

A NEW UPPER BOUND FOR THE EIGENVALUES OF THE CONTINUOUS ALGEBRAIC RICCATI EQUATION*

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Abstract. By using majorization inequalities, we propose a new upper bound for summations of eigenvalues (including the trace) of the solution of the continuous algebraic Riccati equation. The bound extends some of the previous results under certain conditions. Finally, we give a numerical example to illustrate the effectiveness of our results.

Key words. Eigenvalue, Majorization inequality, Trace inequality, Algebraic Riccati equation.

AMS subject classifications. 15A24.

1. Introduction. We consider the continuous algebraic Riccati equation (CARE)

$$(1.1) \quad A^T K + KA - KRK = -Q,$$

where $Q \geq 0, R > 0$. The CARE has a maximal positive semidefinite solution K , which is the solution of practical interest (see Theorem 9.4.4 in Lancaster and Rodman [1]). Various bounds for this solution have been presented in [2-12]. These include norm bounds, eigenvalue bounds and matrix bounds. In this paper, we apply majorization inequality methods in Marshall and Olkin [13] to obtain a new upper bound for summations of eigenvalues (including the trace) of the solution of the CARE. The bound is a refinement of an upper bound presented in [10] under certain conditions.

2. A new upper bound for summations of eigenvalues for the solution of the continuous algebraic Riccati equation. Throughout this paper, we use $\mathbb{R}^{n \times n}$ to denote the set of $n \times n$ real matrices. Let $\mathbb{R}^n = \{(x_1, \dots, x_n) : x_i \in \mathbb{R} \text{ for all } i\}$, $\mathbb{R}_{++}^n = \{(x_1, \dots, x_n) : x_i > 0 \text{ for all } i\}$, $D = \{(x_1, \dots, x_n) : x_1 \geq \dots \geq x_n\}$, $D_+ =$

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$\{(x_1, \dots, x_n) : x_1 \geq \dots \geq x_n \geq 0\}$. Suppose $x = (x_1, x_2, \dots, x_n)$ is a real n -element array which is reordered so that its elements are arranged in non-increasing order. i.e., $x_{[1]} \geq x_{[2]} \geq \dots \geq x_{[n]}$. Let $|x| = (|x_1|, |x_2|, \dots, |x_n|)$. For $A = (a_{ij}) \in \mathbb{R}^{n \times n}$, denote by $d(A) = (d_1(A), d_2(A), \dots, d_n(A))$ and $\lambda(A) = (\lambda_1(A), \lambda_2(A), \dots, \lambda_n(A))$ the diagonal elements and the eigenvalues of A , respectively. Let $\text{tr}(A)$ and A^T denote the trace and the transpose of A , respectively, and define $(A)_{ii} = a_{ii} = d_i(A)$ and $\bar{A} = \frac{A+A^T}{2}$. The notation $A > 0$ ($A \geq 0$) is used to denote that A is a symmetric positive definite (semi-definite) matrix.

Let α and β be two real n -element arrays.

If they satisfy

$$\sum_{i=1}^k \alpha_{[i]} \leq \sum_{i=1}^k \beta_{[i]}, \quad k = 1, 2, \dots, n,$$

then it is said that α is weakly majorized by β , which is denoted by $\alpha \prec_w \beta$.

If they satisfy

$$\sum_{i=1}^k \alpha_{[n-i+1]} \geq \sum_{i=1}^k \beta_{[n-i+1]}, \quad k = 1, 2, \dots, n,$$

then it is said that α is weakly submajorized by β , which is denoted by $\alpha \prec^w \beta$.

If $\alpha \prec_w \beta$ and

$$\sum_{i=1}^n \alpha_{[i]} = \sum_{i=1}^n \beta_{[i]},$$

then it is said that α is majorized by β , which is denoted by $\alpha \prec \beta$.

The following lemmas are used to prove the main results.

LEMMA 2.1. [13, p. 141, A.3] *If $x_i, y_i, i = 1, 2, \dots, n$, are two sets of numbers, then*

$$\sum_{i=1}^n x_{[i]} y_{[n-i+1]} \leq \sum_{i=1}^n x_i y_i \leq \sum_{i=1}^n x_{[i]} y_{[i]}.$$

LEMMA 2.2. [13, p. 95, H.3.b] *If $x, y \in D$ and $x \prec_w y$, then for any $k = 1, 2, \dots, n$,*

$$\sum_{i=1}^k x_{[i]} u_{[i]} \leq \sum_{i=1}^k y_{[i]} u_{[i]}, \quad \forall u \in D_+.$$

LEMMA 2.3. [13, p. 95, H.3.c] *If $x, y \in D_+$, $a, b \in D_+$, $(x_1, \dots, x_n) \prec_w (y_1, \dots, y_n)$ and $(a_1, \dots, a_n) \prec_w (b_1, \dots, b_n)$, then*

$$(a_1x_1, \dots, a_nx_n) \prec_w (b_1y_1, \dots, b_ny_n).$$

LEMMA 2.4. [13, p. 218, B.1] *If $A = A^T \in \mathbb{R}^{n \times n}$, then*

$$d(A) \prec \lambda(A),$$

which is equivalent to

$$d(A) \prec_w \lambda(A) \quad \text{and} \quad d(A) \prec^w \lambda(A).$$

LEMMA 2.5. [13, p. 118, A.2.h] *If $x, y \in \mathbb{R}_{++}^n$ and $x \prec_w y$, then for $r < 0$,*

$$(x_1^r, \dots, x_n^r) \prec_w (y_1^r, \dots, y_n^r).$$

REMARK 2.6. If $A = A^T \in \mathbb{R}^{n \times n}$, then we have

$$(2.1) \quad d(A) \prec^w \lambda(A)$$

from Lemma 2.4. Combining (2.1) with Lemma 2.5, we obtain for $A > 0$,

$$(2.2) \quad \left(\frac{1}{d_1(A)}, \dots, \frac{1}{d_n(A)} \right) \prec_w \left(\frac{1}{\lambda_1(A)}, \dots, \frac{1}{\lambda_n(A)} \right).$$

LEMMA 2.7. (Cauchy-Schwartz inequality) *For real numbers a_i and b_i , $i = 1, 2, \dots, n$,*

$$(2.3) \quad \sum_{i=1}^n a_i b_i \leq \left(\sum_{i=1}^n a_i^2 \right)^{\frac{1}{2}} \left(\sum_{i=1}^n b_i^2 \right)^{\frac{1}{2}}.$$

We are now ready to give a new upper bound for summations of eigenvalues (including the trace) of the solution of the continuous algebraic Riccati equation, which improves under certain conditions the following bound in Komaroff [10]: Let K be the positive semi-definite solution of the CARE (1.1). Then for any $l = 1, 2, \dots, n$,

$$(2.4) \quad \sum_{j=1}^l \lambda_{[j]}(K) \leq \frac{l\lambda_{[1]}(\bar{A}) + l\sqrt{\lambda_{[1]}^2(\bar{A}) + \frac{\lambda_{[n]}(R)}{l} \sum_{j=1}^l \lambda_{[j]}(Q)}}{\lambda_{[n]}(R)}.$$

THEOREM 2.8. *Let K be the positive semi-definite solution of the CARE (1.1) and assume that $\bar{A} \geq 0$. Then for any $l = 1, 2, \dots, n$, we have*

$$(2.5) \quad \sum_{j=1}^l \lambda_{[j]}(K) \leq \sum_{j=1}^l \frac{\lambda_{[j]}(\bar{A})}{\lambda_{[n-j+1]}(R)} + l^{\frac{1}{2}} \sqrt{\sum_{j=1}^l \frac{\lambda_{[j]}^2(\bar{A})}{\lambda_{[n-j+1]}^2(R)} + \sum_{j=1}^l \frac{\lambda_{[j]}(Q)}{\lambda_{[n-j+1]}(R)}}.$$

Proof. Since K is the positive semi-definite solution of the CARE (1.1), we have

$$K = U \text{diag}(\lambda_{[1]}(K), \lambda_{[2]}(K), \dots, \lambda_{[n]}(K)) U^T,$$

where $U \in \mathbb{R}^{n \times n}$ is orthogonal. Thus, (1.1) can be written as

$$(2.6) \quad \Lambda_k \tilde{R} \Lambda_k = \Lambda_k \tilde{A} + \tilde{A}^T \Lambda_k + \tilde{Q},$$

where $\tilde{R} = U^T R U$, $\tilde{A} = U^T A U$, $\tilde{Q} = U^T Q U$, $\Lambda_k = \text{diag}(\lambda_{[1]}(K), \lambda_{[2]}(K), \dots, \lambda_{[n]}(K))$. Then from (2.6), for $j = 1, \dots, l$, we have

$$\lambda_{[j]}^2(K) d_j(\tilde{R}) = d_j(\Lambda_k \tilde{R} \Lambda_k) = d_j(\Lambda_k \tilde{A} + \tilde{A}^T \Lambda_k) + d_j(\tilde{Q}) = 2\lambda_{[j]}(K) d_j(\tilde{A}) + d_j(\tilde{Q}).$$

Hence,

$$(2.7) \quad \sum_{j=1}^l \lambda_{[j]}^2(K) = 2 \sum_{j=1}^l \lambda_{[j]}(K) \frac{d_j(\tilde{A})}{d_j(\tilde{R})} + \sum_{j=1}^l \frac{d_j(\tilde{Q})}{d_j(\tilde{R})}.$$

Obviously, we have $d_{[l-j+1]}(\tilde{R}) \geq d_{[n-j+1]}(\tilde{R})$. Furthermore, we have

$$\left(\frac{1}{d_1(\tilde{R})}, \dots, \frac{1}{d_l(\tilde{R})} \right) \prec_w \left(\frac{1}{d_{[n]}(\tilde{R})}, \dots, \frac{1}{d_{[n-l+1]}(\tilde{R})} \right),$$

$$(d_1(\tilde{A}), \dots, d_l(\tilde{A})) \prec_w (d_{[1]}(\tilde{A}), \dots, d_{[l]}(\tilde{A})).$$

Applying Lemma 2.1 and Lemma 2.3 yields

$$(2.8) \quad \sum_{j=1}^l \frac{d_j(\tilde{Q})}{d_j(\tilde{R})} \leq \sum_{j=1}^l \frac{d_{[j]}(\tilde{Q})}{d_{[l-j+1]}(\tilde{R})} \leq \sum_{j=1}^l \frac{d_{[j]}(\tilde{Q})}{d_{[n-j+1]}(\tilde{R})},$$

$$(2.9) \quad \sum_{j=1}^l \frac{d_j(\tilde{A})}{d_j(\tilde{R})} \leq \sum_{j=1}^l \frac{d_{[j]}(\tilde{A})}{d_{[l-j+1]}(\tilde{R})} \leq \sum_{j=1}^l \frac{d_{[j]}(\tilde{A})}{d_{[n-j+1]}(\tilde{R})}.$$

According to Lemma 2.2 and (2.9), it is evident that

$$(2.10) \quad \sum_{j=1}^l \lambda_{[j]}(K) \frac{d_j(\tilde{A})}{d_j(\tilde{R})} \leq \sum_{j=1}^l \lambda_{[j]}(K) \frac{d_{[j]}(\tilde{A})}{d_{[n-j+1]}(\tilde{R})}.$$

By (2.8), (2.10), (2.2), Lemma 2.3 and Lemma 2.4, (2.7) leads to

$$(2.11) \quad \begin{aligned} \sum_{j=1}^l \lambda_{[j]}^2(K) &\leq 2 \sum_{j=1}^l \lambda_{[j]}(K) \frac{d_{[j]}(\bar{A})}{d_{[n-j+1]}(\tilde{R})} + \sum_{j=1}^l \frac{d_{[j]}(\tilde{Q})}{d_{[n-j+1]}(\tilde{R})} \\ &\leq 2 \sum_{j=1}^l \lambda_{[j]}(K) \frac{\lambda_{[j]}(\bar{A})}{\lambda_{[n-j+1]}(R)} + \sum_{j=1}^l \frac{\lambda_{[j]}(Q)}{\lambda_{[n-j+1]}(R)}. \end{aligned}$$

Consequently,

$$\begin{aligned} &\sum_{j=1}^l \lambda_{[j]}^2(K) - 2 \sum_{j=1}^l \lambda_{[j]}(K) \frac{\lambda_{[j]}(\bar{A})}{\lambda_{[n-j+1]}(R)} + \sum_{j=1}^l \frac{\lambda_{[j]}^2(\bar{A})}{\lambda_{[n-j+1]}^2(R)} \\ &\leq \sum_{j=1}^l \frac{\lambda_{[j]}^2(\bar{A})}{\lambda_{[n-j+1]}^2(R)} + \sum_{j=1}^l \frac{\lambda_{[j]}(Q)}{\lambda_{[n-j+1]}(R)}, \end{aligned}$$

which is equivalent to

$$(2.12) \quad \sum_{j=1}^l \left(\lambda_{[j]}(K) - \frac{\lambda_{[j]}(\bar{A})}{\lambda_{[n-j+1]}(R)} \right)^2 \leq \sum_{j=1}^l \frac{\lambda_{[j]}^2(\bar{A})}{\lambda_{[n-j+1]}^2(R)} + \sum_{j=1}^l \frac{\lambda_{[j]}(Q)}{\lambda_{[n-j+1]}(R)}.$$

By the Cauchy-Schwartz inequality (2.3), it can be shown that

$$(2.13) \quad \begin{aligned} \sum_{j=1}^l \left(\lambda_{[j]}(K) - \frac{\lambda_{[j]}(\bar{A})}{\lambda_{[n-j+1]}(R)} \right)^2 &\geq \frac{1}{l} \left(\sum_{j=1}^l \left| \lambda_{[j]}(K) - \frac{\lambda_{[j]}(\bar{A})}{\lambda_{[n-j+1]}(R)} \right| \right)^2 \\ &\geq \frac{1}{l} \left| \sum_{j=1}^l \left(\lambda_{[j]}(K) - \frac{\lambda_{[j]}(\bar{A})}{\lambda_{[n-j+1]}(R)} \right) \right|^2. \end{aligned}$$

Combining (2.12) with (2.13) implies that

$$\left| \sum_{j=1}^l \lambda_{[j]}(K) - \sum_{j=1}^l \frac{\lambda_{[j]}(\bar{A})}{\lambda_{[n-j+1]}(R)} \right| \leq l^{\frac{1}{2}} \sqrt{\sum_{j=1}^l \frac{\lambda_{[j]}^2(\bar{A})}{\lambda_{[n-j+1]}^2(R)} + \sum_{j=1}^l \frac{\lambda_{[j]}(Q)}{\lambda_{[n-j+1]}(R)}}.$$

Therefore,

$$\sum_{j=1}^l \lambda_{[j]}(K) \leq \sum_{j=1}^l \frac{\lambda_{[j]}(\bar{A})}{\lambda_{[n-j+1]}(R)} + l^{\frac{1}{2}} \sqrt{\sum_{j=1}^l \frac{\lambda_{[j]}^2(\bar{A})}{\lambda_{[n-j+1]}^2(R)} + \sum_{j=1}^l \frac{\lambda_{[j]}(Q)}{\lambda_{[n-j+1]}(R)}}. \quad \square$$

COROLLARY 2.9. Let K be the positive semi-definite solution of the CARE (1.1) and assume that $\bar{A} \geq 0$. The trace of matrix K has the bound given by

$$\text{tr}(K) \leq \sum_{j=1}^n \frac{\lambda_{[j]}(\bar{A})}{\lambda_{[n-j+1]}(R)} + n^{\frac{1}{2}} \sqrt{\sum_{j=1}^n \frac{\lambda_{[j]}^2(\bar{A})}{\lambda_{[n-j+1]}^2(R)} + \sum_{j=1}^n \frac{\lambda_{[j]}(Q)}{\lambda_{[n-j+1]}(R)}}.$$

REMARK 2.10. We point out that (2.5) improves (2.4) when $\bar{A} \geq 0$. Actually, if $\bar{A} \geq 0$, noting that for $j = 1, 2, \dots, l$, $\frac{1}{\lambda_{[n-j+1]}(R)} \leq \frac{1}{\lambda_{[n]}(R)}$, then we have

$$\begin{aligned} & \sum_{j=1}^l \frac{\lambda_{[j]}(\bar{A})}{\lambda_{[n-j+1]}(R)} + l^{\frac{1}{2}} \sqrt{\sum_{j=1}^l \frac{\lambda_{[j]}^2(\bar{A})}{\lambda_{[n-j+1]}^2(R)} + \sum_{j=1}^l \frac{\lambda_{[j]}(Q)}{\lambda_{[n-j+1]}(R)}} \\ & \leq l \max_{1 \leq j \leq l} \frac{\lambda_{[j]}(\bar{A})}{\lambda_{[n-j+1]}(R)} + l^{\frac{1}{2}} \sqrt{l \max_{1 \leq j \leq l} \frac{\lambda_{[j]}^2(\bar{A})}{\lambda_{[n-j+1]}^2(R)} + \sum_{j=1}^n \frac{\lambda_{[j]}(Q)}{\lambda_{[n-j+1]}(R)}} \\ & = \frac{l\lambda_{[1]}(\bar{A})}{\lambda_{[n]}(R)} + l \sqrt{\left(\frac{\lambda_{[1]}(\bar{A})}{\lambda_{[n]}(R)}\right)^2 + \frac{1}{l} \sum_{j=1}^l \frac{\lambda_{[j]}(Q)}{\lambda_{[n-j+1]}(R)}} \\ & \leq \frac{l\lambda_{[1]}(\bar{A})}{\lambda_{[n]}(R)} + l \sqrt{\left(\frac{\lambda_{[1]}(\bar{A})}{\lambda_{[n]}(R)}\right)^2 + \frac{1}{l\lambda_{[n]}(R)} \sum_{j=1}^l \lambda_{[j]}(Q)} \\ & = \frac{l\lambda_{[1]}(\bar{A})}{\lambda_{[n]}(R)} + \frac{l}{\lambda_{[n]}(R)} \sqrt{\lambda_{[1]}^2(\bar{A}) + \frac{\lambda_{[n]}(R)}{l} \sum_{j=1}^l \lambda_{[j]}(Q)}. \end{aligned}$$

This implies that (2.5) is better than (2.4) when $\bar{A} \geq 0$.

3. A numerical example. In this section, we present a numerical example to illustrate the effectiveness of the main results.

EXAMPLE 3.1. Let

$$A = \begin{pmatrix} 11 & 2 & 8 & 7 \\ 1 & 9 & 2 & 3 \\ -2 & -1 & 2 & -5 \\ 1 & 4 & 3 & 12 \end{pmatrix}, \quad R = \begin{pmatrix} 10 & -1 & 6 & 2 \\ -1 & 9 & 4 & 3 \\ 6 & 4 & 16 & -5 \\ 2 & 3 & -5 & 12 \end{pmatrix},$$

$$Q = \begin{pmatrix} 2283 & 809 & 1003 & 1022 \\ 809 & 2693 & 1170 & 1423 \\ 1003 & 1170 & 1119 & 374 \\ 1022 & 1423 & 374 & 1458 \end{pmatrix}.$$

Obviously, $\bar{A} \geq 0$.

Case 1: Choose $l = 3$. Using (2.4) yields

$$(3.1) \quad \sum_{j=1}^3 \lambda_{[j]}(K) \leq 183.45.$$

By (2.5), we have

$$\sum_{j=1}^3 \lambda_{[j]}(K) \leq 130.85,$$

which is better than that of (3.1).

Case 2: Choose $l = n = 4$. Using (2.4) yields

$$(3.2) \quad \text{tr}(K) \leq 244.82.$$

By (2.5), we have

$$\text{tr}(K) \leq 148.75,$$

which is better than that of (3.2).

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