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SPECTRAL VERSUS CLASSICAL NEVANLINNA-PICK INTERPOLATION IN DIMENSION TWO*

HARI BERCOVICI†

Abstract. A genericity condition is removed from a result of Agler and Young which reduces the spectral Nevanlinna-Pick problem in two dimensions to a family of classical Nevanlinna-Pick problems. Unlike the original approach, the argument presented here does not involve state-space methods.

Key words. Interpolation, Spectral radius, Analytic function, Similarity.

AMS subject classifications. 47A57, 15A18, 30E05, 47A48.

1. Introduction and Preliminaries. Consider points $\lambda_1, \lambda_2, \dots, \lambda_n$ in the unit disk $\mathbb{D}$ of the complex plane, and matrices $W_1, W_2, \dots, W_n \in M_N(\mathbb{C})$, where $M_N(\mathbb{C})$ denotes the C^* algebra of $N \times N$ complex matrices. The matricial Nevanlinna-Pick problem asks for equivalent conditions to the existence of an analytic function $F : \mathbb{D} \to M_N(\mathbb{C})$ which interpolates the data, i.e. $F(\lambda_j) = W_j$ for $j = 1, 2, \dots, n$, with $\|F(\lambda)\| \le 1$ for $\lambda \in \mathbb{D}$. An elegant answer to this problem was given by G. Pick (for the case $N = 1$; the extension to $N > 1$ was noted later — we refer to [4] for an account of classical interpolation theory from a modern viewpoint). Pick's condition is simply that the block matrix $[(I - W_i^* W_j)/(1 - \overline{\lambda_i}\lambda_j)]_{i,j=1}^n$ be nonnegative semidefinite:

$$\left[\frac{I - W_i^* W_j}{1 - \overline{\lambda_i}\lambda_j}\right]_{i,j=1}^n \ge 0.$$

The spectral version of the Nevanlinna-Pick problem asks for equivalent conditions to the existence of a bounded analytic function $F : \mathbb{D} \to M_N(\mathbb{C})$ which interpolates the data, with spectral radius of $F(\lambda)$ bounded by one, i.e. $|F(\lambda)|_{\mathrm{sp}} \le 1$ for $\lambda \in \mathbb{D}$. A result analogous to Pick's theorem was proved in [3], and it involves the positivity of a matrix constructed from data W_j' similar to W_j, i.e. $W_j' = X_j W_j X_j^{-1}$ for invertible operators $X_j \in M_N(\mathbb{C})$. In other words, this solution requires a search involving $N^2 n$ parameters. The case $N = 2$ of the spectral Nevanlinna-Pick problem has been studied quite thoroughly by J. Agler and N. J. Young; see for instance [1], [2], and the references quoted therein. They related this problem with questions of complex analysis in two variables, dilation theory, and with state-space methods in control theory. In particular, [2] contains a result which reduces the search required for a solution from $4n$ to $2n$ parameters. Their result requires a genericity condition: none of the W_j can be a scalar multiple of the identity matrix.

The purpose of this note is to remove the genericity condition in the main result of [2], and to provide a simplified proof. An important part of the proof we present

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†Department of Mathematics, Indiana University, Bloomington, Indiana 47405, USA (bercovic@indiana.edu). Supported in part by a grant from the National Science Foundation.

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is already contained in [2], and it is based on an idea due to Petrović. This idea was also introduced in relation with the spectral Nevanlinna-Pick problem (cf. [6]). The result is as follows. We will denote by tr and det the usual trace and determinant functions defined on $M_N(\mathbb{C})$.

THEOREM 1.1. *Given a natural number n, points $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{D}$, and matrices $W_1, W_2, \ldots, W_n \in M_2(\mathbb{C})$, the following conditions are equivalent.*

1. *There exists an analytic function $F : \mathbb{D} \to M_2(\mathbb{C})$ such that $F(\lambda_j) = W_j$, $j = 1, 2, \ldots, n$, and $|F(\lambda)|_{sp} \leq 1$ for $\lambda \in \mathbb{D}$.*
2. *There exists a bounded analytic function satisfying the conditions in* (1).
3. *There exist numbers $b_1, b_2, \ldots, b_n, c_1, c_2, \ldots, c_n \in \mathbb{C}$ such that $b_j = c_j = 0$ when W_j is a scalar multiple of the identity and, upon setting $a_j = \mathrm{tr}(W_j)/2$ and $W_j' = \begin{bmatrix} a_j & b_j \\ c_j & a_j \end{bmatrix}$, we have $\det(W_j') = \det(W_j)$, and*

$$\left[\frac{I - W_i'^* W_j'}{1 - \overline{\lambda_i}\lambda_j}\right]_{i,j=1}^n \geq 0.$$

The reader will notice that, as stated, this theorem does not extend the main result of [2]. Namely, that result reformulates the problem in terms of matrices with zero trace. The relationship becomes clear if we note that the matrix inequality in (3) is equivalent to

$$\left[\frac{I - W_i''^* W_j''}{1 - \overline{\lambda_i}\lambda_j}\right]_{i,j=1}^n \geq 0,$$

where $W_j'' = \begin{bmatrix} a_j & b_j \\ -c_j & -a_j \end{bmatrix}$; in fact $W_j'' = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} W_j'$ so that $W_i''^* W_j'' = W_i'^* W_j'$.

Let us note at this point that the matrices W_j' in part (3) of the above statement are not necessarily similar to W_j. Indeed, let $W, W' \in M_2(\mathbb{C})$ be such that $\mathrm{tr}(W) = \mathrm{tr}(W')$, $\det(W) = \det(W')$, and W' is of the form $W' = \begin{bmatrix} a & b \\ c & a \end{bmatrix}$. Denote by μ_1, μ_2 the eigenvalues of W, which are aslo the eigenvalues of W'. If $\mu_1 \neq \mu_2$ then clearly W and W' are similar. However, if $\mu_1 = \mu_2$ then

$$0 = (\mu_1 - \mu_2)^2 = (\mu_1 + \mu_2)^2 - 4\mu_1\mu_2 = (\mathrm{tr}(W))^2 - 4\det(W) = 4bc.$$

Thus either b or c must be zero. If both are zero then W' is a constant multiple of the identity matrix, while if only one of them is zero, W' is a single Jordan cell.

2. Classical Vs. Spectral Interpolation. We start with a simple case of spectral interpolation which can be treated in arbitrary dimension N.

PROPOSITION 2.1. *Fix a natural number n, points $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{D}$, and matrices $W_1, W_2, \ldots, W_n \in M_N(\mathbb{C})$ such that each W_j has a unique eigenvalue ω_j. The following are equivalent.*

1. *There exists an analytic function $F : \mathbb{D} \to M_N(\mathbb{C})$ such that $F(\lambda_j) = W_j$, $j = 1, 2, \ldots, n$, and $|F(\lambda)|_{sp} \leq 1$ for $\lambda \in \mathbb{D}$.*

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2. *There exists a bounded analytic function satisfying the conditions in* (1).
3. $\left[\frac{1-\overline{\omega_i}\omega_j}{1-\overline{\lambda}_i\lambda_j}\right]_{i,j=1}^n \geq 0.$

Proof. As seen in [3] (see (8) in that paper), in the spectral Nevanlinna-Pick problem we can always replace the matrices W_j by similar matrices. We may, and shall, assume that each W_j is upper triangular, with diagonal entries ω_j. Assume that (3) is satisfied. By the classical Pick theorem, there exists an analytic function $u : \mathbb{D} \to \mathbb{C}$ such that $u(\lambda_j) = \omega_j$, $j = 1, 2, \ldots, n$, and $|u(\lambda)| \leq 1$ for $\lambda \in \mathbb{D}$. For $1 \leq k < \ell \leq N$ consider a polynomial $p_{k\ell}$ such that $p_{k\ell}(\lambda_j)$ is the (k, ℓ) entry of W_j; these polynomials can be constructed by Lagrange interpolation. Define now an upper triangular matrix $F(\lambda)$ with diagonal entries $u(\lambda)$, and entries $p_{k\ell}(\lambda)$ above the diagonal. Clearly F satisfies the conditions in (2) since $f(\lambda)$ is the unique eigenvalue of $F(\lambda)$. This proves the implication (3)⇒(2). The implication (2)⇒(1) is obvious, so it remains to prove that (1)⇒(3). Indeed, let F satisfy condition (1), and set $f(\lambda) = \mathrm{tr}(F(\lambda))/N$. We have then $|f(\lambda)| \leq 1$ ($f(\lambda)$ is the average of the eigenvalues of $F(\lambda)$), and $f(\lambda_j) = \omega_j$. Thus (3) follows from Pick's theorem. □

Observe that Theorem 1.1 follows from Proposition 2.1 in case the W_j have a single eigenvalue. Indeed, in this case one can choose $b_j = c_j = 0$ for all j in condition (3) of Theorem 1.1. When at least one of the W_j has distinct eigenvalues, we have a more precise result.

THEOREM 2.2. *Fix a natural number n, points $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{D}$, and matrices $W_1, W_2, \ldots, W_n \in M_2(\mathbb{C})$ such that at least one of the W_j has distinct eigenvalues. The following are equivalent.*

1. *There exists an analytic function $F : \mathbb{D} \to M_N(\mathbb{C})$ such that $F(\lambda_j) = W_j$, $j = 1, 2, \ldots, n$, and $|F(\lambda)|_{sp} \leq 1$ for $\lambda \in \mathbb{D}$.*
2. *There exists a bounded analytic function satisfying the conditions in* (1).
3. *There exists an analytic function $G : \mathbb{D} \to M_2(\mathbb{C})$ such that $G(\lambda_j)$ is similar to W_j, $j = 1, 2, \ldots, n$, and $\|G(\lambda)\| \leq 1$ for $\lambda \in \mathbb{D}$.*
4. *There exists an analytic function G satisfying the conditions in* (3) *such that* $G(\lambda) = \begin{bmatrix} a(\lambda) & b(\lambda) \\ c(\lambda) & a(\lambda) \end{bmatrix}$ *for* $\lambda \in \mathbb{D}$.
5. *There exist matrices W'_j similar to W_j, $j = 1, 2, \ldots, n$, such that*
$$\left[\frac{I - W'^*_i W'_j}{1 - \overline{\lambda}_i\lambda_j}\right]_{i,j=1}^n \geq 0.$$
6. *There exist complex numbers $b_1, b_2, \ldots, b_n, c_1, c_2, \ldots, c_n \in \mathbb{C}$ with the following properties:*
   (a) $b_jc_j = \frac{1}{4}\mathrm{tr}(W_j)^2 - \det(W_j)$;
   (b) *if W_j is a scalar multiple of the identity, then $b_j = c_j = 0$;*
   (c) *if $\frac{1}{4}\mathrm{tr}(W_j)^2 - \det(W_j) = 0$ but W_j is not a scalar multiple of the identity, then $b_j = 0 \neq c_j$; and*
   (d) *we have*
$$\left[\frac{I - W'^*_i W'_j}{1 - \overline{\lambda}_i\lambda_j}\right]_{i,j=1}^n \geq 0,$$

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where $W'_j = \begin{bmatrix} a_j & b_j \\ c_j & a_j \end{bmatrix}$, *with* $a_j = \frac{1}{2}\mathrm{tr}(W_j)$.

Proof. As noted earlier, a matrix W'_j satisfying the conditions in (6) is similar to W_j. Thus (4) $\Rightarrow$ (6) by Pick's theorem. Clearly (6) $\Rightarrow$ (5), and (5) $\Rightarrow$ (3) by Pick's theorem. The implications (3) $\Rightarrow$ (2) $\Rightarrow$ (1) are immediate, so it remains to prove that (1) $\Rightarrow$ (4). Let us assume therefore that F satisfies condition (1). The functions $a(\lambda) = \frac{1}{2}\mathrm{tr}(F(\lambda))$ and $d(\lambda) = \det(F(\lambda))$ are bounded by one in $\mathbb{D}$. We will find now analytic functions $b(\lambda)$ and $c(\lambda)$ so that $G(\lambda) = \begin{bmatrix} a(\lambda) & b(\lambda) \\ c(\lambda) & a(\lambda) \end{bmatrix}$ is similar to $F(\lambda)$ and $\|G(\lambda)\| \le 1$ for every $\lambda \in \mathbb{D}$. The similarity of $G(\lambda)$ to $F(\lambda)$ amounts to the following three conditions:

(i) $b(\lambda)c(\lambda) = a(\lambda)^2 - d(\lambda)$;
(ii) if $F(\lambda)$ is a scalar multiple of the identity then $b(\lambda) = c(\lambda) = 0$; and
(iii) if $a(\lambda)^2 - d(\lambda) = 0$ but $F(\lambda)$ is not a scalar multiple of the identity, then $b(\lambda) = 0 \neq c(\lambda)$.

For condition (ii) to be realizable, we must show that

$$a(\lambda)^2 - d(\lambda) = \frac{1}{4}[\mathrm{tr}(F(\lambda))]^2 - \det(F(\lambda))$$

has a double zero at λ_0 if $F(\lambda_0)$ is a scalar multiple of the identity. Indeed, if $F(\lambda) = \omega I + (\lambda - \lambda_0)F_1(\lambda)$, we have

$$\frac{1}{4}[\mathrm{tr}(F(\lambda))]^2 - \det(F(\lambda)) = (\lambda - \lambda_0)^2 \left[\frac{1}{4}[\mathrm{tr}(F_1(\lambda))]^2 - \det(F_1(\lambda))\right],$$

as desired. Observe also that $a(\lambda)^2 - d(\lambda)$ is not identically zero because at least one of the W_j has distinct eigenvalues. By classical factorization results (cf. Chapter 5 of [5]), there exist a Blaschke product B, and an analytic function G such that $a^2 - d = Be^G$. Functions b and c can now be defined by $b = B_1 e^{G/2}$, $c = B_2 e^{G/2}$, where B_1, B_2 are Blaschke products and $B_1 B_2 = B$. Conditions (i), (ii), and (iii) are realized by judicious choice of B_1 and B_2, and in addition we have

$$|b(\zeta)|^2 = |c(\zeta)|^2 = |a(\zeta)^2 - d(\zeta)|$$

for almost every ζ with $|\zeta| = 1$. It remains to prove that $\|G(\lambda)\| \le 1$ for $\lambda \in \mathbb{D}$, and for that it suffices to show that $\|G(\zeta)\| \le 1$ for almost every ζ, $|\zeta| = 1$. We know that $|G(\lambda)|_{\mathrm{sp}} \le 1$ for $\lambda \in \mathbb{D}$, and continuity of the spectral radius on $M_2(\mathbb{C})$ shows that $|G(\zeta)|_{\mathrm{sp}} \le 1$ almost everywhere. The proof is concluded by the observation that $G(\zeta)$ is a normal operator for almost every ζ, hence its norm equals the spectral radius. In fact, every matrix of the form $\begin{bmatrix} a & b \\ c & a \end{bmatrix}$ is normal when $|b| = |c|$ since it can be written as $aI + bU$, where $U = \begin{bmatrix} 0 & 1 \\ c/b & 0 \end{bmatrix}$ is a unitary operator (set $c/b = 1$ if $b = c = 0$). ∎

The above proof may fail if each W_j has a single eigenvalue, and in fact the result is not true in that case. An example is obtained for $n = 2$, $\lambda_1 = 0$, $\lambda_2 = \frac{1}{2}$,

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$W_1 = 0$, and $W_2 = \begin{bmatrix} 1/2 & 1/2 \\ 0 & 1/2 \end{bmatrix}$. The function $F(\lambda) = \begin{bmatrix} \lambda & \lambda \\ 0 & \lambda \end{bmatrix}$ satisifies condition (1) in the theorem. We claim that no function G satisfies (3). Assume indeed that $G(0) = 0$, $G(1/2)$ is similar to W_2, and $\|G(\lambda)\| \le 1$ for $\lambda \in \mathbb{D}$. We can then write $G(\lambda) = \lambda G_1(\lambda)$, and a comparison of boundary values will show that G_1 also has norm bounded by one. Now, $\det(G_1(1/2)) = 1$, and we deduce easily that $G_1(1/2)$ is in fact a unitary operator. In particular, $G(1/2)$ must be a normal operator, and hence not similar to W_2, a contradiction.

REFERENCES

[1] J. Agler and N.J. Young. The two-point spectral Nevanlinna-Pick problem. *Integral Equations Operator Theory*, 37:375–385, 2000.
[2] J. Agler and N.J. Young. The two-by-two spectral Nevanlinna-Pick problem. *Preprint*, 2001.
[3] H. Bercovici, C. Foias, and A. Tannenbaum. A spectral commutant lifting theorem. *Transactions of the Amer. Math. Soc.*, 325:7141–763, 1991.
[4] C. Foias and A. Frazho. *The commutant lifting approach to interpolation problems*. Birkhäuser, Basel, 1990.
[5] K. Hoffman. *Banach spaces of analytic functions*. Prentice Hall, Englewood Cliffs, New Jersey, 1962.
[6] S. Petrović. An extremal problem in interpolation theory. *Houston J. Math.*, 26:165–181, 2000.