$ whenever the real numbers ρ, σ satisfy

$$\tau_{rs}\rho\sigma + \tau_r\rho + \tau_s\sigma = 0 \tag{25}$$

and

$$\rho\sigma \leq 0.$$

This solution $x = h^{rs}(x^o, \rho)$ is an analytic function of ρ in its definition interval $(-\tau_s/\tau_{rs}, \infty)$.

Proof. Consider the matrix

$$C = \text{diagonal}\,[f_1(x^o), f_2(x^o), \ldots, f_n(x^o)] - A.$$

Let $C_i(x)$ denote the $(n-1)$-square submatrix obtained by omission of the ith row and column, C_{ij} the $(n-2)$-square submatrix resulting after omission of the ith and jth rows and columns of $C(i, j \in N, i \neq j)$, and $\tau_i = \det(C_i), \tau_{ij} = \det(C_{ij})$ the

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corresponding principle minors. Here $-C$ is essentially positive and its positive eigenvector is x^o, with corresponding eigenvalue $\Lambda[-C] = 0$. By the theorem of Frobenius [9], $\tau_i = \det(C_i) > 0$ and $\tau_{ij} = \det(C_{ij}) > 0$ under these circumstances (see also [10], p. 70 and [8]). If $x \in P$, then the system (24) is equivalent to

$$Fx \equiv (C + \rho\Delta^r + \sigma\Delta^s)x = 0, \tag{26}$$

where Δ^p is the $n \times n$ matrix $\Delta^p_{ij} = \delta_{pi}\delta_{pj}$ and δ_{pi} is the Kronecker delta. Equation (26) has a nontrivial solution if and only if

$$\det(F) = \det(C) + \rho\tau_r + \sigma\tau_s + \rho\sigma\tau_{rs} = 0.$$

Since $\Lambda(C) = 0$, C is singular and therefore the last equation is equivalent to (25). If $\rho = 0$ and $\sigma = 0$ then the (unique) solution of (24) is $x = x^o$. The principle minors of F associated with the (rr) and (ss) elements are

$$F\begin{pmatrix} 1,2 & r-1, r+1 & n \\ 1,2 & r-1, r+1 & n \end{pmatrix} = \tau_s + \sigma\tau_{rs},$$

$$F\begin{pmatrix} 1,2 & s-1, s+1 & n \\ 1,2 & s-1, s+1 & n \end{pmatrix} = \tau_r + \rho\tau_{rs}.$$

If $\rho\sigma < 0$, at least one of these is positive. Therefore, the rank of F is $n-1$ and the solution of (26) is determined uniquely up to a scalar factor.

By (25),

$$\sigma = -\tau_r/(\tau_{rs}\rho + \tau_s) \stackrel{\text{def}}{=} \sigma_{rs}(\rho). \tag{27}$$

Therefore, together with σ, all subdeterminants of F are analytic functions of ρ in the interval (γ, ∞), where $\gamma = -\tau_s/\tau_{rs}$. Therefore, (26) has a normalized solution $t(\rho) = h^{rs}(x^o, \rho)$, an analytic function of ρ in an open domain of the complex plane containing the real interval $(-\gamma, \infty)$, and $t(0) = x^o$. It will be shown next that $M[t(\rho)]$ is bounded in $(-\gamma, \infty)$. In fact, if $x = t(\rho)$, then, by definition of C and by (26),

$$(Ax)_p = [\phi_p + \rho\delta_{rp} + \sigma\delta_{sp}]x_p \ \ (p = 1, 2, 3, \ldots, n).$$

It follows easily from this identity that

$$M(x) \leq M(x^o) + \max\{\rho, \sigma\}, \tag{28}$$

where $\sigma = \sigma_{rs}(\rho)$. Now suppose that (α, β) is the component of the open set $\Omega = \{\rho \in (\gamma, \infty) | t(\rho) \in P\}$ that contains $\rho = 0$. If $\alpha > \gamma$, then by continuity $t(\alpha) \in P$. But, by Lemma 2.1, $M[t(\rho)]$ would be unbounded as $\rho \downarrow \alpha$, contrary to (28). A similar contradiction arises if $\beta < \infty$ is assumed. Hence $\Omega = (\gamma, \infty)$ and $T^{rs}(x_o, \rho) \in P$ for $\gamma < \rho < \infty$. $\square$

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THEOREM 3.2. *Let $n > 3$. Then, with the notation of the corollary of Theorem 2.2,*

(i) $\psi(\mu) < \phi^{-1}(\mu)$ *if* $\mu > \lambda$.

(ii) *The locus L of all (μ, κ) pairs, with $\kappa = m(x)$, $\mu = M(x)$, and $x \in P$, is precisely the closed curvilinear wedge-shaped domain L^* between the curves Γ_1 and Γ_2; i.e.,*

$$L = \{(\mu, \kappa) | \phi^{-1}(\mu) \leq \kappa \leq \psi(\mu)\}$$

(see Fig. 1).

Proof. With the notation of Theorem 2.2, let $\mu > \lambda$ and $r, s \in N$ such that

$$\mu = \phi(\tau) = \theta_r(\tau),$$

$$\psi(\mu) = \theta_s(\mu).$$

Further, let $y = g^s(\mu), x = g^r(\tau)$. Then x, y satisfy the equations

$$f_i(y) = \mu \quad \text{if} \quad i \neq s,$$

$$f_s(y) = \psi(\mu),$$

$$f_i(x) = \tau = \phi^{-1}(\mu) \text{ if } i \neq r, \tag{29a}$$

$$f_r(x) = \mu. \tag{29b}$$

In particular, note that $M(x) = M(y) = \mu$.

For the proof of (i) it is sufficient to show a single vector $\mu \in P$ such that $m(\mu) < \tau$, whereas $M(\mu) = \mu$.

Let $j, k \in N$ and $j, k \neq r$. Then, by (29a), $f_j(x) = f_k(x) = \tau < \mu$.

The function $t(\xi) = h^{jk}(x, \xi)$ introduced in Lemma 3.1 is continuous for $\xi \geq 0$ and satisfies

$$f_j[t(\xi)] = \tau + \xi,$$

$$f_k[t(\xi)] = \tau + \sigma_{jk}(\xi),$$

where $\sigma_{jk}(\xi)$ is the function (27) and is negative for $\xi > 0$. The other ratios $f_i[t^{jk}(\xi)]$ are constants. Setting $\mu = t^{jk}(\mu - \tau)$, we find by Lemma 3.1 that

$$f_k(u) < \tau, M(u) = f_j(u) = \mu.$$

Hence proposition (i) is proved.

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For the proof of (ii) it suffices to show that a continuous function $w : [0,1] \to P$ can be found such that

$$w(0) = x, w(1) = y, \tag{30a}$$

$$M[w(\xi)] = \mu \quad (0 \le \xi \le 1). \tag{30b}$$

Then, by the intermediate value theorem, for any η in the interval $(\phi^{-1}(\mu), \psi(\mu))$, $\eta = m[w(\xi^*)]$ for some $\xi^* \in P$ and thus (η, μ) is an inclusion interval.

Let $p(1), p(2), \ldots, p(n)$ denote a permutation of the numbers $1, 2, \ldots, n$ such that (a) $p(2) \neq r$ and (b) $p(n) = s$. (Such a permutation can be found if $n \geq 3$.) Let the function $v^1, v^2, \ldots, v^{n-1} : [0,1] \to P$ be defined recursively by the following rules:

$$v^1(0) = x \stackrel{\text{def}}{=} z^1,$$

$$v^i(0) = v^{i-1}(1) \stackrel{\text{def}}{=} z^i (i > 1),$$

and

$$v^i(\xi) = h^{p(i)p(i+1)}(z^i, \delta_i \xi),$$

where $i = 1, 2, \ldots, n-1$ and $\delta_i = \mu - f_{p(i)}(z^i)$. By Lemma 3.1,

$$f_{p(i)}[v^i(\xi)] = f_{p(i)}(z^i) + \delta_i \xi = \mu\xi + f_{p(i)}(z^i)(1 - \xi), \tag{31}$$

$$f_{p(i+1)}[v^i(\xi)] = f_{p(i+1)}(z^i) + \sigma_{p(i)p(i+1)}(\delta_i \xi), \tag{32}$$

whereas

$$f_j[v^i(\xi)] = f_j(z^i) \text{ if } j \neq p(i), p(i+1). \tag{33}$$

It may be observed that the function (31) is nondecreasing and (32) is nonincreasing if $\delta_j > 0$. (The function (33) is constant.)

Since $M(x) = \mu$, clearly $\delta_1 \geq 0$. Hence, by (31), (32), and (33), $f_j[v^1(\xi)] \leq \mu$ for $0 \leq \xi \leq 1$. Furthermore, $f_r[v^1(\xi)] = \mu$ since $f_r(z^1) = f_r(x) = \mu$, and, by the assumption $r \neq p(2)$, either (31) or (33) applies. By (31), then, $f_{p(1)}(z^2) = \mu$. Furthermore, among the functions $f_i[v^1(\xi)]$, all but (possibly) the one with $i = p(1)$ are nonincreasing. Therefore, letting $H[t]$ denote the index set

$$H[t] = \{i \in N | f_i(t) = \mu\}$$

for any $t \in P$, we obtain

$$\{p(1)\} \subseteq H[z^2] \subseteq \{p(1)\} \cup \{r\}.$$

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It can be proved similarly for $i = 2, 3, \ldots, n-1$ by induction by means of (31), (32), and (33) that

$$\delta_i \equiv \mu - f_{p(i)}(z^i) \geq 0,$$

$$M[v^i(\xi)] = \mu (0 \leq \xi \leq 1),$$

$$\{p(1), \ldots, p(i)\} \subseteq H[z^{i+1}] \subseteq \{p(1), \ldots, p(i)\} \cup \{r\} \tag{34}$$

($z^n \equiv v^{n-1}(1)$). By (34), then, in particular, $H[z^n]$ contains at least the $(n-1)$-element set $\{p(1), \ldots, p(n-1)\} = N - \{s\}$. It may not contain n elements, in which case $\mu > \Lambda[A]$ would be an eigenvalue of A, contrary to the Perron–Frobenius theorem. Therefore, $H[z^n] = N - \{s\}$. Then, by Theorem 2.2, $Z^n = g^s(\mu) = y$ is a result. It is now easy to see that the function w defined by

$$w(\xi) = v^i[(n-1)\xi - (i-1)] \text{ if } \frac{i-1}{n-1} \leq \xi \leq \frac{i}{n-1},$$

$$w(1) = v^{n-1}(1) = z^n$$

($i = 1, 2, \ldots, n-1$) is continuous in $[0, 1]$ and satisfies the relations in (30). Thus, the proof of proposition (ii) is completed. □

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