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PRODUCTS OF M-MATRICES AND NESTED SEQUENCES OF PRINCIPAL MINORS*

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Abstract. The question of whether or not the product of two nonsingular n-by-n M-matrices has a nested sequence of positive principal minors (abbreviated to a nest) is considered. For $n = 2, 3, 4$ such a product always has a nest, and this is conjectured for $n = 5$. For general n, examples of products of two M-matrices with specified structure are identified as having a leading or trailing nest. For $n = 4$, it is shown that the cube of an M-matrix need not have a nest.

Key words. M-matrix, Nested sequence of principal minors, P-matrix.

AMS subject classifications. 15A15.

1. Introduction.

For C an n-by-n matrix and $\alpha, \beta \subseteq \{1, 2, \dots, n\}$, we denote by $C[\alpha, \beta]$ the submatrix of C in rows α and columns β. The *principal submatrix* $C[\alpha, \alpha]$ is abbreviated $C[\alpha]$; $\det C[\alpha]$ is a *principal minor*. The order of such a principal submatrix or minor is $\mid \alpha \mid$, the cardinality of α. By a *nested sequence* of principal submatrices (minors) in C, we mean those corresponding to a nested sequence of distinct subsets $\alpha_1 \subseteq \alpha_2 \subseteq \dots \subseteq \alpha_n = \{1, \dots, n\}$, in which $\mid \alpha_k \mid = k$. As each subset in the nest brings in an additional index, we identify a nest via the sequence of indices $i_1, i_2, \dots, i_n$ where $\alpha_k = \{i_1, i_2, \dots, i_k\}$. By a *leading* (*trailing*) nest, we mean the sequence $1, 2, \dots, n$ $(n, n-1, \dots, 1)$. We say that a real n-by-n matrix C has a *nested sequence of positive principal minors* (a *nest*, for short) if there is a nested sequence $i_1, i_2, \dots, i_n$ such that $\det C[\{i_1, i_2, \dots, i_k\}] > 0$ for $k = 1, \dots, n$. Note that $C[i_1, i_2, \dots, i_k]$ is the principal submatrix of C with its rows and columns in the same order as they occur in C. A nest clearly requires that $\det C$ be positive.

A matrix with nonpositive off-diagonal entries is called a Z-matrix. An M-matrix is a square Z-matrix that has a nest. Furthermore, an M-matrix is a P-matrix, i.e., all of its principal minors are positive; see [2, 5] for this and many other equivalent characterizations of M-matrices. We are mainly interested here in products of two nonsingular M-matrices and whether such a product necessarily has a nest. For $n = 4$, such a product need not be a P-matrix [8, Example 3], but a nest is not ruled out by the example there. It has been shown that the product of a nonsingular M-matrix and an inverse M-matrix *does* have a nest [9, Theorem 4.6], even though such a product also need not be a P-matrix. (This is so for either order of the factors in such a product and one order follows from the other by transposition, not by inversion as stated in [9, proof of Theorem 4.6].) A similar question may be asked for a product

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of two positive definite matrices, but the answer is negative as we show by example later (Example 6.1).

The question of the existence of a nest in the product of two M-matrices is quite different from that of an M-matrix and an inverse M-matrix, and it seems generally quite subtle. Here, our purpose is to popularize this question and to give quite a number of particular results about nests in general, principal minors in the product of two M-matrices and special situations in which a product of two M-matrices does have a nest. In the process, some interesting questions arise. A number of informative examples are also given.

Part of the motivation for our questions about a nest lies in the fact, due to Fisher and Fuller [3] and Ballantine [1], that if C has a nest, then there is a positive diagonal matrix D so that DC is positive stable. For applications of this fact to negative stability, see [6].

2. Nest Preserving Transformations. If an n-by-n matrix C has the nest $i_1, i_2, \ldots, i_n$, then this nest is preserved under transposition and positive diagonal equivalence. In addition, by Jacobi's theorem, C^{-1} has the nest $i_n, i_{n-1}, \ldots, i_1$, and the permutation similarity $P^T CP$ has a nest as determined by the permutation matrix P.

Note that if C_1C_2 has a nest, then it is not in general true that C_2C_1 has a nest. This is illustrated by the products

$$C_1C_2 = \begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 3 & -2 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ -3 & 1 \end{bmatrix},$$

$$C_2C_1 = \begin{bmatrix} 3 & -2 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & -2 \\ 0 & -1 \end{bmatrix}$$

in which C_1C_2 has the nest 2,1 but C_2C_1 has no nest.

The following result shows that if a matrix has a nest, then so does its product with a triangular matrix having positive diagonal entries.

Lemma 2.1. *Let L and U be lower and upper triangular matrices, respectively, with all diagonal entries positive. If C has a leading nest, then LC and CU have leading nests, whereas UC and CL have trailing nests.*

Proof. The first statement can be seen by partitioning the matrices so that

$$LC = \begin{bmatrix} L_{11} & 0 \\ l_{21}^T & l_{22} \end{bmatrix} \begin{bmatrix} C_{11} & c_{12} \\ c_{21}^T & c_{22} \end{bmatrix} = \begin{bmatrix} L_{11}C_{11} & L_{11}c_{12} \\ l_{21}^T C_{11} + l_{22}c_{21}^T & l_{21}^T c_{12} + l_{22}c_{22} \end{bmatrix}$$

and using induction. The other statements follow by transposition and/or permutation similarity with the backward identity permutation matrix. □

3. Principal Minors in the Product of Two M-matrices. Let A and B be nonsingular n-by-n M-matrices. Since the sign of every principal minor is preserved by positive diagonal equivalence, in considering the question of whether or not the product AB of two M-matrices has a nest, without loss of generality it can be assumed

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that all of the main diagonal entries of A and B are one. By the remark in Section 2 that inversion preserves a nest, this question is equivalent to the existence of a nest in the product of two inverse M-matrices.

If A is an M-matrix, then there exist positive diagonal matrices D_1, D_2 such that D_1A is column diagonally dominant and AD_2 is row diagonally dominant. Thus, for positive diagonal matrices D_i, the product $(D_1AD_2^{-1})(D_2BD_3)$ shows that without loss of generality the M-matrix A can be assumed to be both row and column diagonally dominant and the M-matrix B row diagonally dominant. Since every M-matrix has an LU factorization in which both the lower and upper triangular factors are M-matrices, it follows that $AB = L_AU_AL_BU_B$, where these four triangular factors are all M-matrices. Furthermore, if A is both row and column diagonally dominant and B is row diagonally dominant, then L_A is column dominant and both of U_A and U_B are row dominant. Consequently, if U_AL_B has a leading nest, then by Lemma 2.1 so does AB. It should be noted that although there is no certainty as to whether a nest exists in $AB = L_AU_AB$ or to the sequence of indices in such a nest, by Lemma 2.1 there does exist a trailing nest in U_ABL_A, which is triangularly similar to AB.

If the graphs associated with the matrices A and B are restricted, then some sufficient conditions for AB to be a P-matrix are given in [7, 8]. The following result identifies some minors that are positive in the product of two arbitrary M-matrices.

Proposition 3.1. *If A, B are nonsingular n-by-n M-matrices, then all principal minors of orders 1, $n-1$ and n of AB are positive.*

Proof. The Z-matrix sign pattern combined with the positivity of the diagonal entries of A and B shows that all order 1 principal minors of AB are positive. As $\det A$ and $\det B$ are both positive, it follows that the order n principal minor of AB is positive. Lastly, by the positivity of the diagonal entries in $(AB)^{-1}$ and the positivity of $\det AB$, it follows that all order $n-1$ principal minors in AB are positive. □

Corollary 3.2. *If $n \le 3$ then the product AB of two nonsingular n-by-n M-matrices is a P-matrix.*

For $n = 2$, the product AB is in fact an M-matrix. For $n = 4$, it is not necessarily true that AB is a P-matrix; see [8, Example 3] in which one minor of order 2 is negative.

We now consider order 2 principal minors in the product for general n, and first prove two lemmas that are of independent interest. In the first lemma the inequality is entrywise.

Lemma 3.3. *If A, B are M-matrices, then $AB[\alpha] \ge A[\alpha]B[\alpha]$.*

Proof. Without loss of generality let α = 1,2,...,k. Let

$$A = \begin{bmatrix} A_{11} & -A_{12} \\ -A_{21} & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} B_{11} & -B_{12} \\ -B_{21} & B_{22} \end{bmatrix}$$

where A_{11}, B_{11} are M-matrices of order k and A_{12}, A_{21}, B_{12}, $B_{21} \ge 0$. Then the leading principal submatrix of order k in AB is $AB[\alpha] = A_{11}B_{11} + A_{12}\ B_{21} \ge A_{11}B_{11} = A[\alpha]B[\alpha]$. □

Lemma 3.4. *If A, B are nonsingular M-matrices and $AB[\alpha]$ is a Z-matrix, then $AB[\alpha]$ is a nonsingular M-matrix.*

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Proof. By Lemma 3.3, $AB[\alpha] \geq A[\alpha]B[\alpha]$. Thus if $AB[\alpha]$ is a Z-matrix, then the product $A[\alpha]B[\alpha]$ must also be a Z-matrix. However, since $A[\alpha]$ and $B[\alpha]$ are both nonsingular M-matrices, they are inverse nonnegative and thus $A[\alpha]B[\alpha]$ is a nonsingular M-matrix. By [5, Theorem 2.5.4(a)], it follows that $AB[\alpha]$ is a nonsingular M-matrix. □

THEOREM 3.5. *Let $n \geq 2$ and A, B be nonsingular M-matrices. Then the product AB has at least $\lceil \frac{n}{2} \rceil$ positive principal minors of order 2.*

Proof. Every principal submatrix of order 2 in AB has one of the following forms:

$$\begin{bmatrix} + & 0 \\ * & + \end{bmatrix}, \begin{bmatrix} + & * \\ 0 & + \end{bmatrix}, \begin{bmatrix} + & - \\ + & + \end{bmatrix}, \begin{bmatrix} + & + \\ - & + \end{bmatrix}, \begin{bmatrix} + & + \\ + & + \end{bmatrix} \text{ or } \begin{bmatrix} + & - \\ - & + \end{bmatrix}$$

where $*$ denotes an arbitrary entry. The determinant of a matrix of any of the first four forms is positive. Not all of the principal submatrices of order 2 can be of the fifth form since AB cannot have all entries positive (as $(AB)^{-1}$ has every entry nonnegative). As the last form is a Z-matrix, by Lemma 3.4 it must be a nonsingular M-matrix and therefore has a positive determinant. Thus AB has at least one positive minor of order 2. The lower bound on the number of positive principal minors of order 2 occurs when A or B is irreducible (in which case $(AB)^{-1}$ has every entry positive and thus every row and column of AB has at least one negative entry) and all principal submatrices of order 2 that contain a negative entry in fact contain two negative entries. Thus AB has at least $\lceil \frac{n}{2} \rceil$ positive principal minors of order 2. □

From numerical evidence, the above lower bound of $\lceil \frac{n}{2} \rceil$ is very conservative. For example, if $n = 5$, we know of no such product AB with fewer than eight positive principal minors of order 2.

COROLLARY 3.6. *If A, B are nonsingular M-matrices of order 4, then AB has a nest.*

Proof. Theorem 3.5 shows that there is at least one positive principal minor of order 2 in the product AB. By Proposition 3.1, all principal minors of orders 1, 3 and 4 are positive in AB. Therefore AB has a nest. □

In fact, by Theorem 3.5, since for $n = 4$ such a product AB must have at least two positive principal minors of order 2, in this case AB must have at least eight nests. Lemma 3.4 also gives the following result.

THEOREM 3.7. *If A, B are nonsingular M-matrices and $AB[\alpha]$ is a Z-matrix with $\mid \alpha \mid= n - 2$, then AB has a nest.*

Proof. By Lemma 3.4, $AB[\alpha]$ is an M-matrix. The positivity of all principal minors of order $n-1$ and the determinant of AB complete a nest with the first $n-2$ indices from α. □

EXAMPLE 3.8. Let A and B be the 5-by-5 matrices

$$A = \begin{bmatrix} 1 & -0.1 & -0.1 & -0.1 & -0.1 \\ -0.1 & 1 & -0.1 & -0.1 & -2 \\ -0.1 & -0.1 & 1 & -1 & -0.1 \\ -0.1 & -0.1 & -0.1 & 1 & -0.1 \\ -0.1 & -0.1 & -0.1 & -0.1 & 1 \end{bmatrix},$$

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$$B = \begin{bmatrix} 1 & -0.1 & -0.1 & -0.1 & -0.1 \\ -0.1 & 1 & -0.1 & -0.1 & -0.1 \\ -0.1 & -0.1 & 1 & -0.1 & -0.1 \\ -0.1 & -1 & -0.1 & 1 & -0.1 \\ -0.1 & -0.1 & -2 & -0.1 & 1 \end{bmatrix}.$$

Then A and B are both nonsingular M-matrices and

$$AB = \begin{bmatrix} 1.04 & -0.08 & 0.02 & -0.17 & -0.17 \\ 0.02 & 1.32 & 3.82 & 0.02 & -2.07 \\ -0.08 & 0.82 & 1.32 & -1.07 & -0.08 \\ -0.17 & -1.07 & 0.02 & 1.04 & -0.17 \\ -0.17 & -0.08 & -2.07 & -0.17 & 1.04 \end{bmatrix}.$$

Since $AB[145]$ is a Z-matrix of order 3, by Theorem 3.7 AB has a nest. In particular AB has the nest 1,4,5,2,3. Note that $\det AB[2,3] < 0$, so AB is not a P-matrix.

4. Nests in Products of Two M-Matrices with a Specified Structure. We now prove that a nest in the product of two M-matrices is guaranteed if one of the factors is a Hessenberg matrix.

THEOREM 4.1. *Let A, F, G be nonsingular M-matrices where F is a lower Hessenberg matrix and G is an upper Hessenberg matrix. Then FA and AG contain a leading nest and GA and AF contain a trailing nest.*

Proof. Let

$$A = \begin{bmatrix} A_{11} & -A_{12} \\ -A_{21} & A_{22} \end{bmatrix}, \quad F = \begin{bmatrix} F_{11} & -F_{12} \\ -F_{21} & F_{22} \end{bmatrix}, \quad G = \begin{bmatrix} G_{11} & -G_{12} \\ -G_{21} & G_{22} \end{bmatrix}$$

where $A_{12},\ A_{21},\ F_{12},\ F_{21},\ G_{12},\ G_{21} \geq 0$; $A_{11},\ F_{11},\ G_{11}$ are M-matrices of order k; $A_{22},\ F_{22},\ G_{22}$ are M-matrices of order $n-k$ and $2 \leq k \leq n-2$. As well F_{12} is k-by-$(n-k)$ and G_{21} is $(n-k)$-by-k with

$$F_{12} = \begin{bmatrix} 0 & \cdots & & 0 \\ \vdots & & \iddots & \vdots \\ 0 & 0 & & \\ f & 0 & \cdots & 0 \end{bmatrix}, \quad G_{21} = \begin{bmatrix} 0 & \cdots & 0 & g \\ \vdots & & 0 & 0 \\ & \iddots & & \vdots \\ 0 & \cdots & & 0 \end{bmatrix}$$

where $f, g \geq 0$. Therefore, the leading principal submatrix of order k in FA is $F_{11}A_{11} + F_{12}A_{21}$. The structure of F_{12} ensures that $F_{12}A_{21}$ is a rank one matrix that can be written as xy^T, where x is the first column of F_{12}, y^T is the first row of A_{21} and $x, y \geq 0$. Therefore, by a well known fact for a rank one perturbation of a matrix,

$$\det(F_{11}A_{11} + F_{12}A_{21}) = \det(F_{11}A_{11} + xy^T) = \det F_{11}A_{11}(1 + y^T[F_{11}A_{11}]^{-1}x).$$

Since $F_{11},\ A_{11}$ are nonsingular M-matrices, it follows that $\det F_{11},\ \det A_{11} > 0$ and therefore $\det F_{11}A_{11} > 0$. As well $[F_{11}A_{11}]^{-1} \geq 0$. Thus $(1 + y^T[F_{11}A_{11}]^{-1}x) > 0$ and consequently $\det(F_{11}A_{11} + F_{12}A_{21}) > 0$. As this is true for all k and by Proposition

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3.1 all principal minors of orders 1, $n-1$ and n of FA are positive, it follows that FA contains a leading nest. The other statements follow by transposition and/or permutation similarity with the backward identity permutation matrix. □

The following example shows that a product FA (as in Theorem 4.1) need not be a P-matrix.

Example 4.2.

$$F = \begin{bmatrix} 1 & -0.1 & 0 & 0 \\ -0.1 & 1 & -0.1 & 0 \\ -0.1 & -2 & 1 & -0.1 \\ -2 & -0.1 & -0.1 & 1 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & -0.1 & -2 & -0.1 \\ -0.1 & 1 & -0.1 & -2 \\ -0.1 & -0.1 & 1 & -0.1 \\ -0.1 & -0.1 & -0.1 & 1 \end{bmatrix}$$

are both M-matrices, but

$$FA = \begin{bmatrix} 1.01 & -0.2 & -1.99 & 0.1 \\ -0.19 & 1.02 & 0 & -1.98 \\ 0.01 & -2.08 & 1.41 & 3.81 \\ -2.08 & 0.01 & 3.81 & 1.41 \end{bmatrix}$$

is not a P-matrix as $\det FA[3,4] < 0$.

Corollary 3.2 and induction are now used to identify another M-matrix product FG that has a nest. The matrix F is lower Hessenberg with an extra diagonal immediately above the superdiagonal and G is upper Hessenberg with an extra diagonal immediately below the subdiagonal. Such a product FG includes the product of two pentadiagonal M-matrices.

Theorem 4.3. *Let $n \geq 3$ and $F = [f_{ij}]$, $G = [g_{ij}]$ be nonsingular n-by-n M-matrices such that $f_{ij} = 0$ if $j - i \geq 3$ and $g_{ij} = 0$ if $i - j \geq 3$. Then FG contains a leading nest and GF contains a trailing nest.*

Proof. We use a proof by induction to show that FG contains a leading nest. By Corollary 3.2, when $n = 3$, FG is a P-matrix, and thus contains a leading nest.

Now suppose $k \geq 4$ and the statement is true for all $m \in \mathbb{Z}^+$ such that $3 \leq m < k$. For $n = k$, we know that the order $k-1$ principal minors of FG are positive and $\det FG > 0$. We can partition F and G as follows:

$$F = \begin{bmatrix} F_{k-1} & -a \\ -b^T & f_{kk} \end{bmatrix}, \quad G = \begin{bmatrix} G_{k-1} & -c \\ -d^T & g_{kk} \end{bmatrix}$$

where $a,\ d \geq 0$, $a^T = [0, 0, \ldots, a_{k-2}, a_{k-1}]$ and $d^T = [0, 0, \ldots, d_{k-2},\ d_{k-1}]$. Therefore the leading principal minor of order $k-1$ in FG is $F_{k-1}G_{k-1} + ad^T$ and

$$ad^T = \begin{bmatrix} 0 & \cdots & & 0 \\ \vdots & \ddots & & \vdots \\ & & a_{k-2}d_{k-2} & a_{k-2}d_{k-1} \\ 0 & \cdots & a_{k-1}d_{k-2} & a_{k-1}d_{k-1} \end{bmatrix}.$$

Since $F_{k-1}G_{k-1}$ contains a leading nest by the induction hypothesis, all leading principal minors of orders 1 to $k-3$ in $F_{k-1}G_{k-1} + ad^T$ and consequently in FG are

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positive. As well, since $F_{k-1}G_{k-1} + ad^T$ is an order $k-1$ principal submatrix of FG, its determinant is positive. It is thus only the leading principal minor of order $k-2$ in FG that must be considered, namely $\det FG[1,\ldots,k-2]$. The only change from the order $k-2$ leading principal submatrix of $F_{k-1}G_{k-1}$ to obtain the order $k-2$ leading principal submatrix of $F_{k-1}G_{k-1}+ad^T$ is the nonnegative addition of $a_{k-2}d_{k-2}$ to the last main diagonal entry of the order $k-2$ leading principal submatrix in $F_{k-1}G_{k-1}$. By the induction hypothesis it follows that the complementary minor of this diagonal entry in the leading principal submatrix of order $k-2$ in $F_{k-1}G_{k-1}$ must be positive since it is a leading principal minor of order $k-3$ in FG. Similarly, the induction hypothesis ensures that the leading principal minor of order $k-2$ in $F_{k-1}G_{k-1}$ is positive because it is a leading principal minor of order less than k. By linearity of the determinant, the nonnegative addition to the diagonal entry leaves the sign of the order $k-2$ minor in $F_{k-1}G_{k-1} + ad^T$ positive. Therefore, FG contains a leading nest for $n = k$. Thus by induction, FG contains a leading nest for $n \geq 3$. The other statement follows by permutation similarity with the backward identity permutation matrix. ☐

The next example shows that if the structure of G is slightly more general than that of Theorem 4.3, then FG need not have a leading nest.

Example 4.4. Consider the 4-by-4 M-matrices

$$F = \begin{bmatrix} 1 & -0.1 & -2 & 0 \\ -0.1 & 1 & -0.1 & -2 \\ -0.1 & -0.1 & 1 & -0.1 \\ -0.1 & -0.1 & -0.1 & 1 \end{bmatrix}, \quad G = \begin{bmatrix} 1 & -0.1 & -0.1 & -0.1 \\ -0.1 & 1 & -0.1 & -0.1 \\ -0.1 & -2 & 1 & -0.1 \\ -2 & -0.1 & -0.1 & 1 \end{bmatrix}.$$

Then

$$FG = \begin{bmatrix} 1.21 & 3.8 & -2.09 & 0.11 \\ 3.81 & 1.41 & 0.01 & -2.08 \\ 0.01 & -2.08 & 1.03 & -0.18 \\ -2.08 & 0.01 & -0.18 & 1.03 \end{bmatrix}$$

does not contain a leading nest as $FG[1,2] < 0$. This example does, however, have a trailing nest.

5. Products of More than Two M-Matrices.

The previous two sections discuss nests in the product of two M-matrices, and we now consider more general products. If A_i are 2-by-2 nonsingular M-matrices, then $\prod_{i=1}^{k} A_i$ is a nonsingular M-matrix (and hence has a nest) for all positive integers k. We have a more restrictive result for 3-by-3 matrix products.

Theorem 5.1. *If A is a 3-by-3 nonsingular M-matrix, then A^3 has a nest.*

Proof. Since $\det A^3 > 0$ and $(A^3)^{-1}$ has positive diagonal entries, all order 2 and order 3 principal minors of A^3 are positive. It remains to show that A^3 has at least one positive diagonal entry, which is certainly true if trace A^3 is positive. This is now proved by considering eigenvalues. Since A is a nonsingular M-matrix, it has eigenvalues $a,\ b,\ c$ or $a,\ be^{\pm i\theta}$ where $a,\ b,\ c > 0$, and $0 < \theta < \frac{\pi}{2} - \frac{\pi}{3} = \frac{\pi}{6}$ [5, Theorem 2.5.9(b)]. In the first case, A^3 has eigenvalues $a^3,\ b^3,\ c^3$; thus trace

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$A^3 = a^3 + b^3 + c^3 > 0$. In the second case, A^3 has eigenvalues a^3, $b^3e^{\pm 3i\theta}$, giving trace $A^3 = a^3 + 2b^3 \cos 3\theta$. By the bounds on θ, $0 < 3\theta < \frac{\pi}{2}$, thus $\cos 3\theta > 0$ and trace $A^3 > 0$. ∎

However, if A is as in Theorem 5.1, then A^3 need not be a P-matrix [8, Example 2]. In fact, the following example shows that A^3 may have two negative diagonal entries.

Example 5.2.

$$A = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}, \quad A^3 = \begin{bmatrix} -1 & 10 & -3 \\ -13 & 25 & 5 \\ 10 & -26 & -1 \end{bmatrix}.$$

The next example shows that the result of Theorem 5.1 need not be true if A is a 4-by-4 M-matrix.

Example 5.3. Consider the 4-by-4 circulant M-matrix

$$A = \begin{bmatrix} 1 & -0.9 & 0 & 0 \\ 0 & 1 & -0.9 & 0 \\ 0 & 0 & 1 & -0.9 \\ -0.9 & 0 & 0 & 1 \end{bmatrix}.$$

Then

$$A^3 = \begin{bmatrix} 1 & -2.7 & 2.43 & -0.729 \\ -0.729 & 1 & -2.7 & 2.43 \\ 2.43 & -0.729 & 1 & -2.7 \\ -2.7 & 2.43 & -0.729 & 1 \end{bmatrix}$$

does not contain a nest as all the principal minors of order 2 in A^3 are negative. If A is a symmetric or tridiagonal M-matrix, then A^k is a P-matrix for all positive integers k [8, Theorem 1 and Lemma 2].

6. Discussion. As stated in Corollary 3.2, the product of two nonsingular 3-by-3 M-matrices is a P-matrix (and thus has a nest). In contrast, the following example shows that the product of two 3-by-3 positive definite matrices may not even have a nest.

Example 6.1. The matrix

$$C = \begin{bmatrix} 6 & -1 & 0 \\ 0 & 0 & -1 \\ -60 & 11 & 0 \end{bmatrix}$$

has three distinct positive eigenvalues, namely $\{1, 2, 3\}$, and thus is the product of two positive definite matrices (see, e.g., [4, Problem 9, p. 468]). Note that C does not have a nest.

The product of two nonsingular 4-by-4 M-matrices has a nest (Corollary 3.6) but is not necessarily a P-matrix. Extensive numerical calculations on the product of two 5-by-5 nonsingular M-matrices lead us to conjecture that such a product has a nest.

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However, we do not know how to ensure the existence of a positive 3-by-3 minor, nor how to determine the positions of the positive 2-by-2 principal minors in the product of the two M-matrices.

As mentioned in the Introduction, if a matrix has a nest then it can be stabilized by premultiplication with a positive diagonal matrix. Our results thus give products of certain M-matrices that can be stabilized in this way. Example 4.2 illustrates this, as FA has two eigenvalues with negative real parts. However, since FA has a leading nest, there exists a positive diagonal matrix D so that DFA is positive stable.

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