Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 10, pp. 320-340, December 2003

ELA
http://math.technion.ac.il/iic/ela

AN ALGORITHM THAT CARRIES A SQUARE MATRIX INTO ITS TRANSPOSE BY AN INVOLUTORY CONGRUENCE TRANSFORMATION*

D.Ž. ĐOKOVIĆ†, F. SZECHTMAN‡, AND K. ZHAO§

Abstract. For any matrix X let X' denote its transpose. It is known that if A is an n-by-n matrix over a field F, then A and A' are congruent over F, i.e., $XAX' = A'$ for some $X \in \mathrm{GL}_n(F)$. Moreover, X can be chosen so that $X^2 = I_n$, where I_n is the identity matrix. An algorithm is constructed to compute such an X for a given matrix A. Consequently, a new and completely elementary proof of that result is obtained.

As a by-product another interesting result is also established. Let G be a semisimple complex Lie group with Lie algebra $\mathfrak{g}$. Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a $\mathbf{Z}_2$-gradation such that $\mathfrak{g}_1$ contains a Cartan subalgebra of $\mathfrak{g}$. Then L.V. Antonyan has shown that every G-orbit in $\mathfrak{g}$ meets $\mathfrak{g}_1$. It is shown that, in the case of the symplectic group, this assertion remains valid over an arbitrary field F of characteristic different from 2. An analog of that result is proved when the characteristic is 2.

Key words. Congruence of matrices, Transpose, Rational solution, Symplectic group.

AMS subject classifications. 11E39, 15A63, 15A22.

1. Introduction. Let F be a field and $M_n(F)$ the algebra of n-by-n matrices over F. For $X \in M_n(F)$, let X' denote the transpose of X. In a recent paper [5], the following theorem is proved.

THEOREM 1.1. *If $A \in M_n(F)$, then there exists $X \in \mathrm{GL}_n(F)$ such that*

$$XAX' = A'. \tag{1.1}$$

Subsequently, the first author of that paper was informed that this result was not new. Indeed, R. Gow [7] proved in 1979 the following result.

THEOREM 1.2. *If $A \in \mathrm{GL}_n(F)$, then there exists $X \in \mathrm{GL}_n(F)$ such that $XAX' = A'$ and $X^2 = I_n$.*

The latter theorem is much stronger than the former except that A is required to be nonsingular. This restriction was removed in [3], yielding

THEOREM 1.3. *If $A \in M_n(F)$, then there exists $X \in \mathrm{GL}_n(F)$ such that*

$$XAX' = A', \quad X^2 = I_n. \tag{1.2}$$

*Received by the editors on 12 June 2003. Accepted for publication on 8 December 2003. Handling Editor: Robert Guralnick.

†Department of Pure Mathematics, University of Waterloo, Waterloo, Ontario, N2L 3G1, Canada (djokovic@uwaterloo.ca). Research was supported in part by the NSERC Grant A-5285.

‡Department of Pure Mathematics, University of Waterloo, Waterloo, Ontario, N2L 3G1, Canada. Current Address: Department of Mathematics and Statistics, University of Regina, Regina, Saskatchewan, S4S 0A2, Canada (szechtf@math.uregina.ca).

§Institute of Mathematics, Academy of Mathematics and System Sciences, Chinese Academy of Sciences, Beijing 100080, P.R. China, and Department of Mathematics, Wilfrid Laurier University, Waterloo, Ontario, N2L 3C5, Canada (kzhao@mail2.math.ac.cn). Research supported by the NSF of China (Grant 10371120).

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 10, pp. 320-340, December 2003

We point out that the proof of Theorem 1.1 in [5] and that of Theorem 1.2 in [7] are based on the previous work of C. Riehm [9]. In section 3 we shall indicate how Theorem 1.3 can be derived from Theorem 1.2 by means of a result of P. Gabriel [6] (as restated by W.C. Waterhouse in [11]).

In a certain sense, Theorem 1.1 is quite surprising (and so is Theorem 1.3). Indeed the matrix equation (1.1) is equivalent to a system of n^2 quadratic equations in n^2 variables x_{ij}, the entries of the matrix $X = [x_{ij}]$. There is no apparent reason why this system of quadratic equations should have a nonsingular rational solution, i.e., a solution $X \in \mathrm{GL}_n(F)$. (Note that if A is nonsingular then (1.1) implies that $\det(X) = \pm 1$.)

Let us illustrate this point with an example. Say, $n = 3$ and the given matrix is

$$A = \begin{bmatrix} a & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad a \neq 0.$$

Writing the unknown matrix X as

$$X = \begin{bmatrix} x & y & z \\ u & v & w \\ p & q & r \end{bmatrix},$$

the above mentioned system of quadratic equations is:

$$\begin{aligned} &ax^2 + xy + yz = a, \quad axu + xv + yw = 0, \\ &axp + xq + yr = 0, \quad axu + yu + zv = 1, \\ &au^2 + uv + vw = 0, \quad aup + uq + vr = 0, \\ &axp + yp + zq = 0, \quad aup + vp + wq = 1, \\ &ap^2 + pq + qr = 0. \end{aligned}$$

It is not obvious that this system has a nonsingular rational solution. Nevertheless such a solution exists, for instance the matrix

$$X = \begin{bmatrix} 2 & -a & 1 \\ 2a^{-1} & -1 & 2a^{-1} \\ -1 & a & 0 \end{bmatrix}$$

with determinant -1. In fact, we have $X^2 = I_3$.

The proofs of the first two theorems above are rather complicated and they neither explain why a nonsingular rational solution exists nor do they provide a simple method for finding such a solution. The main objective of this paper is to construct an algorithm for solving this problem, i.e., to prove the following theorem.

Theorem 1.4. *For any field F, there exists an algorithm which solves the system* (1.2). *More precisely, the input of the algorithm is a positive integer n and an arbitrary matrix $A \in M_n(F)$, and the output is a matrix $X \in \mathrm{GL}_n(F)$ which is a solution of the system* (1.2).

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 10, pp. 320-340, December 2003

ELA
http://math.technion.ac.il/iic/ela

The proof is given in section 4. We point out that we do not assume that there is an algorithm for factoring polynomials in $F[t]$ into product of irreducible polynomials. The GCD-algorithm is sufficient. Our algorithm is applicable to arbitrary F and n and so we obtain a new proof of Theorem 1.3. This proof is completely elementary in the sense that it is independent from the work of Riehm and Gabriel and it uses only the standard tools of Linear Algebra and some elementary facts about the symplectic group.

We shall see that Theorem 1.3 is closely related to (the rational version of) a special case of a theorem of L.V. Antonyan on $\mathbf{Z}_2$-graded complex semisimple Lie algebras, which may seem surprising. Let us first state Antonyan's theorem [2, Theorem 2]:

THEOREM 1.5. *Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a $\mathbf{Z}_2$-graded complex semisimple Lie algebra and G a connected complex Lie group with Lie algebra $\mathfrak{g}$. Then the following are equivalent:*

(i) *$\mathfrak{g}_1$ contains a Cartan subalgebra of $\mathfrak{g}$.*

(ii) *Every G-orbit in $\mathfrak{g}$ (under the adjoint action) meets $\mathfrak{g}_1$.*

As a motivation for his theorem, Antonyan mentions the following well known fact: Every complex (square) matrix is similar to a symmetric one. On the other hand, the corresponding statement is utterly false for real matrices. We shall be concerned with another special case of Antonyan's theorem, namely the one dealing with the symplectic group. As we shall prove, in this case the rational version of his result is valid.

A matrix $A = [a_{ij}] \in M_n(F)$ is said to be an *alternate* matrix if $A' = -A$ and all diagonal entries a_{ii} are 0. Of course, the latter condition follows from the former if the characteristic of F is not two.

In the concrete matrix style, let us define the symplectic group $\mathrm{Sp}_n(F)$, $n = 2m$ even, over any field F by:

$$\mathrm{Sp}_n(F) = \{X \in \mathrm{GL}_n(F) : \ X'JX = J\}, \tag{1.3}$$

where $J \in M_n(F)$ is a fixed nonsingular alternate matrix. Recall that $\mathrm{Sp}_n(F)$ acts on its Lie algebra

$$\mathfrak{sp}_n(F) = \{Z \in M_n(F) : \ Z'J + JZ = 0\}$$

via the adjoint action $(X, Z) \to XZX^{-1}$. It also acts on the space $\mathrm{Sym}_n(F)$ of symmetric matrices $S \in M_n(F)$ via the congruence action $(X, S) \to XSX'$. These two modules are isomorphic. An explicit isomorphism is given by $S \to Z = -J^{-1}S$.

In order to state our result it is convenient to fix

$$J = \begin{bmatrix} 0 & I_m \\ -I_m & 0 \end{bmatrix}. \tag{1.4}$$

Then we shall prove the following result.

THEOREM 1.6. *Let F be any field and let $n = 2m$ be even.*

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 10, pp. 320-340, December 2003

ELA
http://math.technion.ac.il/iic/ela

(i) *If the characteristic is not* 2 *and* $A \in \mathrm{Sym}_n(F)$, *then there exists* $X \in \mathrm{Sp}_n(F)$ *such that*

$$XAX' = \begin{bmatrix} B & 0 \\ 0 & C \end{bmatrix}, \tag{1.5}$$

where $B, C \in \mathrm{Sym}_m(F)$.

(ii) *If the characteristic is* 2 *and* $A \in M_n(F)$ *satisfies* $A + A' = J$, *then there exists* $X \in \mathrm{Sp}_n(F)$ *such that*

$$XAX' = \begin{bmatrix} B & I_m \\ 0 & C \end{bmatrix}, \tag{1.6}$$

where B *is invertible and* $B, C \in \mathrm{Sym}_m(F)$.

When $F = \mathbf{C}$, the assertion (i) is a special case of Theorem 1.5. We leave to the reader the task of reformulating part (i) of our result in terms of the adjoint action of $\mathrm{Sp}_n(F)$. One should point out that for special fields there exist more precise results. For instance, if $F = \mathbf{C}$ or $F = \mathbf{R}$, then the canonical forms (under simultaneous congruence) are known for pairs consisting of a symmetric and a skew-symmetric matrix. We refer the reader to the important survey paper of R.C. Thompson [10] and the extensive bibliography cited there.

In the last section we state two open problems concerning the congruence action of $\mathrm{SL}_n(F)$ on $M_n(F)$.

2. Preliminaries. As usual, we set $F^* = F \setminus \{0\}$. We denote by I_n the identity matrix of order n. As in Linear Algebra, we say that $E \in \mathrm{GL}_n(F)$ is an *elementary matrix* if it is obtained from I_n by one of the following operations:

(i) Multiply a row by a nonzero scalar different from 1.
(ii) Add a nonzero scalar multiple of a row to another row.
(iii) Interchange two rows.

If E is an elementary matrix, then $A \to EA$ is an *elementary row transformation* and $A \to AE'$ is an *elementary column transformation*. We shall refer to $A \to EAE'$ as an *elementary congruence transformation* or ECT for short.

For later use, we state the following trivial lemma concerning an arbitrary matrix $A \in M_n(F)$.

Lemma 2.1. *Let* $A \in M_n(F)$. *If* $B = PAP'$ *with* $P \in \mathrm{GL}_n(F)$ *and* $YBY' = B'$ *for some* $Y \in \mathrm{GL}_n(F)$, *then* $X = P^{-1}YP$ *is a solution of* (1.1). *Moreover, if* $Y^2 = I_n$ *then also* $X^2 = I_n$.

This lemma shows that, when considering the problem of finding rational nonsingular solutions of equation (1.1) or the system (1.2), we may without any loss of generality replace the matrix A with any matrix $B = PAP'$, where $P \in \mathrm{GL}_n(F)$.

Let V be a vector space over F and assume that V is equipped with a nondegenerate alternate bilinear form f. The group of all $u \in \mathrm{GL}(V)$ such that $f(u(x), u(y)) = f(x, y)$ for all $x, y \in V$ is the symplectic group of (V, f) and will be denoted by $\mathrm{Sp}(V, f)$ or $\mathrm{Sp}_n(F)$ if $\dim(V) = n$ and V and f are fixed. Note that n must be even. In this paper, f will be given usually by its matrix. If $f(v, w) = 1$, then we say that (v, w)

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 10, pp. 320-340, December 2003

is a *symplectic pair*. We shall need the following well known fact about symplectic groups.

PROPOSITION 2.2. *The symplectic group* $\mathrm{Sp}(V, f)$ *is transitive on the set of nonzero vectors of* V. *More generally, it is transitive on sequences*

$$(v_1, w_1, v_2, w_2, \ldots, v_k, w_k)$$

of orthogonal symplectic pairs (v_i, w_i).

We denote by J_m the direct sum of m blocks

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{2.1}$$

and by N_r the nilpotent lower triangular Jordan block of size r.

For the sake of convenience, we shall say that a matrix $A \in M_n(F)$ *splits* if we can construct $P \in \mathrm{GL}_n(F)$ such that PAP' is a direct sum of two square matrices of size $< n$.

In the proof of the main result we shall use the square-free factorization algorithm for the polynomial ring $F[t]$. A nonzero polynomial is *square-free* if it is not divisible by the square of any irreducible polynomial. Let $p \in F[t]$ be a monic polynomial. By using the GCD-algorithm, one can find the factorization $p = p_1 p_2 \cdots p_k$, where p_i's are monic square-free polynomials of positive degree and such that $p_i | p_{i-1}$ for $1 < i \leq k$. Such an algorithm is described in [4, Appendix 3]. We say that $p = p_1 p_2 \cdots p_k$ is the *square-free factorization* of p and that p_1 is the *square-free part* of p. We wish to remind the reader that we do not assume the existence of a prime factorization algorithm in $F[t]$, and this is the main reason for using the square-free factorization.

3. Proofs of Theorems 1.3 and 1.6. The first of these theorems is an easy consequence of the following important proposition, which will be used also later in the proof of our main result.

PROPOSITION 3.1. *Let* $A \in M_n(F)$, $n \geq 1$, *and* $\det(A) = 0$. *Then there is a recursive algorithm which constructs* $P \in \mathrm{GL}_n(F)$ *such that* $PAP' = N_r \oplus B$ *for some* r $(1 \leq r \leq n)$ *and some* $B \in M_{n-r}(F)$.

Proof. Let us write $A = [a_{ij}]$ and let d be the defect of A, i.e., the dimension of the nullspace of A. By hypothesis, $d \geq 1$. Without any loss of generality, we may assume that the first d rows of A are 0.

Assume that the first d columns of A are linearly dependent. By performing suitable ECT's on the first d rows and columns, we may assume that the first column of A is also 0. Then $A = N_1 \oplus B$ with $N_1 = [0]$ and we are done.

Otherwise we have $n \geq 2d$ and by performing suitable ECT's on the last $n - d$ rows and columns, we may assume that

$$A = \begin{bmatrix} 0 & 0 & 0 \\ A_{21} & A_{22} & A_{23} \\ 0 & * & * \end{bmatrix},$$

where $A_{21} = I_d$ and the diagonal blocks are square. By subtracting suitable linear combinations of the first d columns from the other columns (using ECT's), we may

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 10, pp. 320-340, December 2003

ELA
http://math.technion.ac.il/iic/ela

further simplify A and assume that the blocks A_{22} and A_{23} are 0. Thus

$$A = \begin{bmatrix} 0 & 0 & 0 \\ I_d & 0 & 0 \\ 0 & * & Z \end{bmatrix}.$$

If $n = 2d$, then A splits as the direct sum of d copies of N_2 and we are done.

We may now assume that $n > 2d$. Consider first the case where Z is nonsingular. By subtracting suitable linear combinations of the last $n - 2d$ columns (via ECT's) from the previous d columns, we may assume that the starred block is 0. As a side-effect, the blocks A_{22} and A_{23} may be spoiled. These blocks can be again converted to zero by adding suitable linear combinations of the first d columns. Then A splits as the direct sum of Z and d copies of N_2.

Next we consider the case where Z is singular. Since Z is of size $n - 2d < n$, we may apply our recursive algorithm to it and so we may assume that A has the form:

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ I_d & 0 & 0 & 0 \\ 0 & A_{32} & N_s & 0 \\ 0 & A_{42} & 0 & A_{44} \end{bmatrix},$$

where $s \geq 1$. By subtracting suitable linear combinations of the s columns containing the block N_s from the columns containing A_{32} (using ECT's), we may assume that all the rows of A_{32} but the first are 0. As a side-effect, the zero blocks in the second block-row may be spoiled but we can convert them back to 0 as before. Note that the first row of A_{32} must be nonzero. By using ECT's whose matrices have the form $Y \oplus (Y')^{-1} \oplus I_{n-2d}$, we may assume that the first entry of the first row of A_{32} is 1, while all other entries are 0.

Assume that $n = 2d + s$. If $d = 1$, then $A = N_n$ and we are done. If $d > 1$, then A splits, i.e., by permuting (simultaneously) rows and columns we can transform A into a direct sum $N_2 \oplus B$, where N_2 comes from the principal submatrix occupying the positions d and $2d$.

From now on we assume that $n > 2d+s$. Let X be the $n-2d-s$-by-$n-d-s-1$ matrix obtained from (A_{42}, A_{44}) by deleting its first column v. We leave to the reader to check that X has rank $n - 2d - s$. Hence by adding a suitable linear combination of the columns of A containing the submatrix X to the $d+1$-column (via ECT's), we may assume that the first column v of A_{42} is 0. That might affect the blocks in the second block-row but A_{21} will remain nonsingular. As before, we can convert to 0 the blocks of A in the second block-row except A_{21} itself. Additionally, we may assume that $A_{21} = I_d$. It is now easy to see that A splits, i.e., by permuting (simultaneously) rows and columns we can transform A into a direct sum $N_{s+2} \oplus B$. (This Jordan block comes from the principal submatrix occupying the positions 1, $d+1$, and those of the block N_s.) $\square$

Proof of Theorem 1.3. On the basis of the above proposition, we see that in order to extend Gow's theorem to obtain Theorem 1.3, it suffices to observe that, for each positive integer r, there exists a permutation matrix P_r such that $P_r^2 = I_r$ and

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 10, pp. 320-340, December 2003

$P_r N_r P_r' = N_r'$. We can take P_r to be the permutation matrix with 1's at the positions $(i, r+1-i)$ with $1 \le i \le r$. This completes the proof of Theorem 1.3. □

Proof of Theorem 1.6. Let F, m, n, and A be as in the statement of the theorem and let J be as in (1.4).

(i) By hypothesis, the characteristic of F is not 2. By Theorem 1.3, there exists $Y \in \mathrm{GL}_n(F)$ such that $Y'(A+J)Y = A - J$ and $Y^2 = I_n$. Thus we have

$$Y'JY = -J, \quad Y'AY = A. \tag{3.1}$$

Let $V = F^n$ be the space of column vectors. Denote by E^+ (resp. E^-) the eigenspace of Y for the eigenvalue $+1$ (resp. -1). We note that $J^2 = -I_n$. Hence, for $v, w \in E^+$,

$$v'Jw = (Yv)'Jw = v'Y'Jw = -v'JYw = -v'Jw.$$

As the characteristic of F is not 2, $v'Jw = 0$. Thus E^+ is totally isotropic with respect to the nondegenerate skew-symmetric bilinear form defined by J. The same is true for E^-. Since $V = E^+ \oplus E^-$, we conclude that each of these eigenspaces has dimension m. By Proposition 2.2, there exists a $T \in \mathrm{Sp}_n(F)$ which maps E^+ (resp. E^-) onto the subspace spanned by the first (resp. last) m standard basis vectors of V. Equivalently, we have

$$P := TYT^{-1} = \begin{bmatrix} I_m & 0 \\ 0 & -I_m \end{bmatrix}.$$

Then $X := (T')^{-1} \in \mathrm{Sp}_n(F)$ and the second equality in (3.1) gives $PXAX' = XAX'P$ and, consequently, (1.5) holds. Thus (i) is proved.

(ii) Now suppose the characteristic of F is 2. By Theorem 1.3, there exists $Y \in \mathrm{GL}_n(F)$ such that $Y'AY = A'$ and $Y^2 = I_n$. Thus we have $Y'A'Y = A$ and $Y'JY = J$. Denote by E the eigenspace of Y for the eigenvalue 1. As $Y^2 = I_n$, we have $\dim(E) \ge m$. For $v, w \in E$, we have $v'Jw = v'(A+A')w = v'(A+Y'AY)w = 0$. We conclude that E is totally isotropic with respect to the nondegenerate alternate bilinear form defined by J. Therefore $\dim(E) \le m$. The two inequalities for $\dim(E)$ imply that $\dim(E) = m$.

By Proposition 2.2, there is a $T \in \mathrm{Sp}_n(F)$ which maps E onto the subspace spanned by the first m standard basis vectors of V. Equivalently, we have

$$P := TYT^{-1} = \begin{bmatrix} I_m & S \\ 0 & I_m \end{bmatrix},$$

for some invertible $S \in \mathrm{Sym}_m(F)$. Then $Q := (T')^{-1} \in \mathrm{Sp}_n(F)$ and $Y'AY = A'$ gives $P'QAQ' = QA'Q'P$. As $QAQ' + (QAQ')' = J$, we can write

$$QAQ' = \begin{bmatrix} B & I_m + Z \\ Z' & W \end{bmatrix}$$

with $B, W \in \mathrm{Sym}_m(F)$ and deduce that $B = S^{-1}$ and $SZ \in \mathrm{Sym}_m(F)$. Consequently,

$$R = \begin{bmatrix} I_m & 0 \\ Z'S & I_m \end{bmatrix} \in \mathrm{Sp}_n(F).$$

Then $X = RQ$ satisfies (1.6) with $B = S^{-1}$ and $C = W + Z'S + Z'SZ$. This concludes the proof of Theorem 1.6. □

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 10, pp. 320-340, December 2003

ELA
http://math.technion.ac.il/iic/ela

4. The description of the algorithm. In this section we prove our main result, Theorem 1.4. Our algorithm operates recursively, i.e., we reduce the problem for matrices A of size n to the case of matrices of smaller size.

We now begin the description of our algorithm. Let $A = [a_{ij}] \in M_n(F)$ be given. Throughout this section we shall use the following notation: $A_0 := A + A'$ and $A_1 := A - A'$. The first of these matrices is symmetric and the second one is alternate. If the characteristic is 2, then $A_0 = A_1$. The rank of A_1 is even, say $2m$.

By Lemma 2.1, we may replace A by any matrix congruent to it. Hence without any loss of generality we may assume that A_1 is *normalized*, i.e.,

$$A_1 = \begin{bmatrix} J_m & 0 \\ 0 & 0 \end{bmatrix}.$$

Let G denote the subgroup of $\mathrm{GL}_n(F)$ that preserves the matrix A_1, i.e.,

$$G = \{X \in \mathrm{GL}_n(F) : X'A_1X = A_1\}.$$

For $S \in G$, we say that $A \to SAS'$ is a *symplectic congruence transformation* or SCT. An ECT can be an SCT only if $2m < n$. If it is not an SCT, we can compose it with another ECT to obtain an SCT. For instance, if $m > 0$ and we multiply the first row and column by a nonzero scalar $\lambda \neq 1$, then we also have to multiply the second row and column by λ^{-1}. An *elementary* SCT is an SCT which is either an ECT or a product of two ECT's none of which is an SCT by itself.

The main idea of the algorithm is to find $P \in \mathrm{GL}_n(F)$ such that, when we replace A with PAP', the system (1.2) has an obvious solution Y. Then Lemma 2.1 provides a solution X for the original system.

We distinguish four cases:

(a) $\det(A_1) = 0$ and the characteristic is not 2.

(b) $\det(A_1) \neq 0$, $\det(A_0) = 0$ and the characteristic is not 2.

(c) $\det(A_1) = 0$ and the characteristic is 2.

(d) $\det(A_0A_1) \neq 0$.

Each of these cases will be treated separately. We set $V = F^n$, considered as the space of column vectors, and we shall use its standard basis $\{e_1, e_2, \dots, e_n\}$.

4.1. Algorithm for case (a). The characteristic of F is not 2, $2m < n$, and we set $k = n - 2m$.

In this case, our recursive algorithm will construct an involutory matrix $Y \in \mathrm{GL}_n(F)$ and a sequence of ECT's with the following properties: After transforming A with this sequence of ECT's, Y and the new A satisfy the following conditions:

(i) A_1 is normalized, i.e., $A_1 = J_m \oplus 0$.

(ii) $YAY' = A'$.

(iii) All entries of the last k rows and columns of Y are 0 except the diagonal entries (which are ± 1).

We remark that if $A = B \oplus C$, where B and C are square matrices of smaller size, and if our algorithm works for B and C, then it also works for A.

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 10, pp. 320-340, December 2003

ELA
http://math.technion.ac.il/iic/ela

If $m = 0$, we take $Y = I_n$. From now on we assume that $m \geq 1$. Let us partition the symmetric matrix A_0 into four blocks:

$$A_0 = \begin{bmatrix} B & C \\ C' & D \end{bmatrix},$$

where B is of size $2m$. Assume that $D \neq 0$. Then we may assume that its last diagonal entry is not 0. By elementary row operations and corresponding column operations, we can make all other entries in the last row and column of A_0 vanish. (These operations do not affect A_1.) Hence A splits.

From now on we assume that $D = 0$. If the rank of C is smaller than k, then we may assume that the last column of C is zero and so A splits. Thus we may assume that C has rank k. Our next goal is to simplify the block $C = [c_{ij}]$. Note that if $X \in G$ is block-diagonal:

$$X = \begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix}, \quad X_1 \in \mathrm{Sp}_{2m}(F),\ X_2 \in \mathrm{GL}_k(F),$$

then the effect of the SCT: $A \to XAX'$ on the block C is given by $C \to X_1 C X_2'$.

Let v_j denote the j-th column of A.

Assume that there exist p, q such that $2m < p, q \leq n$ and $v_p' A_1 v_q \neq 0$. By applying a suitable SCT (of the type mentioned above), we may further assume that $c_{2m-1,k-1} = c_{2m,k} = 1$ and all other entries of the last two rows and columns of C vanish. Next by subtracting multiples of the last two columns of A from the first $2m-2$ columns (via ECT's), we can assume that also the first $2m-2$ entries of the last two rows of B vanish. If $n > 4$, then A splits. Otherwise, $n = 4$, we can assume that $B = 0$ and then take $Y = \mathrm{diag}(1, -1, 1, -1)$.

It remains to consider the case where $v_p' A_1 v_q = 0$ for all $p, q > 2m$, i.e., the columns of C form a basis of a k-dimensional totally isotropic space (with respect to J_m). Since $\mathrm{Sp}_{2m}(F)$ acts transitively on such bases, without any loss of generality, we may assume that

$$c_{2m,k} = c_{2m-2,k-1} = \cdots = c_{2m-2k,1} = 1 \tag{4.1}$$

while all other entries of C are 0. Since the column-space of C is totally isotropic, we must have $k \leq m$.

By subtracting suitable multiples of the last k columns from the first $2m$ columns (via ECT's), we may assume that each of the rows $2m-2k+2, 2m-2k+4, \ldots, 2m$ of A_0 has a single nonzero entry. (The corresponding columns have the same property.)

In order to help the reader visualize the shape of the matrix A_0 at this point, we give an example. We take $m = 5$ and $k = 2$. Then A_0 has the form:

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 10, pp. 320-340, December 2003

ELA
http://math.technion.ac.il/iic/ela

$$A_0 = \left[\begin{array}{cccccccccc|cc}
 & & & & & & & 0 & & 0 & 0 & 0 \\
 & & & & & & & 0 & & 0 & 0 & 0 \\
 & & & & & & & 0 & & 0 & 0 & 0 \\
 & & & & & & & 0 & & 0 & 0 & 0 \\
 & & & & & & & 0 & & 0 & 0 & 0 \\
 & & & & & & & 0 & & 0 & 0 & 0 \\
 & & & & & & & 0 & & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
 & & & & & & & 0 & & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0
\end{array}\right],$$

where the blank entries have not been specified.

In order to give a simple formula for the matrix P (which provides the solution of our problem for the matrix A), it will be convenient to perform a congruence transformation which is not an SCT. For that purpose we just rearrange the rows of A so that the rows $2m-2k+1, 2m-2k+3, \ldots, 2m-1$ come before the rows $2m-2k+2, 2m-2k+4, \ldots, 2m$ (and similarly the columns). We continue (as in programming) to refer to this new matrix as the matrix A. Now A_1 is no longer normalized. The matrices A_0 and A_1 have the following form

$$A_0 = \begin{bmatrix} R_1 & R_2 & 0 & 0 \\ R_2' & R_3 & 0 & 0 \\ 0 & 0 & 0 & I_k \\ 0 & 0 & I_k & 0 \end{bmatrix}, \quad A_1 = \begin{bmatrix} J_{m-k} & 0 & 0 & 0 \\ 0 & 0 & I_k & 0 \\ 0 & -I_k & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

where all the blocks, except those in the first row and column, are square of size k.

We now introduce a truncated version of the problem, in which we replace A with its principal submatrix $\bar{A}$ obtained by deleting the last $2k$ rows and columns. Define $\bar{A}_0$ and $\bar{A}_1$ similarly. These truncated matrices have all size $n-2k(=2m-k)$. Note that $\bar{A}_1$ is already normalized and has rank $2(m-k)$. Hence, by using recursion, our algorithm can compute a matrix $\bar{P} \in \mathrm{GL}_{n-2k}(F)$ and an involutory matrix $\bar{Y}$ satisfying the conditions (i–iii). In particular,

$$\bar{Y}\bar{P}\bar{A}_0(\bar{P})'\bar{Y} = \bar{P}\bar{A}_0(\bar{P})', \quad \bar{P}\bar{A}_1(\bar{P})' = \bar{A}_1 \quad \text{and} \quad \bar{Y}\bar{A}_1\bar{Y} = -\bar{A}_1.$$

We now show that we can use $\bar{P}$ and $\bar{Y}$ to construct $P \in \mathrm{GL}_n(F)$ and an involutory matrix Y, such that

$$YPA_0P'Y = PA_0P', \quad PA_1P' = A_1 \quad \text{and} \quad YA_1Y = -A_1.$$

Let us partition $\bar{P}$ as follows:

$$\bar{P} = \begin{bmatrix} P_1 & P_2 \\ P_3 & P_4 \end{bmatrix},$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 10, pp. 320-340, December 2003

ELA
http://math.technion.ac.il/iic/ela

where P_4 is of size k. The matrix $\bar{A}_1$ has the form $J_{m-k} \oplus 0$. The equation $\bar{P}\bar{A}_1(\bar{P})' = \bar{A}_1$ implies that $P_1 J_{m-k} P_1' = J_{m-k}$ and $P_3 = 0$.

Our matrix P is now given by the following formula:

$$P = \begin{bmatrix} P_1 & P_2 & 0 & Q_1 \\ 0 & P_4 & 0 & Q_2 \\ P_5 & P_6 & (P_4')^{-1} & Q_3 \\ 0 & 0 & 0 & P_4 \end{bmatrix},$$

where

$$\begin{aligned}
P_5 &= -(P_1^{-1} P_2 P_4^{-1})' J_{m-k}, \\
P_6 &= -\frac{1}{2}(P_4')^{-1} P_2' J_{m-k} P_2, \\
Q_1 &= -(P_1 R_1 + P_2 R_2') P_5' P_4 - (P_1 R_2 + P_2 R_3) P_6' P_4, \\
Q_2 &= -P_4 (R_2' P_5' + R_3 P_6') P_4, \\
Q_3 &= -\frac{1}{2}(P_5 R_1 P_5' + P_5 R_2 P_6' + P_6 R_2' P_5' + P_6 R_3 P_6') P_4.
\end{aligned}$$

Clearly P is invertible. It is easy to verify that $PA_1P' = A_1$ and that PA_0P' has the same shape as A_0 except that the blocks R_1, R_2 and R_3 may be different from those in A_0. Recall that $\bar{P}\bar{A}_1(\bar{P})' = \bar{A}_1$ and that $\bar{Y} = \Delta \oplus \Lambda$, where Λ is a diagonal matrix of size k. Moreover, $\bar{Y}$ commutes with $\bar{P}\bar{A}_0(\bar{P})'$ and anti-commutes with $\bar{A}_1$. Set $Y = \bar{Y} \oplus (-\Lambda) \oplus (-\Lambda)$. It is easy to verify that Y commutes with PA_0P' and anti-commutes with A_1. Consequently, the conditions (ii) and (iii) are satisfied. By transforming A with a suitable permutation matrix, we may also satisfy the condition (i).

This completes the treatment of case (a).

4.2. Algorithm for case (b). We recall that the characteristic is not 2, $n = 2m$, $A_1 = J_m$, and A_0 is singular. We define here the symplectic group, $\mathrm{Sp}_n(F)$, by using definition (1.3) with $J = J_m$. Let N be the nullspace of A_0, i.e., $N = \{v \in V : A_0 v = 0\}$ and let d be its dimension. Since $\det(A_0) = 0$, we have $d > 0$.

In this case, our recursive algorithm will construct an involutory matrix $Y \in \mathrm{GL}_n(F)$ and a sequence of ECT's with the following properties: After transforming A with this sequence of ECT's, Y and the new A satisfy the following conditions:

(i) A_1 is normalized, i.e., $A_1 = J_m$.
(ii) $YAY' = A'$.
(iii) Exactly d rows and d columns of A_0 are 0, and the corresponding rows and columns of Y have all entries 0 except the diagonal entries (which are ± 1).

Again we remark that if $A = B \oplus C$, where B and C are square matrices of smaller size, and if our algorithm works for B and C, then it also works for A.

Assume that there exist $v, w \in N$ such that $v' A_1 w = 1$. Then it is easy to construct a matrix $P \in \mathrm{Sp}_n(F)$ having v and w as its first two columns. Hence the first two columns (and rows) of $P' A_0 P$ are zero. If $m = 1$, then $Y = \mathrm{diag}(1, -1)$ works, otherwise $P'AP$ splits. Thus we may assume that $v' A_1 w = 0$ for all vectors

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 10, pp. 320-340, December 2003

ELA
http://math.technion.ac.il/iic/ela

$v, w \in N$. Since $\det(A_1) \neq 0$, we deduce that $d \leq m$. Then we can construct $P \in \mathrm{Sp}_n(F)$ such that its columns in positions $n, n-2, \ldots, n-2d+2$ form a basis of N. We replace A with $P'AP$.

If $d = m$, then $Y = \mathrm{diag}(1, -1, \ldots, 1, -1)$ satisfies (ii) and (iii) and we are done.

Now assume that $d < m$. Recall that $\{e_n, e_{n-2}, \ldots, e_{n-2d+2}\}$ is a basis of N. We set $\bar{m} = m - d$ and define $\bar{A}_0$ to be the submatrix of A_0 of size $\bar{n} = 2\bar{m}$ in the upper left hand corner. We denote by $\bar{N}$ the nullspace of $\bar{A}_0$ and by $\bar{d}$ its dimension.

Assume that $\bar{d} = 0$, i.e., $\bar{A}_0$ is nonsingular. Then, by applying a suitable sequence of elementary SCT's, we may assume that the $n - \bar{n}$-by-$\bar{n}$ submatrix of A_0 just below the submatrix $\bar{A}_0$ is zero. This means that A splits.

Now assume that $\bar{d} > 0$. By using recursion, we may assume that we already have an involutory matrix $\bar{Y} \in \mathrm{GL}_{\bar{n}}(F)$ and that $\bar{Y}$ and $\bar{A}$ satisfy the conditions (i-iii) above.

For convenience, we partition the set of the first $\bar{n}$ rows (and similarly columns) of A_0 in two parts: We say that one of these rows or columns is of the *first kind* if it contains a nonzero entry of the submatrix $\bar{A}_0$ and otherwise it is of the *second kind*. The sequence of elementary SCT's that we are going to construct has the additional property that it will not alter the submatrix $\bar{A}_0$.

Denote by B the $\bar{d}$-by-d submatrix of A_0 in the intersection of the rows of the second kind and the columns in positions $n-1, n-3, \ldots, n-2d+1$. Since d is the dimension of N, B must have rank $\bar{d}$. By using elementary SCT's which act only on the last $2d$ columns (and rows), we can modify B without spoiling the zero entries of A_0 which were established previously and assume that $B = (I_{\bar{d}}, 0)$, i.e., B consists of the identity matrix of size $\bar{d}$ followed by $d - \bar{d}$ zero columns.

Let us illustrate the shape of the matrix A_0 at this stage by an example where $n = 2m = 18$, $d = 5$, and $\bar{d} = 4$. We point out that the submatrix made up of the starred entries is nonsingular. Hence a row or column of A_0 is of the first kind if and only if it contains a star entry. The submatrix $\bar{A}_0$ is the block of size 8 in the upper left hand corner.

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 10, pp. 320-340, December 2003

ELA
http://math.technion.ac.il/iic/ela

$$A_0 = \left[\begin{array}{cccccccc|cccccccccc}
\star & \star & \star & 0 & \star & 0 & 0 & 0 & & 0 & & 0 & & 0 & & 0 & & 0 \\
\star & \star & \star & 0 & \star & 0 & 0 & 0 & & 0 & & 0 & & 0 & & 0 & & 0 \\
\star & \star & \star & 0 & \star & 0 & 0 & 0 & & 0 & & 0 & & 0 & & 0 & & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\star & \star & \star & 0 & \star & 0 & 0 & 0 & & 0 & & 0 & & 0 & & 0 & & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
\hline
 & & & 1 & & 0 & 0 & 0 & & 0 & & 0 & & 0 & & 0 & & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & 0 & & 1 & 0 & 0 & & 0 & & 0 & & 0 & & 0 & & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & 0 & & 0 & 1 & 0 & & 0 & & 0 & & 0 & & 0 & & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & 0 & & 0 & 0 & 1 & & 0 & & 0 & & 0 & & 0 & & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & 0 & & 0 & 0 & 0 & & 0 & & 0 & & 0 & & 0 & & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right].$$

By subtracting suitable multiples of the columns of A_0 of the first kind (using elementary SCT's) from the columns in positions $n-1, n-3, \ldots, n-2d+1$, we may assume that all entries of A_0 in the intersection of the latter columns and the rows of the first kind are zero. In the above example this means that all blank entries in the first eight rows (and columns) are being converted to zero.

We can now use elementary SCT's to convert to zero all entries in the $2d$-by-$2d$ submatrix of A_0 in the lower right hand corner, except those in the $2(d-\bar{d})$-by-$2(d-\bar{d})$ submatrix in the same corner. Similarly, if $d > \bar{d}$, we can diagonalize the (nonsingular) square submatrix of size $d - \bar{d}$ in the intersection of rows and columns in positions $n-1, n-3, \ldots, n-2(d-\bar{d})+1$.

We extend $\bar{Y}$ to Y as follows. Let $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{\bar{d}}$ be the entries in the diagonal of $\bar{Y}$ occurring in the rows of A_0 of the second kind. We set the diagonal entries of Y in positions $\bar{n}{+}1, \bar{n}{+}3, \ldots, \bar{n}{+}2\bar{d}{-}1$ to be $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{\bar{d}}$. Furthermore, we set the diagonal entries of Y in positions $\bar{n}+2, \bar{n}+4, \ldots, \bar{n}+2\bar{d}$ to be equal to $-\varepsilon_1, -\varepsilon_2, \ldots, -\varepsilon_{\bar{d}}$. Finally, the last $2(d-\bar{d})$ diagonal entries of Y are set to be $1, -1, \ldots, 1, -1$. One verifies that Y, A_0 and A_1 satisfy the conditions (i-iii) above.

This completes the treatment of case (b).

4.3. Algorithm for case (c). In this case, the characteristic of F is 2, $2m < n$, and we set $k = n - 2m$. We recall that $A_1 = J_m \oplus 0$. Let us partition A into four blocks:

$$A = \left[\begin{array}{cc} B & C \\ C' & D \end{array}\right],$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 10, pp. 320-340, December 2003

ELA
http://math.technion.ac.il/iic/ela

where $B + B' = J_m$ and $D' = D$ is of size k. In this case our recursive algorithm will produce a solution X of (1.2) of the form

$$X = \begin{bmatrix} X_1 & X_2 \\ 0 & I_k \end{bmatrix}.$$

Assume that D is non-alternate. Then we can assume that its last entry $a_{nn} \neq 0$. By adding suitable multiples of the last row of A to other rows (via ECT's), we may assume that a_{nn} is the sole nonzero entry in the last row and column. If $n = 1$, i.e., $m = 0$ and $k = 1$, then we can take $X = I_1$. Otherwise A splits and we can use recursion.

Next assume that D is alternate and nonzero. Then we may assume that it is the direct sum of a symmetric matrix of size $k - 2$ and the block J_1. We can now proceed in the same way as above to convert to 0 the last two columns of C. If $m = 0$ and $k = 2$, then $n = 2$ and we can take $X = I_2$. Otherwise A splits and we can use recursion.

Hence, we may now assume that $D = 0$. We can also split A if the rank of C is less than k. Thus we may assume that C has rank k.

Assume that the k-dimensional space spanned by the columns of C is not totally isotropic (with respect to J_m). If $n > 4$, we can split A as in subsection 4.1. Otherwise $n = 4$ and we may assume that $C = I_2$. Then

$$X = \begin{bmatrix} I_2 & B \\ 0 & I_2 \end{bmatrix}$$

is a solution of (1.2) and has the desired form. Hence we may now assume that the above space is totally isotropic. Consequently, $k \leq m$. As in the previous section, we may also assume that (4.1) holds and all other entries of C are 0.

By adding suitable multiples of the last k columns to the first $2m - 2k$ columns (via ECT's), we may assume that the first $2m-2k$ entries of the rows $2m-2k+2, 2m-2k+4, \ldots, 2m$ of B are 0. The corresponding columns have the same property. By using the same argument, we can also assume that all entries in the intersection of the rows $2m-2k+2, 2m-2k+4, \ldots, 2m$ and columns $2m-2k+1, 2m-2k+3, \ldots, 2m-1$ of B are 0. As $A + A' = J$ remains valid, all entries in the intersection of the rows $2m-2k+1, 2m-2k+3, \ldots, 2m-1$ and columns $2m-2k+2, 2m-2k+4, \ldots, 2m$ of B are 0, except that the entries just above the diagonal are equal to 1. This completes the first subroutine of the algorithm.

In order to help the reader visualize the shape of the matrix A at this point, we give an example. We take $m = 6$ and $k = 3$. Then A has the form:

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 10, pp. 320-340, December 2003

$$
A = \left[\begin{array}{cccccc|cccccc|ccc}
 & & & & & & & 0 & & 0 & & 0 & 0 & 0 & 0 \\
 & & & & & & & 0 & & 0 & & 0 & 0 & 0 & 0 \\
 & & & & & & & 0 & & 0 & & 0 & 0 & 0 & 0 \\
 & & & & & & & 0 & & 0 & & 0 & 0 & 0 & 0 \\
 & & & & & & & 0 & & 0 & & 0 & 0 & 0 & 0 \\
 & & & & & & & 0 & & 0 & & 0 & 0 & 0 & 0 \\
\hline
 & & & & & & \bullet & 1 & \bullet & 0 & \bullet & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \star & 0 & \star & 0 & \star & 1 & 0 & 0 \\
 & & & & & & \bullet & 0 & \bullet & 1 & \bullet & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \star & 0 & \star & 0 & \star & 0 & 1 & 0 \\
 & & & & & & \bullet & 0 & \bullet & 0 & \bullet & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \star & 0 & \star & 0 & \star & 0 & 0 & 1 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0
\end{array}\right],
$$

where the blank, bullet, and star entries remain unspecified.

In order to give a simple formula for the matrix X, it will be convenient to perform a congruence transformation which is not an SCT. For that purpose we just rearrange the rows of A so that the rows $2m-2k+1, 2m-2k+3, \ldots, 2m-1$ come before the rows $2m-2k+2, 2m-2k+4, \ldots, 2m$ (and similarly the columns). Now A_1 is no longer normalized. The matrices A and A_1 have the following form

$$
A = \begin{bmatrix} A_{11} & A_{12} & 0 & 0 \\ A'_{12} & A_{22} & I_k & 0 \\ 0 & 0 & A_{33} & I_k \\ 0 & 0 & I_k & 0 \end{bmatrix}, \quad A_1 = \begin{bmatrix} J_{m-k} & 0 & 0 & 0 \\ 0 & 0 & I_k & 0 \\ 0 & I_k & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}
$$

where all the blocks, except those in the first row and column, are square of size k. As $A + A' = A_1$, the matrices A_{22} and A_{33} are symmetric and $A_{11} + A'_{11} = J_{m-k}$.

Let us illustrate these modifications in the example given above. Then the new matrix A has the following shape:

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 10, pp. 320-340, December 2003

$$A = \left[\begin{array}{cccccc|ccc|ccc|ccc}
 & & & & & & & & & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & & & & & & & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & & & & & & & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & & & & & & & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & & & & & & & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & & & & & & & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline
 & & & & & & \bullet & \bullet & \bullet & 1 & 0 & 0 & 0 & 0 & 0 \\
 & & & & & & \bullet & \bullet & \bullet & 0 & 1 & 0 & 0 & 0 & 0 \\
 & & & & & & \bullet & \bullet & \bullet & 0 & 0 & 1 & 0 & 0 & 0 \\ \hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \star & \star & \star & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \star & \star & \star & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \star & \star & \star & 0 & 0 & 1 \\ \hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0
\end{array}\right],$$

where the bullet (resp. star) entries are those of A_{22} (resp., A_{33}).

Assume that A_{22} is non-alternate. By performing congruence transformation on A with a suitable block-diagonal matrix $I_{2(m-k)} \oplus Z \oplus (Z')^{-1} \oplus Z$, we can assume that A_{22} is a diagonal matrix (see [8]) and that its last diagonal entry is nonzero. By using elementary SCT's, we can assume that the last column of A_{12} is 0. As a side-effect of these elementary SCT's, the zero blocks just below A'_{12} and A_{22} may be spoiled (and the block A_{33} may be altered). This damage can be easily repaired by using elementary SCT's which add multiples of the last k columns to the first $2m-k$ columns. By adding suitable multiples of the last k columns (and rows) we may assume that the symmetric matrix A_{33} is diagonal. If $n = 3$, i.e., $m = k = 1$, then we can take

$$X = \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right].$$

Otherwise A splits (with one of the blocks of size 3).

Next assume that A_{22} is alternate and nonzero. Then we may assume that it is the direct sum of a symmetric matrix of size $k-2$ and the block J_1. We can now proceed in the same way as above to convert to 0 the last two columns of A_{12} and to diagonalize A_{33}. If $n = 6$, i.e., $m = k = 2$, then we can take

$$X = \left[\begin{array}{cc|c} I_2 & 0 & I_2 \\ 0 & I_2 & 0 \\ \hline 0 & 0 & I_2 \end{array}\right].$$

Otherwise A splits and we can use recursion.

Hence, we may now assume that $A_{22} = 0$.

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 10, pp. 320-340, December 2003

ELA
http://math.technion.ac.il/iic/ela

We now introduce a truncated version of the problem, in which we replace A with its principal submatrix $\bar{A}$ obtained by deleting the last $2k$ rows and columns. The truncated matrix, $\bar{A}$, is of size $n-2k(=2m-k)$. Let $\bar{A}_1$ be the corresponding submatrix of A_1, i.e., $\bar{A}_1=\bar{A}+(\bar{A})'=J_{m-k}\oplus 0$.

By using recursion, our algorithm can compute a matrix $\bar{X}\in \mathrm{GL}_{n-2k}(F)$ of the form

$$\bar{X}=\begin{bmatrix} X_1 & X_2 \\ 0 & I_k \end{bmatrix}$$

such that $(\bar{X})^2=I_{n-2k}$ and $\bar{X}\bar{A}(\bar{X})'=(\bar{A})'$. The last condition is equivalent to $\bar{X}\bar{A}$ being a symmetric matrix. In terms of the blocks of $\bar{A}$ and $\bar{X}$, we have

$$X_1^2=I_{2(m-k)},\ X_1X_2=X_2,\ X_1A_{12}=A_{12},\ X_1A_{11}+X_2A_{12}'\in \mathrm{Sym}_{2(m-k)}(F).$$

We now use $\bar{X}$ to construct the desired $X\in \mathrm{GL}_n(F)$. Our matrix X is given by the following formula:

$$X=\begin{bmatrix} X_1 & X_2 & 0 & X_3 \\ 0 & I_k & 0 & X_4 \\ X_5 & X_6 & I_k & A_{33} \\ 0 & 0 & 0 & I_k \end{bmatrix},$$

where

$$\begin{aligned} X_3 &= A_{11}J_{m-k}X_2+A_{12}X_2'J_{m-k}A_{11}J_{m-k}X_2,\\ X_4 &= I_k+A_{12}'J_{m-k}X_2,\\ X_5 &= X_2'J_{m-k},\\ X_6 &= X_2'J_{m-k}A_{11}J_{m-k}X_2. \end{aligned}$$

The matrix X_6 is symmetric. Indeed we have:

$$X_6'=X_2'J_{m-k}A_{11}'J_{m-k}X_2=X_2'J_{m-k}(A_{11}+J_{m-k})J_{m-k}X_2=X_6+X_2'J_{m-k}X_2.$$

Since $X_1X_2=X_2$, the column-space of X_2 is contained in the 1-eigenspace of X_1. On the other hand, we know from the proof of Theorem 1.6 that this eigenspace is a maximal totally isotropic subspace (with respect to J_{m-k}). Hence $X_2'J_{m-k}X_2=0$ and so $X_6'=X_6$. It is now straightforward to verify that XA is symmetric.

It remains to verify that $X^2=I_n$. Note that the column-space of $I_{2(m-k)}+X_1$ and also of A_{12} is contained in the 1-eigenspace of X_1. The same argument as above shows that $X_2'J_{m-k}(I_{2(m-k)}+X_1)=0$ and $A_{12}'J_{m-k}X_2=0$. The first of these equalities can be rewritten as $X_1'J_{m-k}X_2=J_{m-k}X_2$. By using these equalities, we

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 10, pp. 320-340, December 2003

ELA
http://math.technion.ac.il/iic/ela

find that

$$\begin{aligned}
X_5X_1 + X_5 &= X_2'J_{m-k}(I_{2(m-k)} + X_1) = 0,\\
X_5X_2 &= X_2'J_{m-k}X_2 = 0,\\
X_1X_3 + X_2X_4 + X_3 &= X_1A_{11}J_{m-k}X_2 + A_{11}J_{m-k}X_2 + X_2 + X_2A_{12}'J_{m-k}X_2\\
&= (A_{11}'X_1' + A_{12}X_2')J_{m-k}X_2 + A_{11}J_{m-k}X_2 + X_2\\
&= A_{11}'X_1'J_{m-k}X_2 + A_{11}J_{m-k}X_2 + X_2\\
&= A_{11}'J_{m-k}X_2 + A_{11}J_{m-k}X_2 + X_2 = 0,\\
X_5X_3 + X_6X_4 &= X_2'J_{m-k}A_{12}X_2'J_{m-k}A_{11}J_{m-k}X_2\\
&\quad +X_2'J_{m-k}A_{11}J_{m-k}X_2A_{12}'J_{m-k}X_2 = 0.
\end{aligned}$$

Thus $X^2 = I_n$ as claimed.

This completes the treatment of case (c).

4.4. Algorithm for case (d). In this case there is no restriction on the characteristic of F, $n = 2m$, both matrices A_0 and A_1 are nonsingular, and we have $A_1 = J_m$. In view of Proposition 3.1, we may also assume that $\det(A) \neq 0$. Thus all three bilinear forms:

$$\begin{aligned}
\langle x, y\rangle &= x'Ay, \quad x, y \in V,\\
\langle x, y\rangle_0 &= x'A_0y, \quad x, y \in V,\\
\langle x, y\rangle_1 &= x'A_1y, \quad x, y \in V
\end{aligned}$$

are nondegenerate. The second form is symmetric and the third is alternate. If W is a subspace of V, there are two kinds of orthogonal complements with respect to the first form: The *right orthogonal complement*

$$W^\perp = \{y \in V : \langle x, y\rangle = 0, \forall x \in W\},$$

and the *left orthogonal complement*

$$^\perp W = \{y \in V : \langle y, x\rangle = 0, \forall x \in W\}.$$

Set $S = J_mA$, an invertible matrix which we also consider as a linear operator on V. We make V into a module over the polynomial algebra $F[t]$ by letting t act as the multiplication by S. We write $f \cdot x$ for the action of $f \in F[t]$ on $x \in V$. We denote by * the involutorial automorphism of the F-algebra $F[t]$ which sends t to $-1 - t$. Thus if $f \in F[t]$, then $f^* \in F[t]$ is defined by $f^*(t) = f(-1 - t)$. One can check without difficulty that $f^* = f$ holds if and only if $f(t) = g(t + t^2)$ for some $g \in F[t]$. If the characteristic of F is not 2, then $f^* = -f$ holds if and only if $f(t) = (2t + 1)g(t + t^2)$ for some $g \in F[t]$ and, consequently, if $f \neq 0$, then $f^* = -f$ implies that f has odd degree.

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 10, pp. 320-340, December 2003

ELA
http://math.technion.ac.il/iic/ela

It is straightforward to verify that for all $x, y \in V$ we have

$$\begin{aligned}
\langle x, t \cdot y\rangle &= -\langle (1+t) \cdot x, y\rangle,\\
\langle x, t \cdot y\rangle_1 &= -\langle (1+t) \cdot x, y\rangle_1,\\
\langle x, t \cdot y\rangle_1 &= -\langle x, y\rangle,\\
\langle x, y\rangle + \langle y, x\rangle &= \langle x, y\rangle_0,\\
\langle x, y\rangle - \langle y, x\rangle &= \langle x, y\rangle_1.
\end{aligned}$$

The first two of these identities imply that

$$\begin{aligned}
\langle x, f \cdot y\rangle &= \langle f^* \cdot x, y\rangle, \quad \forall x, y \in V, \quad \forall f \in F[t];\\
\langle x, f \cdot y\rangle_1 &= \langle f^* \cdot x, y\rangle_1, \quad \forall x, y \in V, \quad \forall f \in F[t].
\end{aligned}$$

The second and third imply that if a subspace W of V is S-invariant (i.e., an $F[t]$-submodule), then

$$^{\perp}W = W^{\perp} = W^{\perp_1},$$

and we shall denote this subspace simply by $W^{\perp}$.

Let f denote the minimal polynomial of S and let $f = f_1 f_2 \cdots f_k$ be its square-free factorization. Since $J_m S' = -J_m A' J_m = -J_m A J_m - J_m = -(I_n + S)J_m$ and S' is similar to S, we conclude that S and $-I_n - S$ are similar matrices. This implies that $f^* = \pm f$, and so each $f_i^* = \pm f_i$. If the characteristic of F is not 2, since $2S + I_n = J_m A_0$ and A_0 is nonsingular, we conclude that $2t + 1$ does not divide f. Hence $f^* = f$ in all cases, and also $f_i^* = f_i$ for all i's.

Let $f_1 = gh$ where $g^* = g$ and $h^* = h$ are monic polynomials. We set

$$V_g = \bigcup_{i \geq 0} \ker(g(S)^i)$$

and define V_h similarly. Then $V = V_g \oplus V_h$ and $V_g^{\perp} = V_h$. Consequently, if $g^* = g$ is a monic divisor of f_1, then

$$V = V_g \oplus V_g^{\perp}. \tag{4.2}$$

Assume that $f_i \neq f_1$ for some i. Then $g = f_i$ and $h = f_1/f_i = h^*$ are relatively prime monic nonconstant polynomials. Hence (4.2) is valid. Clearly, we can construct the subspace V_g and so we obtain a splitting of A.

From now on we assume that $f_i = f_1$ for all $i = 2, \ldots, k$, i.e., $f = f_1^k$. For $0 \leq i \leq k$, we denote by V_i the kernel of the operator $f_1(S)^{k-i}$. In particular, $V_0 = V$ and $V_k = 0$. Set $\overline{V} = V/V_1$ and, for $x \in V$, $\bar{x} = x + V_1$. The bilinear form $V \times V \to F$ defined by $(x, y) \to \langle x, f_1^{k-1} \cdot y\rangle_0$ has V_1 as its (left and right) radical. It induces a nondegenerate bilinear form $\overline{V} \times \overline{V} \to F$ given by $(\bar{x}, \bar{y}) \to \langle x, f_1^{k-1} \cdot y\rangle$. This form is not skew-symmetric because

$$\langle x, f_1^{k-1} \cdot y\rangle + \langle y, f_1^{k-1} \cdot x\rangle = \langle x, f_1^{k-1} \cdot y\rangle + \langle f_1^{k-1} \cdot y, x\rangle = \langle x, f_1^{k-1} \cdot y\rangle_0$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 10, pp. 320-340, December 2003

ELA
http://math.technion.ac.il/iic/ela

and $\langle \cdot, \cdot \rangle_0$ is nondegenerate.

Therefore we can choose $v \in V$ such that $\bar{v}$ is a non-isotropic vector with respect to the above form, i.e., if $w = f_1^{k-1} \cdot v$, then $\langle v, w \rangle \neq 0$. Let $W = F[t] \cdot v$ be the cyclic submodule generated by v and set $U = W \cap W^{\perp}$. Of course, U is also cyclic, say $U = F[t] \cdot u$, where $u = g \cdot v$ and g is a monic divisor of f. Since $0 = \langle u, v \rangle = \langle g \cdot v, v \rangle = \langle v, g^* \cdot v \rangle$, we deduce that $g^* \cdot v \in U$, i.e., g divides g^*. It follows that $g^* = g$. Thus $f = gh$, with $h^* = h$ also monic. As $\langle v, w \rangle \neq 0$, we have $w \notin U$ which implies that f_1 does not divide h. Thus h_1, the square-free part of h, is a proper divisor of f_1 and $h_1^* = h_1$. If $h_1 \neq 1$, then by applying an argument used above we can construct a splitting of A. Otherwise $U = 0$, i.e., $V = W \oplus W^{\perp}$. If $W \neq V$, then again A splits. Hence we may assume that $V = W$.

The vectors $S^i v$ for $0 \leq i < n$ form a basis of V. Hence if W_0 is the subspace of V with basis $\{v_i = (t + t^2)^{i-1} \cdot v : 1 \leq i \leq m\}$ and $W_1 = t \cdot W_0$, then $V = W_0 \oplus W_1$. Since

$$\langle (t + t^2)^i \cdot v, (t + t^2)^j \cdot v \rangle_1 = (-1)^{i+j} \langle t^{i+j} \cdot v, t^{i+j} \cdot v \rangle_1 = 0,$$

we have $W_0^{\perp_1} = W_0$ and similarly $W_1^{\perp_1} = W_1$. Since

$$\begin{aligned} \langle t(t + t^2)^i \cdot v, (t + t^2)^j \cdot v \rangle &= (-1)^{i+j} \langle t^{i+j+1} \cdot v, t^{i+j} \cdot v \rangle \\ &= (-1)^{i+j} \langle t^{i+j+1} \cdot v, t^{i+j+1} \cdot v \rangle_1 = 0, \end{aligned}$$

we have also $W_1^{\perp} = W_0$. Hence there exists a unique ordered basis $\{w_1, \ldots, w_m\}$ of W_1 such that $\{v_1, w_1, \ldots, v_m, w_m\}$ is a symplectic basis of V. Denote by P the matrix whose columns are the vectors $v_1, v_2, \ldots, v_m, w_1, w_2, \ldots, w_m$ (in this order). Since $\langle w_j, v_i \rangle = 0$, we have $\langle v_i, w_j \rangle = \langle v_i, w_j \rangle_1$ for all i, j. We infer that

$$P'AP = \begin{bmatrix} B & I_m \\ 0 & C \end{bmatrix}$$

for some invertible symmetric matrices B and C. It remains to observe that

$$Y = \begin{bmatrix} -I_m & C^{-1} \\ 0 & I_m \end{bmatrix}$$

is involutory and satisfies $YP'APY' = P'A'P$.

This completes the description of our algorithm (and the proof of Theorem 1.4).

5. Two open problems. We denote by F an arbitrary field. The matrices will be denoted by lower case latin letters. We set $M_n = M_n(F)$ and $G = \mathrm{SL}_n(F)$. The congruence action of G on M_n is defined by $a \cdot x = axa'$ ($a \in G$, $x \in M_n$). It is one of the several most important group actions studied in Linear Algebra. (One usually considers the restriction of this action to symmetric or skew-symmetric matrices.)

Let $F[M_n]$ be the algebra of polynomial functions on M_n. The congruence action of G on M_n induces an action of G on $F[M_n]$ by

$$(a \cdot f)(x) = f(a^{-1} \cdot x) = f(a^{-1}x(a')^{-1}),$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 10, pp. 320-340, December 2003

ELA
http://math.technion.ac.il/iic/ela

where $a \in G$, $f \in F[M_n]$, and $x \in M_n$.

When $F = \mathbf{C}$, Adamovich and Golovina [1] have computed the subalgebra, $\mathbf{C}[M_n]^G$, of G-invariants in $\mathbf{C}[M_n]$. More precisely, they have shown that this subalgebra is isomorphic to a polynomial algebra over $\mathbf{C}$ in $[(n+1)/2]$ explicitly given algebraically independent generators.

One can ask many interesting questions about the congruence action of G on M_n but we shall state only two of them.

Problem 1 *Extend the results of Adamovich and Golovina to algebraically closed fields of prime characteristic (or to arbitrary fields).*

Problem 2 *If F is algebraically closed and $x \in M_n$, determine which G-orbits are contained in the Zariski closure of the orbit $G \cdot x$.*

In connection with Proposition 3.1 we would like to make the following remark. This proposition shows that an arbitrary $A \in M_n(F)$ is congruent to the direct sum of various niloptent Jordan blocks N_r and a nonsingular matrix X (where either of these two types may be missing). By applying the well known theory of matrix pencils $\lambda A + \mu B$ to the special pencils $\lambda A + \mu A'$, it is easy to see that the nilpotent Jordan blocks N_r are indecomposable under congruence and that the sizes of Jordan blocks N_r and their multiplicities in the above decomposition are uniquely determined by A. As shown by Gabriel [6], X is unique up to congruence. In the statement of Gabriel's theorem as given by Waterhouse [11], instead of the Jordan blocks N_r, more complicated singular indecomposable blocks B_r are used. Since for each size $r \geq 1$ there is only one (up to congruence) singular indecomposable matrix of size r, the matrices B_r and N_r must be congruent. This can be also shown directly by constructing a permutation matrix P such that $PB_rP' = N_r$.

Acknowledgement. An anonymous referee of an earlier mini-version of this paper is to be thanked for pointing out the references [7] and [11].

REFERENCES

[1] O.M. Adamovich and E.O. Golovina. Invariants of a pair of bilinear forms (in Russian). *Vestnik Moskovskogo Universiteta, Matematika*, 32:15–18, 1977.
[2] L.V. Antonyan. On the classification of homogeneous elements of $\mathbf{Z}_2$-graded semisimple Lie algebras (in Russian). *Vestnik Moskovskogo Universiteta, Seriya I Matematika, Mekhanika*, 29–34. 1982.
[3] C.S. Ballantine and E.L. Yip. Congruence and conjunctivity of matrices to their adjoints. *Linear Algebra and its Applications*, 41: 33–72, 1981.
[4] J.H. Davenport. *On the Integration of Algebraic Functions.* Lecture Notes in Computer Science 102. Springer–Verlag, Berlin, 1981.
[5] D.Ž. Đoković and Kh. Ikramov. A square matrix is congruent to its transpose. *Journal of Algebra* 257:97–105, 2002.
[6] P. Gabriel. Appendix: Degenerate bilinear forms. *Journal of Algebra*, 31:67–72, 1974.
[7] R. Gow. The equivalence of an invertible matrix to its transpose. *Linear and Multilinear Algebra*, 8:329–336, 1980.
[8] I. Kaplansky. *Linear Algebra and Geometry, A second course.* Allyn and Bacon, Boston, 1969.
[9] C. Riehm. The equivalence of bilinear forms. *Journal of Algebra*, 31:45–66, 1974.
[10] R.C. Thompson. Pencils of complex and real symmetric and skew matrices. *Linear Algebra and its Applications*, 147:323–371, 1991.
[11] W.C. Waterhouse. The number of congruence classes in $M_n(\mathbf{F}_q)$. *Finite Fields and their Applications*, 1:57–63, 1995.