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THE Q-PROPERTY OF A MULTIPLICATIVE TRANSFORMATION IN SEMIDEFINITE LINEAR COMPLEMENTARITY PROBLEMS*

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Abstract. The Q-property of a multiplicative transformation AXA^T in semidefinite linear complementarity problems is characterized when A is normal.

Key words. Multiplicative transformations, Q-property, Complementarity.

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1. Introduction. Let S^n be the space of all real symmetric matrices of order n. Suppose that $L : S^n \to S^n$ is a linear transformation and $Q \in S^n$. We write $X \succeq 0$, if X is symmetric and positive semidefinite. The semidefinite linear complementarity problem, $\mathrm{SDLCP}(L, Q)$ is to find a matrix X such that

$$X \succeq 0, \quad Y := L(X) + Q \succeq 0, \quad \text{and} \quad XY = 0.$$

SDLCP has various applications in control theory, semidefinite programming and other optimization related problems. We refer to [2] for details. SDLCP can be considered as a generalization of the standard linear complementarity problem [1]. However many results in the linear complementarity problem cannot be generalized to SDLCP, as the semidefinite cone is nonpolyhedral and the matrix multiplication is noncommutative.

We say that a linear transformation L defined on S^n has the Q-property if $\mathrm{SDLCP}(L, Q)$ has a solution for all $Q \in S^n$. Let $A \in R^{n\times n}$. Then the double sided multiplicative linear transformation $M_A : S^n \to S^n$ is defined by $M_A(X) := AXA^T$. One of the problems in SDLCP is to characterize the Q-property of a multiplicative linear transformation. When A is a symmetric matrix, Sampangi Raman [6] proved that M_A has the Q-property if and only if A is either positive definite or negative definite and conjectured that the result holds when A is normal. In this paper, we prove this conjecture.

The transformation M_A has the following property: $X \succeq 0 \Rightarrow M_A(X) \succeq 0$. In other words, the multiplicative transformation leaves the positive semidefinite cone invariant. Using this interesting property, Gowda et al. [3] derived some specialized results for the multiplicative transformation. However, the problem of characterizing the Q-property of M_A remains open.

We recall a theorem due to Karamardian [5].

THEOREM 1.1. *Let L be a linear transformation on S^n. If* $\mathrm{SDLCP}(L, 0)$ *and* $\mathrm{SDLCP}(L, I)$ *have unique solutions then L has the Q-property.*

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The following theorem is well known, see for example [3].

THEOREM 1.2. *Let A be a $n \times n$ matrix. Then the following are equivalent:*

1. *A is positive definite or negative definite.*
2. *$\mathrm{SDLCP}(M_A, Q)$ has a unique solution for all $Q \in S^n$.*

We mention a few notations. If k is a positive integer, let I_k be the $k \times k$ identity matrix. Let $\mathrm{SOL}(M_A, Q)$ be the set of all solutions to $\mathrm{SDLCP}(M_A, Q)$. Suppose that F is a $n \times n$ matrix. Then f_{ij} will denote the (i, j)-entry of F. Given a vector $x \in R^n$, we let $\mathrm{diag}(x)$ to denote the diagonal matrix with the vector x along its diagonal.

2. Main Result. We introduce the following definitions.

DEFINITION 2.1. Let A be a $k \times k$ matrix. We say that A is of *type*$(*)$, if $A = I + B$ where B is a $k \times k$ skew-symmetric matrix.

EXAMPLE 2.2. Let $A := \begin{pmatrix} 1 & -5 \\ 5 & 1 \end{pmatrix}$. Then A is a *type*$(*)$ matrix.

DEFINITION 2.3. Let $A \in R^{n\times n}$. We say that A is of $form(n_1, n_2)$, if there exist *type*$(*)$ matrices S and T of order n_1 and n_2 respectively such that $n_1 + n_2 = n$ and

$$A = \begin{pmatrix} S & 0 \\ 0 & -T \end{pmatrix}.$$

DEFINITION 2.4. Let $m > 2$ and $A \in R^{m\times m}$. We say that A is of $form(*)$, if there exists a skew-symmetric matrix W of order $k \geq 2$ such that

$$A = \begin{pmatrix} W & 0 \\ 0 & \widehat{A} \end{pmatrix},$$

where $\widehat{A} \in R^{(m-k)\times(m-k)}$.

DEFINITION 2.5. We say that an $n \times n$ symmetric matrix $D = (d_{ij})$ is a *corner* matrix if its rank is one, d_{11}, d_{1n}, d_{n1} and d_{nn} are nonzero real numbers and all the remaining entries are zeros.

DEFINITION 2.6. We say that an $n \times n$ symmetric matrix Q is of *type*(n_1, n_2), if Q is not positive semidefinite and there exist integers n_1 and n_2 and a rank one matrix $Q_1 \in R^{n_1\times n_2}$ such that $n_1 + n_2 = n$ and $Q = \begin{pmatrix} I_{n_1} & Q_1 \\ Q_1^T & I_{n_2} \end{pmatrix}$.

Define

$$\widetilde{Q} := \begin{pmatrix} I_{n-1} & q \\ q^T & 1 \end{pmatrix},$$

where $q := (2, 0, \cdots, 0)^T$.

By the well-known formula of Schur, $\det \widetilde{Q} = -3$. Therefore $\widetilde{Q}$ is not positive semidefinite. It is clear that if n_1 and n_2 are any two positive integers such that $n_1 + n_2 = n$, then $\widetilde{Q}$ can be written as a *type*(n_1, n_2) matrix. Throughout the paper, we use $\widetilde{Q}$ to denote this matrix.

We will make use of the following proposition. The proof is a direct verification.

PROPOSITION 2.7. *Let $A \in R^{n\times n}$. Then the following statements are true.*

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1. *If* $0 \in \mathrm{SOL}(M_A, Q)$, *then* $Q \succeq 0$.
2. *Suppose that* P *is a nonsingular matrix. Then*

$$X \in \mathrm{SOL}(M_A, Q) \Leftrightarrow P^{-1} X P^{-T} \in \mathrm{SOL}(M_{P^T AP}, P^T QP).$$

Thus M_A *has the Q-property iff* M_{PAP^T} *has the Q-property.*

3. *If* M_A *has the Q-property, then* A *must be nonsingular.*

We will use the following property of positive semidefinite matrices.

Theorem 2.8. *Suppose that* $X := \begin{pmatrix} X_1 & Y_1 \\ Y_1^T & Z_1 \end{pmatrix} \succeq 0$. *If* $X_1 = 0$ *or* $Z_1 = 0$, *then* $Y_1 = 0$.

We begin with the following lemma.

Lemma 2.9. *Suppose that* U_1 *and* U_2 *are orthogonal matrices of order* n_1 *and* n_2 *respectively where* $n_1 + n_2 = n$. *Let* $U = \begin{pmatrix} U_1 & 0 \\ 0 & U_2 \end{pmatrix}$. *If* $A \in R^{n \times n}$ *is of* $form(n_1, n_2)$, *then* UAU^T *is of* $form(n_1, n_2)$.

Proof. Let $B := UAU^T$. Then there exist *type*$(*)$ matrices S and T of order n_1 and n_2 respectively such that

$$A = \begin{pmatrix} S & 0 \\ 0 & -T \end{pmatrix}.$$

It is easy to see that

$$B = \begin{pmatrix} S_1 & 0 \\ 0 & -T_1 \end{pmatrix}$$

where $S_1 = U_1 S U_1^T$ and $T_1 = U_2 T U_2^T$.

Let $S = I_{n_1} + W$, where W is a skew-symmetric matrix. Then $W_1 := U_1 W U_1^T$ will be skew-symmetric. Therefore $S_1 = I_{n_1} + W_1$. So S_1 is of *type*$(*)$. Similarly, T_1 is of *type*$(*)$. Thus B is of $form(n_1, n_2)$. □

Lemma 2.10. *Let* $A \in R^{n \times n}$. *Suppose* X *is a solution to* $\mathrm{SDLCP}(M_A, P\widetilde{Q}P^T)$, *where* P *is a permutation matrix. Then rank of* X *must be one.*

Proof. Let $\widehat{Q} := P\widetilde{Q}P^T$ and $Y := AXA^T + \widehat{Q}$. Let K be the leading principal $(n-1) \times (n-1)$ submatrix of Y. Then it can be easily verified that K is positive definite. Therefore the rank of Y must be at least $n-1$.

Since $X \in \mathrm{SOL}(M_A, \widehat{Q})$, $XY = 0$. Suppose that U is a orthogonal matrix which diagonalize X and Y simultaneously. Let $D = UXU^T$ and $E = UYU^T$, where D and E are diagonal. Then $DE = 0$. The rank of E is at least $n-1$. Therefore the rank of D can be at most one. If $D = 0$, then $X = 0$. This implies that $\widehat{Q} \succeq 0$ (Proposition 2.7) which is a contradiction. This means that the rank of X is exactly one. □

Lemma 2.11. *Let* $A \in R^{n \times n}$. *Suppose that* A *is of* $form(n_1, n_2)$. *If* $X \in \mathrm{SOL}(M_A, \widetilde{Q})$ *then there exists a* $form(n_1, n_2)$ *matrix* B *and a* $type(n_1, n_2)$ *matrix* $\widehat{Q}$ *such that* $\mathrm{SDLCP}(M_B, \widehat{Q})$ *has a corner solution.*

Proof. Write

$$X = \begin{pmatrix} X_1 & Y_1 \\ Y_1^T & Z_1 \end{pmatrix},$$

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where $X_1 \in S^{n_1 \times n_1}$ and $Z_1 \in S^{n_2 \times n_2}$. The above lemma implies that rank of X is one. Therefore rank of X_1 can be at most one. We now claim that rank of X_1 is exactly one. Let $Y := AXA^T + \widetilde{Q}$.

Since A is of $form(n_1, n_2)$, there exist $type(*)$ matrices S_1 and S_2 of order n_1 and n_2 respectively such that

$$A = \begin{pmatrix} S_1 & 0 \\ 0 & -S_2 \end{pmatrix}.$$

Now

$$\widetilde{Q} = \begin{pmatrix} I_{n_1} & Q_1 \\ Q_1^T & I_{n_2} \end{pmatrix},$$

where Q_1 is of rank one. Suppose $X_1 = 0$. Then Theorem 2.8 implies that $Y_1 = 0$. Thus,

$$AXA^T = \begin{pmatrix} 0 & 0 \\ 0 & S_2 Z_1 S_2^T \end{pmatrix}$$

and hence

$$Y = \begin{pmatrix} I_{n_1} & Q_1 \\ Q_1^T & S_2 Z_1 S_2^T + I_{n_2} \end{pmatrix}.$$

From the condition $XY = 0$, we see that

$$Z_1 (S_2 Z_1 S_2^T + I_{n_2}) = 0.$$

This implies that $Z_1 = 0$; so $X = 0$. Therefore $\widetilde{Q} \succeq 0$ (Proposition 2.7) which is a contradiction. Thus, X_1 is of rank one. Similarly we can prove that Z_1 and Y_1 are of rank one.

Since X_1 is a rank one matrix, we can find an orthogonal matrix U_1 such that

$$D := U_1 X_1 U_1^T = \begin{pmatrix} d & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & \cdots & 0 \end{pmatrix},$$

where $d > 0$.

Let U_2 be an orthogonal matrix such that

$$R := U_2 Z_1 U_2^T = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & \cdots & r \end{pmatrix},$$

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where $r > 0$. Let $G = U_1 Y_1 U_2^T$. Then rank of G must be one as rank of Y_1 is one. Define

$$U := \begin{pmatrix} U_1 & 0 \\ 0 & U_2 \end{pmatrix}.$$

Then U is orthogonal. Let $Z := UXU^T$. Now

$$Z = \begin{pmatrix} D & G \\ G^T & R \end{pmatrix}.$$

Since $Z \succeq 0$, by Theorem 2.8,

$$Z = \begin{pmatrix} d & 0 & \dots & 0 & e \\ 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ e & 0 & \dots & 0 & r \end{pmatrix}.$$

As G is of rank one, e is nonzero. Thus Z is a *corner* matrix.

Let $B := UAU^T$. Then by Proposition 2.7, Z is a solution to SDLCP$(M_B, \widehat{Q})$, where $\widetilde{Q} := U\widetilde{Q}U^T$. By Lemma 2.9, B must be of $form(n_1, n_2)$. It is direct to verify that $\widehat{Q}$ is of $type(n_1, n_2)$. This completes the proof. □

LEMMA 2.12. *Let Q be a $m \times n$ matrix defined as follows:*

$$Q = \begin{pmatrix} 0 & 0 & \dots & 0 & \pm 1 \\ q_{21} & q_{22} & \cdots & q_{2n-1} & 0 \\ \dots & \dots & \dots & \dots & \dots \\ q_{m1} & q_{m2} & \cdots & q_{mn-1} & 0 \end{pmatrix}.$$

Suppose the rank of Q is one. Then the submatrix of Q obtained by deleting the first row and the last column is a zero matrix.

Proof. We claim that $q_{21} = 0$. Consider the 2×2 submatrix

$$\begin{pmatrix} 0 & \pm 1 \\ q_{21} & 0 \end{pmatrix}.$$

Since Q is of rank one, $q_{21} = 0$. By repeating a similar argument for the remaining entries we get the result. □

LEMMA 2.13. *Suppose that $\widehat{B}$ is of $form(n_1, n_2)$. Let $\widehat{Q}$ be a $type(n_1, n_2)$ matrix. Then a corner matrix cannot be a solution to* SDLCP$(M_{\widehat{B}}, \widehat{Q})$.

Proof. Since $\widehat{B}$ is of $form(n_1, n_2)$, there exist $type(*)$ matrices B and C of order n_1 and n_2 respectively such that

$$\widehat{B} = \begin{pmatrix} B & 0 \\ 0 & -C \end{pmatrix}.$$

Let $B = (b_{ij})$ and $C = (c_{ij})$. Then $b_{ii} = c_{ii} = 1$. Every off-diagonal entry of B and C will now satisfy $b_{ij} + b_{ji} = 0$ and $c_{ij} + c_{ji} = 0$.

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Suppose that X is a *corner* matrix and solves $\mathrm{SDLCP}(M_{\widehat{B}}, \widehat{Q})$. Let

$$X = \begin{pmatrix} d & 0 & ... & 0 & e \\ 0 & 0 & ... & 0 & 0 \\ ... & ... & ... & ... & ... \\ e & 0 & ... & 0 & r \end{pmatrix}.$$

Let $\widehat{Q} = \begin{pmatrix} I_{n_1} & Q_1 \\ Q_1^T & I_{n_2} \end{pmatrix}$ where

$$Q_1 = \begin{pmatrix} q_{11} & q_{12} & \cdots & q_{1n_2} \\ q_{21} & q_{22} & \cdots & q_{2n_2} \\ \cdots & \cdots & \cdots & \cdots \\ q_{n_1 1} & q_{n_1 2} & \cdots & q_{n_1 n_2} \end{pmatrix}.$$

Suppose that $Y := \widehat{B} X \widehat{B}^T + \widehat{Q}$. Then

$$Y = \begin{pmatrix} d+1 & * & ... & * & q_{1n_2} - e \\ -b_{12}d & * & ... & * & b_{12}e + q_{2n_2} \\ ... & ... & ... & * & \\ -b_{1n_1}d & * & ... & * & b_{1n_1}e + q_{n_1 n_2} \\ c_{1n_2}e + q_{11} & * & ... & * & -c_{1n_2}r \\ c_{2n_2}e + q_{12} & * & ... & * & -c_{2n_2}r \\ ... & ... & ... & * & \\ -e + q_{1n_2} & * & ... & * & r+1 \end{pmatrix}.$$

Suppose that $y_1, y_2, ..., y_n$ are the columns of Y and $x_1, x_2, ..., x_n$ are the columns of X. Since X is a solution to $\mathrm{SDLCP}(M_{\widehat{B}}, \widehat{Q})$, $XY = 0$. Therefore for all $i \in \{1, ..., n\}$ and $j \in \{1, ..., n\}$, we must have $y_i^T x_j = 0$.

From the equations $y_1^T x_1 = 0$ and $y_n^T x_n = 0$, we have

$$d(d+1) + e(q_{1n_2} - e) = 0, \tag{2.1}$$

$$r(r+1) + e(q_{1n_2} - e) = 0. \tag{2.2}$$

Equations (2.1) and (2.2) imply that

$$d(d+1) = r(r+1).$$

As d and r are positive, $d = r$. Since X is a *corner* matrix, rank of X must be one and hence

$$d = r = \pm e.$$

Now $d^2 = e^2$, and therefore from (2.1) we have

$$q_{1n_2} = \pm 1.$$

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Let $i \in \{2, \cdots, n_1\}$. Then $y_i^T x_1 = 0$ gives

$$-b_{1i}d^2 + b_{1i}e^2 + q_{in_2}e = 0.$$

As $d^2 = e^2$ and e is nonzero,

$$q_{in_2} = 0.$$

Thus the last column of Q_1 is $(\pm 1, 0, \cdots, 0)^T$.

Let $i \in \{1, \cdots, n_2 - 1\}$. Then

$$c_{in_2}ed + q_{1i}d - c_{in_2}re = 0.$$

Using $r = d$, we have

$$q_{1i} = 0.$$

Thus the first row of Q_1 is $(0, \cdots, 0, \pm 1)$.

Now $\widehat{Q}$ is a *type*(n_1, n_2) matrix and hence Q_1 is of rank one. Thus Q_1 satisfies the conditions of Lemma 2.12 and therefore the submatrix obtained by deleting the first row and last column of Q_1 is a zero matrix. Thus

$$\widehat{Q} = \begin{pmatrix} I_{n-1} & e \\ e^T & 1 \end{pmatrix},$$

where e is the $n-1$ vector $(\pm 1, 0, \cdots, 0)^T$.

If $x \in R^n$, then

$$x^T \widehat{Q} x = (x_1 \pm x_n)^2 + \sum_{i=2}^{n-1} x_i^2 \geq 0.$$

Hence $\widehat{Q} \succeq 0$. This contradicts that $\widehat{Q}$ is a *type*(n_1, n_2) matrix. This completes the proof. $\square$

Lemmas 2.11 and 2.13 now implies the following result.

Lemma 2.14. *Let A be a $form(n_1, n_2)$ matrix. Then M_A cannot have the Q-property.*

We now claim that a skew-symmetric matrix cannot have Q-property.

Lemma 2.15. *If A is a $n \times n$ skew-symmetric matrix, then* SDLCP$(M_A, \widetilde{Q})$ *has no solution.*

Proof. Suppose that X is a solution. Then the rank of X must be one. Therefore $X = xx^T$ for some vector $x \in R^n$. By the skew-symmetry of A, $x^T Ax = 0$; hence $XAX = 0$. Now $X(AXA^T + \widetilde{Q}) = 0$. So $X\widetilde{Q} = 0$. Since $\widetilde{Q}$ is nonsingular, $X = 0$. This implies that $\widetilde{Q} \succeq 0$ (Proposition 2.7) which is a contradiction. $\square$

Lemma 2.16. *Let $A \in R^{n \times n}$. If A is a $form(*)$ matrix, then M_A cannot have the Q-property.*

Proof. Suppose that M_A has the Q-property. Since A is of $form(*)$,

$$A = \begin{pmatrix} W & 0 \\ 0 & B \end{pmatrix},$$

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where W is skew-symmetric of order $k \geq 2$ and B is of order l.

Define a $k \times k$ matrix by

$$Q_{11} = \begin{pmatrix} I_{k-1} & p \\ p^T & 1 \end{pmatrix},$$

where $p := (2, 0, ..., 0)^T$.

Now define

$$Q' = \begin{pmatrix} Q_{11} & 0 \\ 0 & I_l \end{pmatrix}.$$

Note that there exists a permutation matrix P such that $P\widetilde{Q}P^T = Q'$. Suppose that X is a solution to SDLCP(M_A, Q'). Write

$$X = \begin{pmatrix} X_1 & X_2 \\ X_2^T & X_3 \end{pmatrix},$$

where X_1 is of order k.

Suppose that $X_3 = 0$. Then, as $X \succeq 0$, $X_2 = 0$.

Now

$$AXA^T + Q' = \begin{pmatrix} WX_1W^T + Q_{11} & 0 \\ 0 & I_l \end{pmatrix}.$$

It is now easy to verify that X_1 is a solution to SDLCP(M_W, Q_{11}). However by applying the previous lemma, we see that SDLCP(M_W, Q_{11}) has no solution. Thus, we have a contradiction. Therefore X_3 cannot be zero.

In view of Lemma 2.10, rank of X must be one. Hence the rank of X_1 can be at most one and the rank of X_3 is exactly one.

Let U_1 be a orthogonal matrix such that

$$U_1X_1U_1^T = \begin{pmatrix} d & 0 & ... & 0 \\ 0 & 0 & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & 0 \end{pmatrix},$$

and U_2 be a orthogonal matrix such that

$$U_2X_3U_2^T = \begin{pmatrix} 0 & 0 & ... & 0 \\ 0 & 0 & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & r \end{pmatrix}.$$

Define an orthogonal matrix U by

$$U := \begin{pmatrix} U_1 & 0 \\ 0 & U_2 \end{pmatrix}.$$

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Suppose that $Z := UXU^T$. Then by Theorem 2.8

$$Z = \begin{pmatrix} d & 0 & ... & e \\ 0 & 0 & ... & 0 \\ ... & ... & ... & ... \\ e & 0 & ... & r \end{pmatrix}.$$

Note that $r > 0$. Now Z is a solution to SDLCP$(M_{UAU^T}, UQ'U^T)$. Suppose that $Y := M_{UAU^T} + UQ'U^T$.

Now

$$UQ'U^T = \begin{pmatrix} U_1Q_{11}U_1^T & 0 \\ 0 & I_l \end{pmatrix} \text{ and } UAU^T = \begin{pmatrix} U_1WU_1^T & 0 \\ 0 & U_2BU_2^T \end{pmatrix}.$$

Let α be the (n,n)-entry of UBU^T. Clearly, $U_1WU_1^T$ is skew-symmetric. Let the last row of Y be the vector $\mathbf{y} := (y_1, ..., y_n)^T$. Then by a direct verification, $y_1 = 0$ and $y_n = \alpha^2 r + 1$. By the complementarity condition, $\mathbf{y}$ is orthogonal to $(e, 0, ..., 0, r)^T$. Thus, $r(\alpha^2 r + 1) = 0$, which is a contradiction. This completes the proof. ∎

The next result is apparent from Theorem 2.5.8 in Horn and Johnson [4]; hence we omit the proof.

LEMMA 2.17. *Suppose that $A \in R^{n\times n}$ is a nonsingular normal matrix. If A is neither positive definite nor negative definite, then one of the following statements must be true:*

1. *There exists a nonsingular matrix Q and positive integers n_1 and n_2 such that QAQ^T is of $form(n_1, n_2)$.*
2. *There exists a nonsingular matrix Q such that QAQ^T is a $form(*)$ matrix.*
3. *A is skew-symmetric.*

Now the following theorem which is our main result follows from item (2) of Proposition 2.7 and the above results.

THEOREM 2.18. *Let $A \in R^{n\times n}$ be normal. Then the following are equivalent:*

(i) $\pm A$ *is positive definite.*
(ii) SDLCP(M_A, Q) *has a unique solution for all* $Q \in S^n$.
(ii) M_A *has the* Q*-property.*

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