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SUBDIRECT SUMS OF NONSINGULAR M-MATRICES AND OF THEIR INVERSES*

RAFAEL BRU†, FRANCISCO PEDROCHE†, AND DANIEL B. SZYLD‡

Abstract. The question of when the subdirect sum of two nonsingular M-matrices is a nonsingular M-matrix is studied. Sufficient conditions are given. The case of inverses of M-matrices is also studied. In particular, it is shown that the subdirect sum of overlapping principal submatrices of a nonsingular M-matrix is a nonsingular M-matrix. Some examples illustrating the conditions presented are also given.

AMS subject classifications. 15A48.

Key words. Subdirect sum, M-matrices, Inverse of M-matrix, Overlapping blocks.

1. Introduction. Subdirect sum of matrices are generalizations of the usual sum of matrices (a k-subdirect sum is formally defined below in Section 2). They were introduced by Fallat and Johnson in [3], where many of their properties were analyzed. For example, they showed that the subdirect sum of positive definite matrices, or of symmetric M-matrices, are positive definite or symmetric M-matrices, respectively. They also showed that this is not the case for M-matrices: the sum of two M-matrices may not be an M-matrix. One goal of the present paper is to give sufficient conditions so that the subdirect sum of nonsingular M-matrices is a nonsingular M-matrix. We also treat the case of the subdirect sum of inverses of M-matrices.

Subdirect sums of two overlapping principal submatrices of a nonsingular M-matrix appear naturally when analyzing additive Schwarz methods for Markov chains or other matrices [2], [4]. In this paper we show that the subdirect sum of two overlapping principal submatrices of a nonsingular M-matrix is a nonsingular M-matrix.

The paper is structured as follows. In Section 2 we focus on the nonsingularity of the subdirect sum of any pair of nonsingular matrices, giving an explicit expression for the inverse. In Section 2.1 we study the k-subdirect sum of two nonsingular M-matrices and in particular, the case of subdirect sums of overlapping blocks of nonsingular M-matrices. In Section 2.3 we extend some results to the subdirect sum of more than two nonsingular M-matrices. In Section 3 we analyze the subdirect sum of two inverses. Finally, in Section 4 we mention some open questions on subdirect sums of P-matrices. Throughout the paper we give examples which help illustrate the theoretical results.

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†Institut de Matemàtica Multidisciplinar, Universitat Politècnica de València. Camí de Vera s/n. 46022 València. Spain (rbru@mat.upv.es, pedroche@mat.upv.es). Supported by Spanish DGI and FEDER grant MTM2004-02998 and by the Oficina de Ciécia i Tecnologia de la Presidécia de La Generalitat Valenciana under project GRUPOS03/062.

‡Department of Mathematics, Temple University, Philadelphia, PA 19122-6094, U.S.A. (szyld@math.temple.edu). Supported in part by the U.S. National Science Foundation under grant DMS-0207525.

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2. Subdirect sums of nonsingular matrices. Let A and B be two square matrices of order n_1 and n_2, respectively, and let k be an integer such that $1 \leq k \leq \min(n_1, n_2)$. Let A and B be partitioned into 2×2 blocks as follows:

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}, \tag{2.1}$$

where A_{22} and B_{11} are square matrices of order k. Following [3], we call the following square matrix of order $n = n_1 + n_2 - k$,

$$C = \begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{21} & A_{22} + B_{11} & B_{12} \\ 0 & B_{21} & B_{22} \end{bmatrix} \tag{2.2}$$

the k-subdirect sum of A and B and denote it by $C = A \oplus_k B$.

We are interested in the case when A and B are nonsingular matrices. We partition the inverses of A and B conformably to (2.1) and denote its blocks as follows:

$$A^{-1} = \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} \\ \hat{A}_{21} & \hat{A}_{22} \end{bmatrix}, \quad B^{-1} = \begin{bmatrix} \hat{B}_{11} & \hat{B}_{12} \\ \hat{B}_{21} & \hat{B}_{22} \end{bmatrix}, \tag{2.3}$$

where, as before, $\hat{A}_{22}$ and $\hat{B}_{11}$ are square of order k.

In the following result we show that nonsingularity of matrix $\hat{A}_{22} + \hat{B}_{11}$ is a necessary and sufficient condition for the k-subdirect sum C to be nonsingular. The proof is based on the use of the relation $n = n_1 + n_2 - k$ to properly partition the indicated matrices.

THEOREM 2.1. *Let A and B be nonsingular matrices of order n_1 and n_2, respectively, and let k be an integer such that $1 \leq k \leq \min(n_1, n_2)$. Let A and B be partitioned as in* (2.1) *and their inverses be partitioned as in* (2.3). *Let $C = A \oplus_k B$. Then C is nonsingular if and only if $\hat{H} = \hat{A}_{22} + \hat{B}_{11}$ is nonsingular.*

Proof. Let I_m be the identity matrix of order m. The theorem follows from the following relation:

$$\begin{bmatrix} A^{-1} & O \\ O & I_{n-n_1} \end{bmatrix} C \begin{bmatrix} I_{n-n_2} & O \\ O & B^{-1} \end{bmatrix} = \begin{bmatrix} I_{n-n_2} & \hat{A}_{12} & O \\ O & \hat{H} & \hat{B}_{12} \\ O & O & I_{n-n_1} \end{bmatrix}. \quad \square \tag{2.4}$$

2.1. Nonsingular M-matrices. Given $A = \{a_{ij}\} \in \mathbb{R}^{m \times n}$, we write $A > O$ ($A \geq O$), to indicate $a_{ij} > 0$ ($a_{ij} \geq 0$), for $i = 1, \ldots, m$, $j = 1, \ldots, n$, and such matrices are called positive (nonnegative). Similarly, $A \geq B$ when $A - B \geq O$. Square matrices which have nonpositive off-diagonal entries are called Z-matrices. We call a Z-matrix M a nonsingular M-matrix if $M^{-1} \geq O$. We recall some properties of these matrices; see [1], [8]:

(i) The diagonal of a nonsingular M-matrix is positive.

(ii) If B is a Z-matrix and M is a nonsingular M-matrix, and $M \leq B$, then B is also a nonsingular M-matrix. In particular, any matrix obtained from a nonsingular M-matrix by setting certain off-diagonal entries to zero is also a nonsingular M-matrix.

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(iii) A matrix M is a nonsingular M-matrix if and only if each principal submatrix of M is a nonsingular M-matrix.
(iv) A Z-matrix M is a nonsingular M-matrix if and only if there exists a positive vector $x > 0$ such that $Mx > 0$.

We first consider the k-subdirect sum of nonsingular Z-matrices. >From (2.4) we can explicitly write

$$C^{-1} = \begin{bmatrix} I_{n-n_2} & O \\ O & B^{-1} \end{bmatrix} \begin{bmatrix} I_{n-n_2} & -\hat{A}_{12}\hat{H}^{-1} & \hat{A}_{12}\hat{H}^{-1}\hat{B}_{12} \\ O & \hat{H}^{-1} & -\hat{H}^{-1}\hat{B}_{12} \\ O & O & I_{n-n_1} \end{bmatrix} \begin{bmatrix} A^{-1} & O \\ O & I_{n-n_1} \end{bmatrix}$$

from which we obtain

$$C^{-1} = \begin{bmatrix} \hat{A}_{11} - \hat{A}_{12}\hat{H}^{-1}\hat{A}_{21} & \hat{A}_{12} - \hat{A}_{12}\hat{H}^{-1}\hat{A}_{22} & \hat{A}_{12}\hat{H}^{-1}\hat{B}_{12} \\ \hat{B}_{11}\hat{H}^{-1}\hat{A}_{21} & \hat{B}_{11}\hat{H}^{-1}\hat{A}_{22} & -\hat{B}_{11}\hat{H}^{-1}\hat{B}_{12} + \hat{B}_{12} \\ \hat{B}_{21}\hat{H}^{-1}\hat{A}_{21} & \hat{B}_{21}\hat{H}^{-1}\hat{A}_{22} & -\hat{B}_{21}\hat{H}^{-1}\hat{B}_{12} + \hat{B}_{22} \end{bmatrix} \tag{2.5}$$

and therefore we can state the following immediate result.

THEOREM 2.2. *Let A and B be nonsingular Z-matrices of order n_1 and n_2, respectively, and let k be an integer such that $1 \le k \le \min(n_1, n_2)$. Let A and B be partitioned as in* (2.1) *and their inverses be partitioned as in* (2.3). *Let $C = A \oplus_k B$. Let $\hat{H} = \hat{A}_{22} + \hat{B}_{11}$ be nonsingular. Then C is a nonsingular M-matrix if and only if each of the nine blocks of C^{-1} in* (2.5) *is nonnegative.*

We consider now the case where A and B are nonsingular M-matrices. It was shown in [3] that even if $H = A_{22} + B_{11}$ is a nonsingular M-matrix, this does not guarantee that $C = A \oplus_k B$ is a nonsingular M-matrix. We point out that this matrix H is not the matrix $\hat{H}$ obtained from A^{-1} and B^{-1} and used in Theorem 2.1. The fact that H is a nonsingular M-matrix is a necessary but not a sufficient condition for C to be a nonsingular M-matrix. Sufficient conditions are presented in the following result.

THEOREM 2.3. *Let A and B be nonsingular M-matrices partitioned as in* (2.1). *Let $x_1 > 0 \in \mathbb{R}^{(n_1-k)\times 1}$, $y_1 > 0 \in \mathbb{R}^{k\times 1}$, $x_2 > 0 \in \mathbb{R}^{k\times 1}$ and $y_2 > 0 \in \mathbb{R}^{(n_2-k)\times 1}$ be such that*

$$A \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} > 0, \quad B \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} > 0. \tag{2.6}$$

Let $H = A_{22} + B_{11}$ be a nonsingular M-matrix and let

$$y = H^{-1}\left(A_{22}y_1 + B_{11}x_2\right). \tag{2.7}$$

Then if $y \le y_1$ and $y \le x_2$ the k-subdirect sum $C = A \oplus_k B$ is a nonsingular M-matrix.

Proof. We will show that there exists $u > 0$ such that $Cu > 0$. We first note that from (2.6) we get

$$\left.\begin{array}{lcl} A_{11}x_1 + A_{12}y_1 & > & 0 \\ A_{21}x_1 + A_{22}y_1 & > & 0 \end{array}\right\}, \quad \left.\begin{array}{lcl} B_{11}x_2 + B_{12}y_2 & > & 0 \\ B_{21}x_2 + B_{22}y_2 & > & 0 \end{array}\right\}. \tag{2.8}$$

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Taking $u = \begin{bmatrix} x_1 \\ y \\ y_2 \end{bmatrix}$ and partitioning C as in (2.2) we obtain

$$Cu = \begin{bmatrix} A_{11}x_1 + A_{12}y \\ A_{21}x_1 + (A_{22} + B_{11})y + B_{12}y_2 \\ B_{21}y + B_{22}y_2 \end{bmatrix}. \tag{2.9}$$

Since $A_{21} \leq O$ and $B_{12} \leq O$, from (2.8) it follows that $A_{22}y_1 > 0$ and $B_{11}x_2 > 0$. Since $H^{-1} \geq O$, from (2.7) we have that y is positive, and consequently, so is u, i.e., $u > 0$. We will show that $Cu > 0$ one block of rows in (2.9) at a time. If $y \leq y_1$, as $A_{12} \leq 0$, we have that $A_{12}y \geq A_{12}y_1$ and again using (2.8) we obtain that the first block of rows of Cu is positive. In a similar way, the condition $y \leq x_2$ together with the last equation of (2.8) allows to conclude that the third block of rows of Cu is positive. Finally, substituting y given by (2.7) in the second row of Cu and considering (2.8) we conclude that the second block of rows of Cu is also positive. ☐

Note that A and B are nonsingular M-matrices and therefore the positive vectors (x_1, y_1) and (x_2, y_2) of (2.6) always exist. This theorem gives sufficient but not necessary conditions for $C = A \oplus_k B$ to be a nonsingular M-matrix, as illustrated in Example 2.5 further below.

Example 2.4. The matrices

$$A = \left[\begin{array}{c|cc} 3 & -2 & -1 \\ \hline -1/2 & 2 & -3 \\ -1 & -1 & 4 \end{array}\right] \quad \text{and} \quad B = \left[\begin{array}{cc|c} 1 & -2 & -1/3 \\ -3 & 9 & 0 \\ \hline -2 & -1/2 & 6 \end{array}\right],$$

and the vectors

$$\left[\begin{array}{c} x_1 \\ \hline y_1 \end{array}\right] = \left[\begin{array}{c} 1.8 \\ \hline 2 \\ 1 \end{array}\right] \quad \text{and} \quad \left[\begin{array}{c} x_2 \\ \hline y_2 \end{array}\right] = \left[\begin{array}{c} 2.5 \\ 1 \\ \hline 1 \end{array}\right]$$

satisfy the inequalities (2.6), and computing the vector y from (2.7) we get $y \approx (1.95, 0.87)^T$, which satisfy $y \leq y_1$ and $y \leq x_2$. Therefore the 2-subdirect sum

$$C = \left[\begin{array}{c|cc|c} 3 & -2 & -1 & 0 \\ \hline -1/2 & 3 & -5 & -1/3 \\ -1 & -4 & 13 & 0 \\ \hline 0 & -2 & -1/2 & 6 \end{array}\right]$$

is a nonsingular M-matrix in accordance with Theorem 2.3.

Example 2.5. The matrices

$$A = \left[\begin{array}{c|cc} 5 & -1/2 & -1/3 \\ \hline -1 & 4 & -2 \\ -1 & -6 & 10 \end{array}\right] \quad \text{and} \quad B = \left[\begin{array}{cc|c} 1 & -2 & -1/3 \\ -3 & 9 & 0 \\ \hline -2 & -1/2 & 6 \end{array}\right]$$

and the vectors

$$\left[\begin{array}{c} x_1 \\ \hline y_1 \end{array}\right] = \left[\begin{array}{c} 1 \\ \hline 1 \\ 1 \end{array}\right] \quad \text{and} \quad \left[\begin{array}{c} x_2 \\ \hline y_2 \end{array}\right] = \left[\begin{array}{c} 2.5 \\ 1 \\ \hline 1 \end{array}\right]$$

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satisfy the inequalities (2.6), but computing vector y from (2.7) we obtain $y \approx (1.18, 0.85)^T$, which does not satisfy the conditions of Theorem 2.3. Nevertheless the 2-subdirect sum

$$C = A \oplus_2 B = \left[\begin{array}{c|cc|c} 5 & -1/2 & -1/3 & 0 \\ \hline -1 & 5 & -4 & -1/3 \\ -1 & -9 & 19 & 0 \\ \hline 0 & -2 & -1/2 & 6 \end{array}\right]$$

is a nonsingular M-matrix.

In the special case of A and B block lower and upper triangular nonsingular M-matrices, respectively, the results of Theorems 2.2 and 2.3 are easy to establish. Let

$$A = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} B_{11} & B_{12} \\ 0 & B_{22} \end{bmatrix}, \tag{2.10}$$

with A_{22} and B_{11} square matrices of order k.

THEOREM 2.6. *Let A and B be nonsingular lower and upper block triangular nonsingular M-matrices, respectively, partitioned as in* (2.10)*. Then $C = A \oplus_k B$ is a nonsingular M-matrix.*

Proof. We can repeat the same argument as in the proof of Theorem 2.3 with the advantage of having $A_{12} = O$ and $B_{21} = O$. Note that conditions $y \leq y_1$ and $y \leq x_2$ are not necessary here because the first and last block of rows of Cu in (2.9) are automatically positive in this case. ☐

REMARK 2.7. The expression of C^{-1} is given by (2.5). In this particular case of block triangular matrices we have $\hat{A}_{12} = O$, $\hat{B}_{21} = O$, $\hat{A}_{22} = A_{22}^{-1}$, $\hat{B}_{11} = B_{11}^{-1}$, from which $\hat{H} = A_{22}^{-1} + B_{11}^{-1}$. If, in addition, $A_{22} = B_{11}$, then we obtain

$$C^{-1} = \begin{bmatrix} A_{11}^{-1} & O & O \\ -\frac{1}{2}A_{22}^{-1}A_{21}A_{11}^{-1} & \frac{1}{2}A_{22}^{-1} & -\frac{1}{2}A_{22}^{-1}B_{12}B_{22}^{-1} \\ O & O & B_{22}^{-1} \end{bmatrix} \geq O.$$

EXAMPLE 2.8. The matrices

$$A = \left[\begin{array}{c|cc} 3 & 0 & 0 \\ \hline -1 & 5 & -1 \\ -1 & -9 & 5 \end{array}\right] \quad \text{and} \quad B = \left[\begin{array}{cc|c} 6 & -2 & -1 \\ -4 & 3 & -3 \\ \hline 0 & 0 & 2 \end{array}\right]$$

satisfy the hypotheses of Theorem 2.6. The matrices $C = A \oplus_2 B$ and C^{-1} are

$$C = \left[\begin{array}{c|cc|c} 3 & 0 & 0 & 0 \\ \hline -1 & 11 & -3 & -1 \\ -1 & -13 & 8 & -3 \\ \hline 0 & 0 & 0 & 2 \end{array}\right], \quad C^{-1} = \left[\begin{array}{c|cc|c} 1/3 & 0 & 0 & 0 \\ \hline 11/147 & 8/49 & 3/49 & 17/98 \\ 8/49 & 13/49 & 11/49 & 23/49 \\ \hline 0 & 0 & 0 & 1/2 \end{array}\right]$$

and therefore C is a nonsingular M-matrix as expected.

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In some applications, such as in domain decomposition [6], [7], matrices A and B partitioned as in (2.1) arise with a common block, i.e., $A_{22} = B_{11}$. In the next example we show that even if A and B are nonsingular M-matrices, and so is the common block, we can not ensure that $C = A \oplus_k B$ is a nonsingular M-matrix.

EXAMPLE 2.9. The matrices

$$A = \left[\begin{array}{c|cc} 370 & -342 & -318 \\ \hline -448 & 737 & -107 \\ -46 & -190 & 444 \end{array}\right], \quad B = \left[\begin{array}{cc|c} 737 & -107 & -134 \\ -190 & 444 & -440 \\ \hline -885 & -182 & 603 \end{array}\right]$$

are nonsingular M-matrices with $A_{22} = B_{11}$ an M-matrix, but $C = A \oplus_2 B$ is not an M-matrix, since we have

$$C = \left[\begin{array}{c|cc|c} 370 & -342 & -318 & 0 \\ \hline -448 & 1474 & -214 & -134 \\ -46 & -380 & 888 & -440 \\ \hline 0 & -885 & -182 & 603 \end{array}\right]$$

$$\text{and} \qquad C^{-1} \approx \left[\begin{array}{c|cc|c} -0.0291 & -0.0242 & -0.0204 & -0.0203 \\ \hline -0.0145 & -0.0109 & -0.0098 & -0.0096 \\ -0.0214 & -0.0163 & -0.0132 & -0.0133 \\ \hline -0.0277 & -0.0210 & -0.0183 & -0.0164 \end{array}\right].$$

In the next section we shall see that when A and B share a block and they are submatrices of a given nonsingular M-matrix, the resulting k-subdirect sum is in fact a nonsingular M-matrix.

2.2. Overlapping M-matrices. In this section we restrict A and B to be principal submatrices of a given nonsingular M-matrix and such that they have a common block. Let

$$M = \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix} \tag{2.11}$$

be a nonsingular M-matrix with M_{22} square matrix of order $k \geq 1$ and let

$$A = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \text{ and } B = \begin{bmatrix} M_{22} & M_{23} \\ M_{32} & M_{33} \end{bmatrix} \tag{2.12}$$

be of order n_1 and n_2, respectively. The k-subdirect sum of A and B is thus given by

$$C = A \oplus_k B = \begin{bmatrix} M_{11} & M_{12} & O \\ M_{21} & 2M_{22} & M_{23} \\ O & M_{32} & M_{33} \end{bmatrix}. \tag{2.13}$$

In the following theorem we show that C is a nonsingular M-matrix.

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THEOREM 2.10. *Let M be a nonsingular M-matrix partitioned as in* (2.11), *and let A and B be two overlapping principal submatrices given by* (2.12). *Then the k-subdirect sum $C = A \oplus_k B$ is a nonsingular M-matrix.*

Proof. Let us construct an $n \times n$ Z-matrix T as follows:

$$T = \begin{bmatrix} M_{11} & 2M_{12} & M_{13} \\ M_{21} & 2M_{22} & M_{23} \\ M_{31} & 2M_{32} & M_{33} \end{bmatrix}. \tag{2.14}$$

Then $T = M\,\mathrm{diag}(I, 2I, I)$ and we get $T^{-1} = \mathrm{diag}(I, (1/2)I, I)M^{-1} \geq O$. Then T is a nonsingular M-matrix. Finally since C is a Z-matrix and $C \geq T$ we conclude that C is a nonsingular M-matrix. $\square$

EXAMPLE 2.11. The following nonsingular M-matrix is partitioned as in (2.11):

$$M = \left[\begin{array}{cc|ccc|c} 13/14 & -4/23 & -3/20 & -1/42 & -19/186 & -3/46 \\ -3/7 & 21/23 & -1/5 & -1/21 & -1/93 & -6/23 \\ \hline -1/7 & -7/46 & 17/20 & -1/14 & -1/186 & -2/23 \\ -4/21 & -27/92 & -1/15 & 4/7 & -58/93 & -27/92 \\ -1/14 & -9/46 & -3/10 & -1/7 & 53/62 & -9/46 \\ \hline -2/21 & -9/92 & -2/15 & -2/7 & -7/62 & 83/92 \end{array}\right]. \tag{2.15}$$

Taking overlapping submatrices A and B as in (2.12) the 3-subdirect sum $C = A \oplus_3 B$ is given by

$$C = \left[\begin{array}{cc|ccc|c} 13/14 & -4/23 & -3/20 & -1/42 & -19/186 & 0 \\ -3/7 & 21/23 & -1/5 & -1/21 & -1/93 & 0 \\ \hline -1/7 & -7/46 & 17/10 & -1/7 & -1/93 & -2/23 \\ -4/21 & -27/92 & -2/15 & 8/7 & -116/93 & -27/92 \\ -1/14 & -9/46 & -3/5 & -2/7 & 53/31 & -9/46 \\ \hline 0 & 0 & -2/15 & -2/7 & -7/62 & 83/92 \end{array}\right]$$

and it is a nonsingular M-matrix according to Theorem 2.10. In fact, we have that

$$C^{-1} \approx \left[\begin{array}{cc|ccc|c} 1.3500 & 0.3977 & 0.2624 & 0.1609 & 0.2103 & 0.1232 \\ 0.7628 & 1.4108 & 0.3383 & 0.2085 & 0.2185 & 0.1478 \\ \hline 0.3007 & 0.2845 & 0.7422 & 0.2006 & 0.1824 & 0.1763 \\ 1.1024 & 1.1571 & 0.8927 & 1.6092 & 1.3118 & 0.8940 \\ 0.4854 & 0.5256 & 0.5116 & 0.4379 & 0.9664 & 0.4013 \\ \hline 0.4543 & 0.4743 & 0.4564 & 0.5941 & 0.5634 & 1.4679 \end{array}\right].$$

2.3. k-subdirect sum of p M-matrices. In this section we extend Theorems 2.3 and 2.10 to the subdirect sum of several nonsingular M-matrices. Example 2.14 later in the section illustrates the notation used in the proofs.

THEOREM 2.12. *Let $A_i \in \mathbb{R}^{n_i \times n_i}$, $i = 1, \ldots p$, be nonsingular M-matrices partitioned as*

$$A_i = \begin{bmatrix} A_{i,11} & A_{i,12} \\ A_{i,21} & A_{i,22} \end{bmatrix} \tag{2.16}$$

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with $A_{i,11}$ *a square matrix of order* $k_{i-1} \geq 1$ *and* $A_{i,22}$ *a square matrix of order* $k_i \geq 1$, *i.e.,* $n_i = k_{i-1} + k_i$. *Since* A_i *are nonsingular* M*-matrices we have that there exist* $x_i > 0 \in \mathbb{R}^{(n_i - k_i)\times 1}$ *and* $y_i > 0 \in \mathbb{R}^{k_i \times 1}$ *such that*

$$A_i \begin{bmatrix} x_i \\ y_i \end{bmatrix} > 0, \quad i = 1, \ldots, p. \tag{2.17}$$

Let $C_0 = A_1$ *and define the following* $p-1$ k_i*-subdirect sums:*

$$C_i = C_{i-1} \oplus_{k_i} A_{i+1}, \quad i = 1, \ldots, p-1, \tag{2.18}$$

i.e.,

$$\begin{array}{rcl} C_1 & = & A_1 \oplus_{k_1} A_2, \\ C_2 & = & (A_1 \oplus_{k_1} A_2) \oplus_{k_2} A_3 = C_1 \oplus_{k_2} A_3, \\ & \vdots & \\ C_{p-1} & = & \big(A_1 \oplus_{k_1} A_2 \oplus_{k_2} \cdots \oplus_{k_{p-2}} A_{p-1}\big) \oplus_{k_{p-1}} A_p = C_{p-2} \oplus_{k_p - 1} A_p. \end{array}$$

Each subdirect sum C_i *is of order* m_i, *such that* $m_0 = n_1$ *and*

$$m_i = m_{i-1} + n_{i+1} - k_i = m_{i-1} + k_{i+1}, \quad i = 1, \ldots, p-1.$$

Let us partition C_i *in the form*

$$C_i = \begin{bmatrix} C_{i,11} & C_{i,12} \\ C_{i,21} & C_{i,22} \end{bmatrix}, \quad i = 1, \ldots, p-1, \tag{2.19}$$

with $C_{i,22}$ *a square matrix of order* k_{i+1}. *Let*

$$H_i = C_{i-1,22} + A_{i+1,11}, \quad i = 1, \ldots p-1,$$

be nonsingular M*-matrices and let*

$$z_i = H_i^{-1}(C_{i-1,22}\, y_i + A_{i+1,11}\, x_{i+1}), \quad i = 1, \ldots p-1.$$

Then, if $z_i \leq y_i$ *and* $z_i \leq x_{i+1}$, *the subdirect sums* C_i *given by* (2.18) *are nonsingular* M*-matrices for* $i = 1, \ldots, p-1$.

Proof. It is easy to see that applying Theorem 2.3 to each consecutive pair of matrices C_i we have that $C_1, C_2, \ldots, C_{p-1}$ are nonsingular M-matrices. This can be shown by induction. ☐

We now extend Theorem 2.10 to the sub-direct sum of p submatrices of a given nonsingular M-matrix M. To that end, we first define $M(S)$ a principal submatrix of M with rows and columns with indices in the set of indices $S = \{i, i+1, i+2, \ldots, j\}$. In [2] we call these *consecutive principal submatrices.* For example, matrices A and B given by (2.12) can be expressed as a submatrices of M given by (2.11) as $A = M(S_1)$, $B = M(S_2)$ with $S_1 = \{1, 2\}$ and $S_2 = \{2, 3\}$.

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THEOREM 2.13. *Let M be a nonsingular M-matrix. Let $A_i = M(S_i)$, $i = 1, \ldots, p$, be principal consecutive submatrices of M and consider the $p-1$ k_i-subdirect sums given by*

$$C_i = C_{i-1} \oplus_{k_i} A_{i+1}, \quad i = 1, \ldots, p-1, \tag{2.20}$$

in which $C_0 = A_1$. Then each of the k_i-subdirect sums C_i is a nonsingular M-matrix.

Proof. It is easy to relate the structure of each C_i to that of the submatrices A_i involved. We consider that A_i are overlapping principal submatrices of the form (2.12) but allowing that each A_i has different number of blocks. Let M be partitioned as

$$M = \begin{bmatrix} M_{11} & M_{12} & M_{13} & \cdots & M_{1n} \\ M_{21} & M_{22} & M_{23} & \cdots & M_{2n} \\ M_{31} & M_{32} & M_{33} & \cdots & M_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ M_{n1} & M_{n2} & M_{n3} & \cdots & M_{nn} \end{bmatrix} \tag{2.21}$$

according with the size of the principal submatrices A_i. Each block M_{ij} may be a submatrix of more than one A_m, $m = 1, \ldots, p$. Let $b_{ij}^{(l)} \geq 0$ be the number of matrices A_m such that M_{ij} is a submatrix of A_m, for $m = 1, \ldots, l+1$. Of course we can have $b_{ij}^{(l)} = 0$. Let us consider the lth subdirect sum C_l, $1 \leq l \leq p-1$, which is of the form

$$C_l = \begin{bmatrix} b_{11}^{(l)} M_{11} & b_{12}^{(l)} M_{12} & b_{13}^{(l)} M_{13} & \cdots & b_{1l}^{(l)} M_{1l} \\ b_{21}^{(l)} M_{21} & b_{22}^{(l)} M_{22} & b_{23}^{(l)} M_{23} & \cdots & b_{2l}^{(l)} M_{2l} \\ b_{31}^{(l)} M_{31} & b_{32}^{(l)} M_{32} & b_{33}^{(l)} M_{33} & \cdots & b_{3l}^{(l)} M_{3l} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{l1}^{(l)} M_{l1} & b_{l2}^{(l)} M_{l2} & b_{l3}^{(l)} M_{l3} & \cdots & b_{ll}^{(l)} M_{ll} \end{bmatrix}. \tag{2.22}$$

Observe that C_l is a Z-matrix and that $b_{ii}^{(l)} > 0$. Furthermore, for each column it holds that $b_{ii}^{(l)} \geq b_{ji}^{(l)}, j = 1, \ldots, l$.

The proof proceeds in a manner similar to that of Theorem 2.10. Consider the Z-matrix (partitioned in the same manner as M)

$$T_l = M_l \operatorname{diag}(b_{11}^{(l)} I, b_{22}^{(l)} I, b_{33}^{(l)} I, \ldots, b_{ll}^{(l)} I),$$

where M_l is the principal submatrix of (2.21) with row and column blocks from 1 to l. It follows that $T_l^{-1} \geq O$ and therefore T_l is a nonsingular M-matrix. Finally, since $C_l \geq T_l$, we conclude that C_l is a nonsingular M-matrix, $l = 1, \ldots, p$. ☐

EXAMPLE 2.14. Given the nonsingular M-matrix M of Example 2.11, let us consider the following overlapping blocks

$$A_1 = M(\{1,2,3\}) = \begin{bmatrix} 13/14 & -4/23 & -3/20 \\ -3/7 & 21/23 & -1/5 \\ -1/7 & -7/46 & 17/20 \end{bmatrix},$$

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$$A_2 = M(\{2,3,4,5\}) = \begin{bmatrix} 21/23 & -1/5 & -1/21 & -1/93 \\ -7/46 & 17/20 & -1/14 & -1/186 \\ -27/92 & -1/15 & 4/7 & -58/93 \\ -9/46 & -3/10 & -1/7 & 53/62 \end{bmatrix},$$

$$A_3 = M(\{3,4,5,6\}) = \begin{bmatrix} 17/20 & -1/14 & -1/186 & -2/23 \\ -1/15 & 4/7 & -58/93 & -27/92 \\ -3/10 & -1/7 & 53/62 & -9/46 \\ -2/15 & -2/7 & -7/62 & 83/92 \end{bmatrix}.$$

Then we have the 2-subdirect sum

$$C_1 = A_1 \oplus_2 A_2 = \begin{bmatrix} 13/14 & -4/23 & -3/20 & 0 & 0 \\ -3/7 & 42/23 & -2/5 & -1/21 & -1/93 \\ -1/7 & -7/23 & 17/10 & -1/14 & -1/186 \\ 0 & -27/92 & -1/15 & 4/7 & -58/93 \\ 0 & -9/46 & -3/10 & -1/7 & 53/62 \end{bmatrix},$$

which is a nonsingular M-matrix, and the 3-subdirect sum

$$C_2 = C_1 \oplus_3 A_3 = \begin{bmatrix} 13/14 & -4/23 & -3/20 & 0 & 0 & 0 \\ -3/7 & 42/23 & -2/5 & -1/21 & -1/93 & 0 \\ -1/7 & -7/23 & 51/20 & -1/7 & -1/93 & -2/23 \\ 0 & -27/92 & -2/15 & 8/7 & -116/93 & -27/92 \\ 0 & -9/46 & -3/5 & -2/7 & 53/31 & -9/46 \\ 0 & 0 & -2/15 & -2/7 & -7/62 & 83/92 \end{bmatrix},$$

which is also a nonsingular M-matrix in accordance with Theorem 2.13. Observe that in this example we have $k_1 = 2$ and $k_2 = 3$. Note also that, for example, we have $b_{22}^{(1)} = 2$, $b_{33}^{(1)} = 2$, $b_{14}^{(1)} = 0$, $b_{22}^{(2)} = 2$, $b_{22}^{(3)} = 2$, $b_{33}^{(2)} = 3$, $b_{14}^{(2)} = 0$.

3. Subdirect sums of inverses. Let A and B be nonsingular matrices partitioned as in (2.1). In this section we consider the k-subdirect sum of their inverses. We will establish counterparts to some of results in the previous sections. Let us denote by $G = A^{-1} \oplus_k B^{-1}$, with A^{-1} and B^{-1} partitioned as in (2.3), i.e.,

$$G = \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} & 0 \\ \hat{A}_{21} & \hat{A}_{22} + \hat{B}_{11} & \hat{B}_{12} \\ 0 & \hat{B}_{21} & \hat{B}_{22} \end{bmatrix}. \tag{3.1}$$

As a corollary to, and in analogy to Theorem 2.1, the next statement indicates that the nonsingularity of $A_{22} + B_{11}$ is a necessary condition to obtain G nonsingular.

THEOREM 3.1. *Let A and B be nonsingular matrices partitioned as in* (2.1) *and let their inverses be partitioned as in* (2.3). *Let $G = A^{-1} \oplus_k B^{-1}$ partitioned as in* (3.1) *with $k \geq 1$. Then G is nonsingular if and only if $H = A_{22} + B_{11}$ is nonsingular.*

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We remark that in analogy to the expression (2.5) of C^{-1}, the explicit form of G^{-1} is

$$G^{-1} = \begin{bmatrix} A_{11} - A_{12}H^{-1}A_{21} & A_{12} - A_{12}H^{-1}A_{22} & A_{12}H^{-1}B_{12} \\ B_{11}H^{-1}A_{21} & B_{11}H^{-1}A_{22} & -B_{11}H^{-1}B_{12} + B_{12} \\ B_{21}H^{-1}A_{21} & B_{21}H^{-1}A_{22} & -B_{21}H^{-1}B_{12} + B_{22} \end{bmatrix}. \quad (3.2)$$

Corollary 3.2. *When A and B are nonsingular M-matrices with the common block $A_{22} = B_{11}$ a square matrix of order k, i.e., of the form*

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} A_{22} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}, \quad (3.3)$$

then $H = 2A_{22}$ is nonsingular and therefore $G = A^{-1} \oplus_k B^{-1}$ is nonsingular.

We note that this is the case when A and B are overlapping submatrices of an M-matrix, i.e., of the form (2.12) and (2.11) considered in Section 2.2, where we were interested in the subdirect sum of A and B. Here we conclude that the subdirect sum of their inverses is always nonsingular.

Example 3.3. Let A and B be the matrices of Example 2.11, then according to Corollary 3.2, the 3-subdirect sum of the inverses

$$G = A^{-1} \oplus_3 B^{-1} \approx \left[\begin{array}{cc|ccc|c} 1.5033 & 0.5513 & 0.5547 & 0.2757 & 0.3912 & 0 \\ 0.9540 & 1.5996 & 0.7158 & 0.3635 & 0.4038 & 0 \\ \hline 0.6004 & 0.5636 & 2.9750 & 0.8144 & 0.7407 & 0.3708 \\ 2.0383 & 2.1242 & 3.5729 & 6.5498 & 5.3372 & 2.0139 \\ 0.8953 & 0.9650 & 2.0470 & 1.8025 & 3.9062 & 0.9048 \\ \hline 0 & 0 & 0.8551 & 1.3803 & 1.2652 & 1.9143 \end{array}\right]$$

is a nonsingular matrix.

In the above example a direct computation shows that G^{-1} is not an M-matrix:

$$G^{-1} \approx \left[\begin{array}{cc|ccc|c} 0.8900 & -0.2337 & -0.0750 & -0.0119 & -0.0511 & 0.0512 \\ -0.4682 & 0.8566 & -0.1000 & -0.0238 & -0.0054 & 0.0470 \\ \hline -0.0714 & -0.0761 & 0.4250 & -0.0357 & -0.0027 & -0.0435 \\ -0.0952 & -0.1467 & -0.0333 & 0.2857 & -0.3118 & -0.1467 \\ -0.0357 & -0.0978 & -0.1500 & -0.0714 & 0.4274 & -0.0978 \\ \hline 0.1242 & 0.2045 & -0.0667 & -0.1429 & -0.0565 & 0.7123 \end{array}\right]$$

which is not a Z-matrix. Note that when A and B are M-matrices we have from (3.1) that $G = A^{-1} \oplus B^{-1}$ is nonnegative. Therefore assuming that G^{-1} exists we have $(G^{-1})^{-1} \geq O$. Then it is a natural question to seek conditions so that G^{-1} is a nonsingular M-matrix. We study this question next.

The expressions (3.1) of G and (3.2) of G^{-1}, Theorem 3.1, and the observation that for nonsingular M-matrices we have $(G^{-1})^{-1} \geq O$, imply the following result.

Theorem 3.4. *Let A and B be nonsingular M-matrices partitioned as in* (2.1) *and their inverses partitioned as in* (2.3)*. Let $G = A^{-1} \oplus_k B^{-1}$ with $k \geq 1$, and let $H = A_{22} + B_{11}$ be nonsingular. Then G^{-1} is a nonsingular M-matrix if and only if G^{-1} is a Z-matrix.*

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COROLLARY 3.5. *Let A and B be lower and upper block triangular nonsingular M-matrices, respectively, partitioned as in* (2.10) *with A_{22} and B_{11} square matrices of order k and $H = A_{22} + B_{11}$ nonsingular. Then $G^{-1} = (A^{-1} \oplus_k B^{-1})^{-1}$ is a nonsingular M-matrix if and only if the following conditions hold:*

i) $B_{11}H^{-1}A_{21} \leq O$.
ii) $B_{11}H^{-1}A_{22}$ *is a Z-matrix.*
iii) $-B_{11}H^{-1}B_{12} + B_{12} \leq O$.

Proof. >From (3.2) and (2.10) we have that

$$G^{-1} = \begin{bmatrix} A_{11} & 0 & 0 \\ B_{11}H^{-1}A_{21} & B_{11}H^{-1}A_{22} & -B_{11}H^{-1}B_{12} + B_{12} \\ 0 & 0 & B_{22} \end{bmatrix} \tag{3.4}$$

and therefore G^{-1} is a Z-matrix if and only if the conditions i), ii) and iii) hold. □

Conditions i), ii) and iii) in the corollary are not as stringent as they may appear. For example, let A and B be block triangular nonsingular M-matrices partitioned as in (2.10) with a common block $A_{22} = B_{11}$, a square matrix of order k, i.e.,

$$A = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} A_{22} & B_{12} \\ 0 & B_{22} \end{bmatrix}. \tag{3.5}$$

Then $G^{-1} = (A^{-1} \oplus_k B^{-1})^{-1}$ is a nonsingular M-matrix, since we have from (3.4) that

$$G^{-1} = \begin{bmatrix} A_{11} & O & O \\ \frac{1}{2}A_{21} & \frac{1}{2}A_{22} & \frac{1}{2}B_{12} \\ O & O & B_{22} \end{bmatrix},$$

and therefore G^{-1} is a Z-matrix. In fact, in this case, we have

$$G = \begin{bmatrix} A_{11}^{-1} & O & O \\ -A_{22}^{-1}A_{21}A_{11}^{-1} & 2A_{22}^{-1} & -A_{22}^{-1}B_{12}B_{22}^{-1} \\ O & O & B_{22}^{-1} \end{bmatrix} \geq O.$$

The next example illustrates this situation.

EXAMPLE 3.6. Let A and B be the matrices of Example 2.8, then

$$G = A^{-1} \oplus_2 B^{-1} = \left[\begin{array}{c|cc|c} 1/3 & 0 & 0 & 0 \\ \hline 1/8 & 49/80 & 21/80 & 9/20 \\ 7/24 & 77/80 & 73/80 & 11/10 \\ \hline 0 & 0 & 0 & 1/2 \end{array}\right],$$

and

$$G^{-1} = \left[\begin{array}{c|cc|c} 3 & 0 & 0 & 0 \\ \hline -18/49 & 146/49 & -6/7 & -39/49 \\ -4/7 & -22/7 & 2 & -11/7 \\ \hline 0 & 0 & 0 & 2 \end{array}\right]$$

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is a nonsingular M-matrix in accordance with Corollary 3.5.

Note that if the hypotheses of Corollary 3.5 are satisfied, and recalling Theorem 2.6, we have that each of the matrices $C = A \oplus_k B$ and $G^{-1} = (A^{-1} \oplus_k B^{-1})^{-1}$ are both nonsingular M-matrices.

4. P-matrices. A square matrix is a P-matrix if all its principal minors are positive. As a consequence we have that all the diagonal entries of a P-matrix are positive. It is also follows that a nonsingular M-matrix is a P-matrix. It can also be shown that if A is a nonsingular M-matrix, then A^{-1} is a P-matrix; see, e.g., [5].

In [3] it is shown that the k-subdirect sum (with $k > 1$) of two P-matrices is not necessarily a P-matrix. Our results in Sections 2.1 and 3 hold for nonsingular M-matrices and inverses of M-matrices, respectively. As these two classes of matrices are subsets of P-matrices, it is natural to ask if similar sufficient conditions can be found so that the k-subdirect sum of P-matrices is a P-matrix. The following example indicates that the answer may not be easy to obtain, since even in the simplest case of diagonal submatrices the k-subdirect sum may not be a P-matrix.

EXAMPLE 4.1. Given the P-matrices

$$A = \left[\begin{array}{c|cc} 543 & 388 & 322 \\ \hline 69 & 160 & 0 \\ 368 & 0 & 375 \end{array}\right], \quad B = \left[\begin{array}{cc|c} 136 & 0 & 219 \\ 0 & 225 & 159 \\ \hline 61 & 177 & 230 \end{array}\right]$$

we have that the 2-subdirect sum

$$C = A \oplus_2 B = \left[\begin{array}{c|cc|c} 543 & 388 & 322 & 0 \\ \hline 69 & 296 & 0 & 219 \\ 368 & 0 & 600 & 159 \\ \hline 0 & 61 & 177 & 230 \end{array}\right]$$

is not a P-matrix, since $\det(C) < 0$.

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