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ON $\mathcal{C}-$COMMUTING GRAPH OF MATRIX ALGEBRA*

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Abstract. Let D be a division ring, $n \geqslant 2$ a natural number, and $\mathcal{C} \subseteq M_n(D)$. Two matrices A and B are called $\mathcal{C}-$commuting if there is $C \in \mathcal{C}$ that $AB - BA = C$. In this paper the $\mathcal{C}-$commuting graph of $M_n(D)$ is defined and denoted by $\Gamma_{\mathcal{C}}(M_n(D))$. Conditions are given that guarantee that the $\mathcal{C}-$commuting graph is connected.

Key words. Division ring, Matrix Algebra, Commuting.

AMS subject classifications. 15A27, 15A33, 16P10.

1. Introduction. Given a graph G, a *path* P is a sequence $v_0e_1v_1e_2 \dots e_kv_k$ whose terms are alternately distinct vertices and distinct edges in G, such that for any i, $1 \leq i \leq k$, the ends of e_i are v_{i-1} and v_i. We say u is *connected to* v in G if there exists a path between u and v. The graph G is *connected* if there exists a path between any two distinct vertices of G. For more details see [2].

Let D be a division ring and $M_n(D)$ be the set of all $n \times n$ matrices over D. As is defined in [1], for $S \subseteq M_n(D)$ the commuting graph of S, denoted by $\Gamma(S)$, is the graph with vertex set $S \backslash Z(S)$ such that distinct vertices A and B are adjacent if and only if $AB = BA$, where $Z(S) = \{\, A \,|\, A \in S,\ AB = BA \text{ for every } B \in S \,\}$.

Let $A \in M_n(D)$. If $A^2 = I$, A is called an *involution*, and A is *reducible* if it has a non-trivial invariant subspace in D^n. It is easily seen that if A is reducible, then there are an invertible matrix P and integers k and m so that $(P^{-1}AP)_{ij} = 0$, for all i and j with $k+1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant m$.

Some properties of commuting graph of $M_n(D)$ were considered in [1]. In particular, we proved the following theorems that are useful in this paper.

THEOREM 1.1. [1, Theorem 1] *Let D be a division ring and $n > 2$ a natural number. If $\mathcal{A}$ is the set of all non-invertible matrices in $M_n(D)$, then $\Gamma(\mathcal{A})$ is a connected graph.*

THEOREM 1.2. [1, Theorem 2] *Let D be a division ring with center F and $n > 1$*

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a natural number. If $A \in M_n(D)$ is a non-cyclic matrix, then A is connected to E_{11} in $\Gamma(M_n(D))$.

In the following we extend the definition of commuting graph and define the $\mathcal{C}-$commuting graph of $M_n(D)$. We prove the connectivity of $\mathcal{C}-$commuting graphs for some special cases of $\mathcal{C}$.

Notation. For a division ring D and $a \in D$, we use $C_D(a)$ for the centralizer of a in D. Also the ring of all $m \times n$ matrices over D is denoted by $M_{m\times n}(D)$, and for simplicity we put $D^n = M_{1\times n}(D)$. The zero matrix, the identity matrix, the zero matrix of size r, and the identity matrix of size r, are denoted by 0, I, 0_r and I_r, respectively, and we use X^t for the transpose of X, for every $X \in D^n$.

2. Main Results.

Throughout this section E_{ij} denotes the matrix in $M_n(D)$ whose (i,j)-entry is 1 and other entries are zero, and e_i denotes the element in D^n whose ith entry is 1 and other entries are zero, for i and j with $1 \leq i, j \leq n$. Also we recall that if $A \in M_n(D)$ is a cyclic matrix, then the representation of A in a special basis has the following form

$$\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ a_1 & a_2 & \cdots & a_{n-1} & a_n \end{pmatrix}$$

for some $a_1, \ldots, a_n \in D$.

Definition 2.1. For a division ring D, $n \in \mathbb{N}$, and $\mathcal{C} \subseteq M_n(D)$, a pair of matrices A and B in $M_n(D)$ is called $\mathcal{C}-$*Commuting* if $AB - BA = C$, for some $C \in \mathcal{C}$.

Thus, if A and B commute, then they are $\{0\}-$commuting.

Definition 2.2. For a division ring D with center F, $n \in \mathbb{N}$, and $\mathcal{C} \subseteq M_n(D)$, the $\mathcal{C}-$*Commuting graph* of $M_n(D)$, denoted by $\Gamma_{\mathcal{C}}(M_n(D))$, is a graph with vertex set $M_n(D)\backslash FI$ such that distinct vertices A and B are adjacent if and only if they are $\mathcal{C}-$Commuting, where $FI = \{\, \alpha I \mid \alpha \in F \,\}$.

Note that the $\{0\}-$Commuting graph of $M_n(D)$ is the commuting graph of $M_n(D)$ that was defined in [1].

Now, we are going to establish basic properties of this graph.

Theorem 2.3. *Let D be a division ring with center F and $n \geqslant 3$ a natural number. Then the following hold:*

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(i) *If D is non-commutative and $\mathcal{C}_1$ is the set of all matrices in $M_n(D)$ such that their ranks are at most 1, then $\Gamma_{\mathcal{C}_1}(M_n(D))$ is a connected graph.*
(ii) *If D is commutative and $\mathcal{C}_2$ is the set of all matrices in $M_n(D)$ such that their ranks are at most 2, then $\Gamma_{\mathcal{C}_2}(M_n(D))$ is a connected graph.*

Proof. Since the zero matrix is in $\mathcal{C}_1$ and $\mathcal{C}_2$, then by Theorem 1.1, each pair of non-invertible matrices are joined by a path in $\Gamma_{\mathcal{C}_1}(M_n(D))$ and $\Gamma_{\mathcal{C}_2}(M_n(D))$. So to prove the theorem it suffices to show that for every non-scalar invertible matrix $A \in M_n(D)$, A is joined to a non-zero, non-invertible matrix in $\Gamma_{\mathcal{C}_1}(M_n(D))$ and $\Gamma_{\mathcal{C}_2}(M_n(D))$. By Theorem 1.2, we may assume that A is a cyclic matrix. So there is an invertible matrix P such that

$$B = P^{-1}AP = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ a_1 & a_2 & \cdots & a_{n-1} & a_n \end{pmatrix},$$

where $a_i \in D$, for i, $1 \leq i \leq n$. To prove (i), let D be a non-commutative division ring and $\alpha \in C_D(a_n)\backslash F$. Then we have the path $B — \alpha I — E_{11}$ in $\Gamma_{\mathcal{C}_1}(M_n(D))$, and so is $A — P(\alpha I)P^{-1} — PE_{11}P^{-1}$. To prove (ii), assume D is commutative and put

$$C = \begin{pmatrix} 1 & a_1^{-1}a_2 \\ 0 & 0 \end{pmatrix} \oplus I_{n-4} \oplus I_2.$$

Then C is a non-zero, non-invertible matrix and it is easily seen that

$$BC - CB = \begin{pmatrix} 0 & -1 & -a_1^{-1}a_2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \oplus 0_{n-3} \in \mathcal{C}_2.$$

So $\operatorname{rank}(BC - CB) = 2$ and also $\operatorname{rank}(A(PCP^{-1}) - (PCP^{-1})A) = 2$, and the proof is complete. □

Remark 2.4. Note that in the proof of Theorem 2.3, we have

$$E = B(\alpha I) - (\alpha I)B = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ b_1 & b_2 & \cdots & b_{n-1} & 0 \end{pmatrix},$$

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where $b_i \in D$, for i, $1 \leq i \leq n-1$, and also

$$G = BC - CB = \begin{pmatrix} 0 & -1 & -a_1^{-1}a_2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \oplus 0_{n-3}.$$

So E, G and consequently PEP^{-1}, PGP^{-1} are non-invertible, triangularizable, reducible, and also nilpotent matrices. Therefore we have the following corollaries.

Corollary 2.5. *Let D be a division ring and $n \geqslant 3$ a natural number. If $\mathcal{A}_n$ is the set of all non-invertible matrices in $M_n(D)$, then $\Gamma_{\mathcal{A}_n}(M_n(D))$ is a connected graph.*

Corollary 2.6. *Let D be a division ring and $n \geqslant 3$ a natural number. If $\mathcal{T}_n$ is the set of all triangularizable matrices in $M_n(D)$, then $\Gamma_{\mathcal{T}_n}(M_n(D))$ is a connected graph.*

Corollary 2.7. *Let D be a division ring and $n \geqslant 3$ a natural number. If $\mathcal{R}_n$ is the set of all reducible matrices in $M_n(D)$, then $\Gamma_{\mathcal{R}_n}(M_n(D))$ is a connected graph.*

Corollary 2.8. *Let D be a division ring and $n \geqslant 3$ a natural number. If $\mathcal{N}_n$ is the set of all nilpotent matrices in $M_n(D)$, then $\Gamma_{\mathcal{N}_n}(M_n(D))$ is a connected graph.*

Theorem 2.9. *Let F be a field with $\operatorname{char} F = 0$ and $n \geqslant 3$ a natural number. If $\mathcal{D}_n$ is the set of all diagonalizable matrices in $M_n(F)$, then $\Gamma_{\mathcal{D}_n}(M_n(F))$ is a connected graph.*

Proof. Since the zero matrix is diagonalizable, by Theorem 1.1, we have each pair of non-invertible matrices is joined by a path in $\Gamma_{\mathcal{D}_n}(M_n(F))$. So to prove the theorem it suffices to show that for every non-scalar invertible matrix $A \in M_n(F)$, A is joined to a non-zero, non-invertible matrix in $\Gamma_{\mathcal{D}_n}(M_n(F))$. By Theorem 1.2, we may assume that A is a cyclic matrix. Hence there is an invertible matrix P such that

$$B = P^{-1}AP = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ a_1 & a_2 & \cdots & a_{n-1} & a_n \end{pmatrix},$$

where $a_i \in F$, for i, $1 \leq i \leq n$. First suppose n is not a multiple of 4. Let $C = \sum_{i=2}^{n}(-1)^i(i-1)E_{i(i-1)}$. We show that $BC - CB$ is a lower triangular matrix that has distinct diagonal entries. For i, j, $1 \leqslant i \leqslant n-1$, $1 \leqslant j \leqslant n$, and $i < j$, we have $(BC-CB)_{ij} = (B)_{i(i+1)}(C)_{(i+1)j} - (C)_{i(i-1)}(B)_{(i-1)j}$. By the definition of B and C,

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$(C)_{(i+1)j} = (B)_{(i-1)j} = 0$. So $BC - CB$ is a lower triangular matrix. Now, assume $2 \leqslant i \leqslant n-1$. Then

$$(BC - CB)_{ii} = (B)_{i(i+1)}(C)_{(i+1)i} - (C)_{i(i-1)}(B)_{(i-1)i}$$

$$= (-1)^{i+1}i - (-1)^i(i-1) = (-1)^{i+1}(2i-1).$$

Also

$$(BC - CB)_{11} = (B)_{12}(C)_{21} - 0 = -1,$$

$$(BC - CB)_{nn} = 0 - (C)_{n(n-1)} = (-1)^{n+1}(n-1).$$

It is easily seen that for i, j, $1 \leqslant i, j \leqslant n$, $i \neq j$, we have

$$(BC - CB)_{ii} \neq (BC - CB)_{jj}.$$

Hence by [3, Theorem 6, p. 204], $BC - CB$ is a diagonalizable matrix. Next, assume $n = 4k$ for some positive integer k. Let

$$C = \sum_{i=2}^{n-1} (-1)^i(i-1)E_{i(i-1)} - (n-1)E_{n(n-1)}.$$

Similarly to the previous case, it can be shown that $BC - CB$ is a lower triangular matrix. Now, for $2 \leqslant i \leqslant n-2$,

$$(BC - CB)_{ii} = (A)_{i(i+1)}(C)_{(i+1)i} - (C)_{i(i-1)}(B)_{(i-1)i}$$

$$= (-1)^{i+1}i - (-1)^i(i-1) = (-1)^{i+1}(2i-1).$$

Also $(BC - CB)_{11} = (B)_{12}(C)_{21} - 0 = 1$ and

$$(BC - CB)_{(n-1)(n-1)} = (B)_{(n-1)n}(C)_{n(n_1)} - (C)_{n(n-1)}(B)_{(n-1)n}$$

$$= -(n-1) + (n-1) = 0.$$

It is easily checked that by [3, Theorem 6, p. 204], $BC - CB$ is diagonalizable and the proof is complete. □

THEOREM 2.10. *Let D be a division ring, $n \geq 3$ a natural number, and $\mathcal{C}$ is a set that includes the zero matrix and all involutions in $M_n(D)$. Then we have the following:*

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(i) *If n is an odd number and D is commutative, then $\Gamma_{\mathcal{C}}(M_n(D))$ is a connected graph if and only if $\Gamma(M_n(D))$ is a connected graph.*

(ii) *If n is an even number, then $\Gamma_{\mathcal{C}}(M_n(D))$ is a connected graph.*

Proof. First, suppose that n is odd and D is commutative. If $char\, F \neq 2$, then it is easily check that a matrix $G \in M_n(D)$ is an idempotent if and only if $2G - I$ is an involution. So if C is an involution, then $H = 2^{-1}(C + I)$ is an idempotent. Hence there is an invertible matrix P such that $P^{-1}HP = I_t \oplus 0_{n-t}$, for some $0 \leq t \leq n$. Therefore $P^{-1}CP = I_t \oplus (-I)_{n-t}$. Now, one can easily seen that the trace of $MN - NM$ is equals to zero, for every M, $N \in M_n(D)$. So if C has the form $MN - NM$, for some M, $N \in M_n(D)$, then $t = n - t$; i.e. $n = 2t$. So in this case the $\mathcal{C}-$commuting elements are $0-$commuting matrices. Now suppose that $char\, F = 2$ and C is an involution. So $C^2 = I$ and therefore the minimal polynomial of C is equal to $x^2 + 1 = x^2 - 1 = (x - 1)(x + 1)$. By [3, Theorem 5, p. 203], C is a triangularizable matrix that has only 1 as its eigenvalue. Since only the trace of commutators is equal to 0, then the $\mathcal{C}-$commuting elements are $0-$commuting matrices, and the result follows. Next, suppose that n is an even number. Since the zero matrix is in $\mathcal{C}$, by Theorem 1.1, we have each pair of non-invertible matrices are joined by a path in $\Gamma_{\mathcal{C}}(M_n(D))$. So to prove the theorem it suffices to show that each non-scalar invertible matrix $A \in M_n(D)$, is joined to a non-zero, non-invertible matrix in $\Gamma_{\mathcal{C}}(M_n(D))$. By Theorem 1.2, we may assume that A is a cyclic matrix. Hence there is an invertible matrix P such that

$$B = P^{-1}AP = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ a_1 & a_2 & \cdots & a_{n-1} & a_n \end{pmatrix},$$

where $a_i \in D$, for i, $1 \leq i \leq n$. Now, let $C = \sum_{k=1}^{\frac{n}{2}} E_{(2k)(2k-1)}$. Since C is non-invertible, then it suffices to show that $BC - CB \in \mathcal{C}$. It is easily checked that

$$BC = \sum_{k=1}^{\frac{n}{2}} E_{(2k-1)(2k-1)} + \sum_{k=1}^{\frac{n}{2}} a_{2k} E_{n(2k-1)},$$

and

$$CB = \sum_{k=1}^{\frac{n}{2}} E_{(2k)(2k)}.$$

So

$$BC - CB = \sum_{k=1}^{n} (-1)^k E_{kk} + \sum_{k=1}^{\frac{n}{2}} a_{2k} E_{n(2k-1)}.$$

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To complete the proof, it suffices to show that the last row of $(BC-CB)^2$ equals $(0,\ldots,0,1)$. For k, $1 \le k \le \frac{n}{2}-2$, $(BC-CB)^2_{n(2k)} = 0$ and $(BC-CB)^2_{n(2k-1)} = a_{2k} - a_{2k} = 0$, and $(BC-CB)^2_{nn} = 1$. This completes the proof. $\square$

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