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SPECTRUM LOCALIZATION OF REGULAR MATRIX POLYNOMIALS AND FUNCTIONS[*]

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Abstract. This paper is devoted to the spectrum localization problem for regular matrix polynomials and functions. Sufficient conditions are proposed for spectrum placement in a wide class of regions bounded by analytical curves. These conditions generalize the known linear matrix inequalities (LMI) approaches to stability analysis and pole placement of polynomial matrices. In addition, a method of robust spectrum placement is developed in the form of the LMI systems for a parametric set of matrix polynomials.

Key words. Matrix polynomial, Eigenvalue, Spectrum localization, Linear matrix inequality, Robust stability.

AMS subject classifications. 5A18, 15A22, 26C10.

1. Introduction. Many theoretical and applied problems are associated with the analysis of spectral properties of matrix functions including polynomial matrices. If direct evaluation of the eigenvalues does not leads to desirable outcomes, there are problems of their estimation and localization with respect to certain regions of a complex plane. Numerous works are devoted to solving these problems (see, e.g., [1]–[4]).

In this paper, we study the problem of spectrum localization for matrix polynomials and functions. We establish sufficient conditions for checking location of all eigenvalues of regular matrix functions in specified regions of the complex plane. The results obtained essential complement and generalize the known approaches to pole placement of polynomial matrices reduced to the LMI technics (see, e.g., [5] and [3, Section 2.7]). Moreover, we expand the obtained results to a parametric set of matrix polynomials in the form of the LMI systems.

Notation: $\otimes$, $*$, and T denote the matrix operations of the Kronecker product, complex conjugation and transposition, respectively; I_n is the identity $n \times n$ matrix; O_n and O_{nm} denote the zero $n \times n$ and $n \times m$ matrix, respectively; the matrix inequalities $X > Y$, $X \geq Y$, and $X \succeq Y$ mean that the matrix $X - Y$ is po-

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sitive definite, positive semidefinite, and nonzero positive semidefinite, respectively; $i(X) = \{i_+(X), i_-(X), i_0(X)\}$ is the inertia of the Hermitian matrix $X = X^*$ consisting of its numbers of positive ($i_+(X)$), negative ($i_-(X)$), and zero ($i_0(X)$) eigenvalues (taking into account the multiplicities); $\lambda_{\max}(X)$ ($\lambda_{\min}(X)$) is the maximum (minimum) eigenvalue of the Hermitian matrix X.

2. Matrix function and analytical regions. Consider the regular matrix function

$$(2.1) \qquad F(\lambda) = \sum_{i=0}^{m} f_i(\lambda) A_i, \quad \det F(\lambda) \not\equiv 0, \quad \lambda \in \mathbb{C}^1,$$

where $A_0, \ldots, A_m \in \mathbb{C}^{n \times n}$ are $n \times n$ matrices. Let $\sigma(F)$ be a spectrum of the matrix function that consists of all eigenvalues. Suppose that the scalar functions $f_0(\lambda), \ldots, f_m(\lambda)$ are analytical in a neighbourhood of $\sigma(F)$ and let

$$z_m(\lambda) = [f_0(\lambda), \ldots, f_m(\lambda)] \neq 0, \quad \lambda \in \sigma(F).$$

We study a location of the spectrum with respect to regions in the complex plane of the type

$$(2.2) \qquad \Omega = \left\{ \lambda \in \mathbb{C}^1 : i_+(V(\lambda, \overline{\lambda})) \geq 1 \right\}, \quad \hat{\Omega} = \left\{ \lambda \in \mathbb{C}^1 : V(\lambda, \overline{\lambda}) \leq 0 \right\},$$

where $V(\lambda, \overline{\lambda})$ is a Hermitian matrix-valued function. It is obvious that $\Omega \cap \hat{\Omega} = \emptyset$ and $\Omega \cup \hat{\Omega} = \mathbb{C}^1$.

Let's introduce the block matrices

$$A = \begin{bmatrix} A_0 \\ \vdots \\ A_m \end{bmatrix}, \quad G = \begin{bmatrix} G_0 \\ \vdots \\ G_m \end{bmatrix}, \quad L = \begin{bmatrix} L_{00} & \cdots & L_{0m} \\ \vdots & \ddots & \vdots \\ L_{m0} & \cdots & L_{mm} \end{bmatrix}, \quad W = \begin{bmatrix} L & G \\ G^* & H \end{bmatrix}.$$

Here A_i, G_i, L_{ij}, H are the $n \times n$ blocks, and the matrices L and W are Hermitian.

We define the matrix $A^\perp = [B_0, \ldots, B_m]^*$ of size $n(m+1) \times nm$ as a basis for the right null space of A^*. Then

$$(2.3) \qquad A^{\perp *} A = 0, \quad A^+ A = I_n, \quad \det T \neq 0, \quad T = [A^{+*}, A^\perp],$$

where $A^+ = (A^* A)^{-1} A^*$ is the pseudoinverse matrix. Under regularity condition of the matrix function $F(\lambda)$, it is always possible to construct the matrix $A^\perp$ with specified properties.

Let the regions (2.2) be defined by

$$V(\lambda, \overline{\lambda}) = \sum_{i,j=0}^{m} f_i(\lambda) \overline{f_j(\lambda)} L_{ij}.$$

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LEMMA 2.1. *If*

$$(2.4) \qquad \left[I_{n(m+1)}, A\right] W \left[I_{n(m+1)}, A\right]^* > 0$$

or

$$(2.5) \qquad A^{\perp *} L A^{\perp} > 0,$$

then $\sigma(F) \cap \hat{\Omega} = \emptyset$, i.e. all the eigenvalues of $F(\lambda)$ lie within Ω.

Proof. It is obvious that $F(\lambda) = Z_m(\lambda)A$ and $V(\lambda, \overline{\lambda}) = Z_m(\lambda)LZ_m^*(\lambda)$, where $Z_m(\lambda) = [f_0(\lambda)I_n, \ldots, f_m(\lambda)I_n] = z_m(\lambda) \otimes I_n$. We multiply (2.4) on the left and on the right by $Z_m(\lambda)$ and $Z_m^*(\lambda)$, respectively. Then we obtain

$$F(\lambda)G^*(\lambda) + G(\lambda)F^*(\lambda) + F(\lambda)HF^*(\lambda) + V(\lambda, \overline{\lambda}) > 0,$$

where $G(\lambda)$ is a matrix function. Let $v_0^* \neq 0$ be the left eigenvector of the matrix function $F(\lambda)$ corresponding to an eigenvalue $\lambda_0 \in \sigma(F)$. Multiplying the matrix inequality on the left and on the right by v_0^* and v_0, respectively, at $\lambda = \lambda_0$, we have $v_0^* V(\lambda_0, \overline{\lambda}_0)v_0 > 0$, that means $\lambda_0 \in \Omega$.

Let's multiply the matrix inequality (2.4) on the left and on the right by T^* and T, respectively, and use the known criterion of positive definiteness of a block matrix:

$$(2.6) \qquad \begin{bmatrix} P & N \\ N^* & Q \end{bmatrix} > 0 \iff Q > 0, \quad P > NQ^{-1}N^*.$$

In this case,

$$P = H + G^*A^{+*} + A^+G + A^+LA^{+*}, \quad Q = A^{\perp *}LA^{\perp}, \quad N = (G^* + A^+L)A^{\perp}.$$

Since last inequality in (2.6) may be always satisfied by choosing $H > 0$, for inequality performance (2.4), it is enough to demand that $Q > 0$. Therefore, the inclusion $\sigma(F) \subset \Omega$ also follows from (2.5). ∎

Note that, in Lemma 2.1, it is necessary to have the inequalities

$$n(m+1) \leq i_+(W) < n(m+2), \quad 1 \leq i_+(L) < n(m+1).$$

As H, there may be any positive definite matrix, for example, $H = I_n$. If the inequality (2.4) holds for $H \leq 0$, then all the eigenvalues of the matrix function $G(\lambda)$ as well as the $F(\lambda)$ should belong to Ω.

Supposing in Lemma 2.1 that $L = \Gamma \otimes X$ with the Hermitian matrices Γ and X of sizes $(m+1) \times (m+1)$ and $n \times n$, respectively, we obtain conditions when $\sigma(F)$ belongs to a region Ω described by the scalar Hermitian function $f(\lambda, \overline{\lambda}) = z_m(\lambda)\Gamma z_m^*(\lambda)$.

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THEOREM 2.2. *If*

$$(2.7) \qquad AG^* + GA^* + AHA^* + \Gamma \otimes X > 0, \quad X \succeq 0;$$

or

$$(2.8) \qquad \sum_{i,j=0}^{m} \gamma_{ij} B_i X B_j^* > 0, \quad X \succeq 0;$$

then all the eigenvalues of the matrix function $F(\lambda)$ lie within the region

$$\Omega = \left\{ \lambda \in \mathbb{C}^1 : f(\lambda, \overline{\lambda}) = \sum_{i,j=0}^{m} \gamma_{ij} f_i(\lambda) \overline{f_i(\lambda)} > 0 \right\}.$$

Let $f_0(\lambda) \neq 0$ in a neighbourhood of $\sigma(F)$, and let regions Ω and $\hat{\Omega}$ be defined as in (2.2) with

$$V(\lambda, \overline{\lambda}) = \begin{bmatrix} \Psi^*(\lambda) L \Psi(\lambda) & f_0(\lambda) \Psi^*(\lambda) G \\ \overline{f_0(\lambda)} G^* \Psi(\lambda) & f_0(\lambda) \overline{f_0(\lambda)} H \end{bmatrix}, \ \Psi(\lambda) = \begin{bmatrix} -f_1(\lambda) I_n & \cdots & -f_m(\lambda) I_n \\ f_0(\lambda) I_n & \cdots & O_n \\ \vdots & \ddots & \vdots \\ O_n & \cdots & f_0(\lambda) I_n \end{bmatrix}.$$

LEMMA 2.3. *If*

$$(2.9) \qquad [A^*, I_n] W [A^*, I_n]^* > 0,$$

then $\sigma(F) \cap \hat{\Omega} = \emptyset$, i.e. all the eigenvalues of $F(\lambda)$ lie within Ω.

Proof. Let $v_0 \neq 0$ be the right eigenvector of the matrix function $F(\lambda)$ corresponding to an eigenvalue $\lambda_0 \in \sigma(F)$. Multiplying (2.9) on the left and on the right by $\overline{f_0(\lambda_0)} v_0^*$ and $f_0(\lambda_0) v_0$, respectively, and taking into account the relations

$$f_0(\lambda_0) A_0 v_0 = -\sum_{i=1}^{m} f_i(\lambda_0) A_i v_0, \quad w_0^* = [v_0^* A_1^*, \ldots, v_0^* A_m^*, v_0^*] \neq 0,$$

we have $w_0^* V(\lambda_0, \overline{\lambda}_0) w_0 > 0$, that means $\lambda_0 \in \Omega$. ▯

Note that, in Lemma 2.3, it is necessary the inequality $n \leq i_+(W) < n(m+2)$.

THEOREM 2.4. *If*

$$(2.10) \qquad \sum_{i,j=0}^{m} \gamma_{ij} A_i^* X A_j > 0, \quad X \succeq 0,$$

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then all the eigenvalues of the matrix function $F(\lambda)$ lie within the region

$$\Omega = \left\{ \lambda \in \mathbb{C}^1 : i_+(U(\lambda, \overline{\lambda})) \geq 1 \right\},$$

where

$$U(\lambda, \overline{\lambda}) = \Phi^*(\lambda)\Gamma\Phi(\lambda), \quad \Phi(\lambda) = \begin{bmatrix} -z(\lambda) \\ f_0(\lambda)I_m \end{bmatrix}, \quad z(\lambda) = [f_1(\lambda), \ldots, f_m(\lambda)].$$

Theorem 2.4 follows from Lemma 2.3 when G and H are zero, and $L = \Gamma \otimes X$. Similar statement was formulated in [2] under weaker restrictions (controllability type) to the matrix expressions in (2.10). Lemmas 2.1 and 2.3 develop and complement the known approaches to pole placement of matrix functions (see [2, pp. 94–98] and [3, Section 2.7]) by means of the LMI (2.4), (2.5), (2.9) and their special cases (2.7), (2.8) and (2.10).

REMARK 2.5. If we choose Γ in the form

$$\Gamma = \begin{bmatrix} 0 & 1 & O_{1\,m-1} \\ 1 & q & O_{1\,m-1} \\ O_{m-1\,1} & O_{m-1\,1} & -Q^{-1} \end{bmatrix}, \quad Q = Q^* > 0,$$

then Ω in Theorem 2.4 can be described by the scalar function:

$$\Omega = \left\{ \lambda \in \mathbb{C}^1 : f(\lambda, \overline{\lambda}) = z_m(\lambda)\widetilde{\Gamma}z_m^*(\lambda) > 0 \right\}, \quad \widetilde{\Gamma} = \begin{bmatrix} q & -1 & O_{1\,m-1} \\ -1 & 0 & O_{1\,m-1} \\ O_{m-1\,1} & O_{m-1\,1} & Q \end{bmatrix}.$$

EXAMPLE 2.6. Consider the regular matrix quasipolynomial

$$F(\lambda) = A_0 + \lambda A_1 + e^{-\lambda \tau_1} A_2 + \cdots + e^{-\lambda \tau_{m-1}} A_m, \quad \tau_i \geq 0, \quad i = \overline{1, m-1}.$$

If any numbers $q, q_1 > 0, \ldots, q_{m-1} > 0$ and matrix $X \succeq 0$ satisfy the matrix inequality

$$(2.11) \qquad A_0^* X A_1 + A_1^* X A_0 + q A_1^* X A_1 - \sum_{i=1}^{m-1} \frac{1}{q_i} A_{i+1}^* X A_{i+1} > 0,$$

then according to Theorem 2.4 all the eigenvalues of $F(\lambda)$ lie within the region

$$\Omega = \left\{ \lambda \in \mathbb{C}^1 : \lambda + \overline{\lambda} < q + \sum_{i=1}^{m-1} q_i e^{-(\lambda + \overline{\lambda})\tau_i} \right\}.$$

If $q \leq -q_1 - \cdots - q_{m-1}$, then given region is located in the left half-plane. In this case, we have sufficient stability conditions of quasipolynomial $F(\lambda)$ in the form of LMI (2.11) (see [2, 3]). This result can be used in the absolute stability analysis of the linear time-delay control systems.

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3. Matrix polynomial and algebraic regions. Given the regular matrix polynomial of size $n \times n$ and degree s

$$(3.1) \qquad F(\lambda) = A_0 + \lambda A_1 + \cdots + \lambda^s A_s, \quad \det F(\lambda) \not\equiv 0, \quad \lambda \in \mathbb{C}^1,$$

we consider the class of algebraic regions:

$$(3.2) \qquad \Lambda_k = \left\{ \lambda \in \mathbb{C}^1 : f(\lambda, \overline{\lambda}) = \sum_{i,j=0}^{k} \gamma_{ij} \lambda^i \overline{\lambda}^j > 0 \right\},$$

where γ_{ij} are entries of the Hermitian matrix Γ, and $s \geq 1$, $k \geq 1$. We assume that $\Lambda_k \neq \emptyset$ and $\Lambda_k \neq \mathbb{C}^1$, i.e. $i_{\pm}(\Gamma) \neq 0$.

It is obvious that any region Λ_1 is bounded by a line or a circle. In particular, the matrices

$$(3.3) \qquad \Gamma = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, \quad \Gamma = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

correspond to the left half-plane and the unit disk. The class Λ_2 contains all the regions bounded by algebraic curves of the second order.

Let $m = \max\{s, k\}$ and $r = m - k$. We construct the block matrices

$$A = \begin{bmatrix} A_0 \\ \vdots \\ A_m \end{bmatrix}, \quad G = \begin{bmatrix} G_0 \\ \vdots \\ G_m \end{bmatrix}, \quad X = \begin{bmatrix} X_{00} & \cdots & X_{0r} \\ \vdots & \ddots & \vdots \\ X_{r0} & \cdots & X_{rr} \end{bmatrix},$$

of sizes $n(m+1) \times n$, $n(m+1) \times n$ and $n(r+1) \times n(r+1)$, respectively, and introduce the linear operators

$$(3.4) \qquad L(X) = C(\Gamma \otimes X)C^T, \quad M(X) = D(\Gamma \otimes X)D^*,$$

where

$$C = R \otimes I_n = [C_0, \dots, C_k], \quad D = A^{\perp *}C = [D_0, \dots, D_k],$$

$$R = \left[E, \Delta E, \dots, \Delta^k E \right], \quad \Delta = \begin{bmatrix} O_{1m} & 0 \\ I_m & O_{m1} \end{bmatrix}, \quad E = \begin{bmatrix} I_{r+1} \\ O_{k\,r+1} \end{bmatrix},$$

$A^{\perp *} = [B_0, \dots, B_m]$ is the matrix defined by (2.3). In the case of $k > s$, the blocks A_i $(i > s)$ should be chosen in such a way that the spectrum of the matrix polynomial $F_m(\lambda) = A_0 + \lambda A_1 + \cdots + \lambda^m A_m$ contains $\sigma(F)$. For example, we can suppose that $A_i = 0$ for $i > s$.

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Let $\hat{\Lambda}_k \triangleq \mathbb{C}^1 \backslash \Lambda_k$ be the closed complement of Λ_k, and let $\Lambda_r(X)$ be the closed region defined by the block Hermitian matrix X:

$$\Lambda_r(X) = \left\{ \lambda \in \mathbb{C}^1 : Z_r(\lambda) X Z_r^*(\lambda) = \sum_{i,j=0}^{r} \lambda^i \overline{\lambda}^j X_{ij} \geq 0 \right\}, \quad Z_r(\lambda) = [I_n, \lambda I_n, \ldots, \lambda^r I_n].$$

THEOREM 3.1. *Let for some matrices G, $H = H^*$, $X = X^*$, the inclusion*

(3.5)
$$\hat{\Lambda}_k \subseteq \Lambda_r(X)$$

and one of the matrix inequalities

(3.6)
$$AG^* + GA^* + AHA^* + L(X) > 0$$

or

(3.7)
$$M(X) > 0$$

hold. Then all the eigenvalues of the matrix polynomial $F(\lambda)$ lie within the region Λ_k.

Proof. The matrix R of size $(m+1) \times (k+1)(r+1)$ has the following structure:

$$R = \begin{bmatrix} E, \Delta E, \ldots, \Delta^k E \end{bmatrix} = \left[\begin{array}{c|c|c|c} I_{r+1} & O_{1\,r+1} & & O_{1\,r+1} \\ O_{1\,r+1} & I_{r+1} & \cdots & O_{k-1\,r+1} \\ O_{k-1\,r+1} & O_{k-1\,r+1} & & I_{r+1} \end{array} \right].$$

It is obvious that

$$F_m(\lambda) = Z_m(\lambda) A, \quad f(\lambda, \overline{\lambda}) = z_k(\lambda) \Gamma z_k^*(\lambda),$$

where $Z_m(\lambda) = z_m(\lambda) \otimes I_n$, $z_m(\lambda) = [1, \lambda, \ldots, \lambda^m]$. Using the structure of matrix C and the Kronecker products properties, we have

$$Z_m(\lambda) C = (z_m(\lambda) \otimes I_n)(R \otimes I_n) = z_m(\lambda) R \otimes I_n =$$

$$= [z_r(\lambda), \lambda z_r(\lambda), \ldots, \lambda^k z_r(\lambda)] \otimes I_n = z_k(\lambda) \otimes z_r(\lambda) \otimes I_n = z_k(\lambda) \otimes Z_r(\lambda).$$

Multiplying (3.6) on the left and on the right by the full rank matrices $Z_m(\lambda)$ and $Z_m^*(\lambda)$, respectively, we obtain

$$F_m(\lambda) G_m^*(\lambda) + G_m(\lambda) F_m^*(\lambda) + F_m(\lambda) H F_m^*(\lambda) + [z_k(\lambda) \otimes Z_r(\lambda)](\Gamma \otimes X)[z_k^*(\lambda) \otimes Z_r^*(\lambda)] =$$

$$= F_m(\lambda) G_m^*(\lambda) + G_m(\lambda) F_m^*(\lambda) + F_m(\lambda) H F_m^*(\lambda) + f(\lambda, \overline{\lambda}) \, Z_r(\lambda) X Z_r^*(\lambda) > 0,$$

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where $G_m(\lambda) = G_0 + \lambda G_1 + \cdots + \lambda^m G_m$ is a matrix polynomial.

Let $\lambda = \lambda_0 \in \sigma(F)$ be an arbitrary eigenvalue of $F(\lambda)$. Also we assume that $\lambda_0 \notin \Lambda_k$, i.e. $\lambda_0 \in \hat{\Lambda}_k$. Then, according to (3.5), $f(\lambda_0, \overline{\lambda}_0) Z_r(\lambda_0) X Z_r^*(\lambda_0) \leq 0$. Thus,

$$F_m(\lambda_0) G_m^*(\lambda_0) + G_m(\lambda_0) F_m^*(\lambda_0) + F_m(\lambda_0) H F_m^*(\lambda_0) > 0.$$

However, this is not true, because $\sigma(F) \subseteq \sigma(F_m)$ and

$$v_0^*[F_m(\lambda_0) G_m^*(\lambda_0) + G_m(\lambda_0) F_m^*(\lambda_0) + F_m(\lambda_0) H F_m^*(\lambda_0)]v_0 = 0,$$

where $v_0^* \neq 0$ is the left eigenvector of the matrix polynomial $F_m(\lambda)$ corresponding to $\lambda_0 \in \sigma(F_m)$. From this contradiction, it follows that, under conditions (3.5) and (3.6), it should be $\lambda_0 \in \Lambda_k$.

From (3.5) and (3.7), it also follows that $\sigma(F) \subset \Omega$. It is established by multiplying (3.6) on the left and on the right by T and T^*, respectively, and applying the criterion (2.6) (see the proof of Lemma 2.1). □

Theorem 3.1 develops known approach for the spectrum localization of a matrix polynomial proposed in [5] for the class of regions Λ_1. We do not use any restrictions to the order of the algebraic curves. In addition, condition (3.5) generally does not demand the positive definiteness of solutions of the corresponding matrix inequalities. The results obtained in [4] extend the approach of [5] by considering algebraic regions $\mathcal{D}$ described by the quadratic matrix inequalities and involving the optimization algorithms. Every $\mathcal{D}$-region can always be written as a finite set of Ω-regions. However, the converse procedure is nontrivial and unsolved problem for the present.

REMARK 3.2. The proof of Theorem 3.1 can be obtained as a corollary of Lemma 2.1. Indeed, in the case of the matrix polynomial, we suppose that $f_i(\lambda) = \lambda^i$ for $i = \overline{0, m}$, and, in virtue of (3.5), $\Omega \subseteq \Lambda_k$. The restriction (3.5) holds for arbitrary region Λ_k when $\Lambda_r(X) = \mathbb{C}^1$. For example, we can find X as an algebraically nonnegative defined matrix, in particular, nonnegative defined matrix in usual sense.

DEFINITION 3.3. The block Hermitian matrix X is called algebraically positive (nonnegative) defined if

$$Z_r(\lambda) X Z_r^*(\lambda) = \sum_{i,j=0}^{r} \lambda^i \overline{\lambda}^j X_{ij} > 0 \ (\geq 0), \ \ \forall \lambda \in \mathbb{C}^1.$$

Note that all the matrices of the type

$$X = \begin{bmatrix} X_p & O_{n(p+1)\,n(r-p)} \\ O_{n(r-p)\,n(p+1)} & O_{n(r-p)\,n(r-p)} \end{bmatrix}, \ \ X_p = \begin{bmatrix} X_{00} & \cdots & X_{0p} \\ \vdots & \ddots & \vdots \\ X_{p0} & \cdots & X_{pp} \end{bmatrix} > 0,$$

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are algebraically positive defined. These matrices are positive defined when $p = r$.

REMARK 3.4. If $k \geq s$, then $m = k$, $r = 0$, and $C = I_{n(k+1)}$. In this case, in (3.6) and (3.7), we suppose $L(X) = \Gamma \otimes X$ and $M(X) = A^{\perp *}L(X)A^{\perp}$. In addition, in Theorem 3.1, we use the inequality $X = X^* \succeq 0$ instead of (3.5).

REMARK 3.5. Under certain restrictions, the proposed sufficient conditions for spectrum inclusion $\sigma(F) \subset \Lambda_k$ can become necessary conditions. If the LMI (3.6) holds, then $i_+(L(X)) \geq nm$, and consequently (see [3, Theorem 4.2.1]),

$$i_+(\Gamma)i_+(X) + i_-(\Gamma)i_-(X) \geq nm.$$

This inequality is necessary also for feasibility of the LMI (3.7). Particularly, in the case of $k \geq s$ and $X > 0$, it is necessary that $i(\Gamma) = \{k, 1, 0\}$. The necessary and sufficient solvability conditions of (3.7) can be studied via the general theorems on inertia of Hermitian solutions of transformable matrix equations [2, 3]. Especially, such conditions are connected with constraints for the inertia $i(\Gamma)$ and the property of simultaneous reducibility of the matrix coefficients $D_i = A^{\perp *}C_i$ ($i = \overline{0, k}$) to triangular form through common similarity transformation. When $k = 1$, the matrices D_i are quadratic (see, e.g., Theorem 3.7 and Example 3.8 below).

Notice that if $\det A_0 \neq 0$, then in (2.3), $A^{\perp *}$ always can be chosen as

$$(3.8) \qquad A^{\perp *} = \begin{bmatrix} \hat{A}_1 & I_n & \cdots & O_n \\ \vdots & \vdots & \ddots & \vdots \\ \hat{A}_m & O_n & \cdots & I_n \end{bmatrix}, \quad \hat{A}_i = \begin{cases} -A_i A_0^{-1}, & i \leq s, \\ O_n, & i > s. \end{cases}$$

When $k = s$, the operator $M(X)$ in (3.7), in virtue of (3.8), has the block structure:

$$(3.9) \qquad M(X) = \begin{bmatrix} Y_{11}(Z) & \cdots & Y_{1s}(Z) \\ \vdots & \ddots & \vdots \\ Y_{s1}(Z) & \cdots & Y_{ss}(Z) \end{bmatrix}, \quad X = A_0 Z A_0^*,$$

where $Y_{pq}(Z) = \gamma_{00}A_pZA_q^* - \gamma_{p0}A_0ZA_q^* - \gamma_{0q}A_pZA_0^* + \gamma_{pq}A_0ZA_0^*$, $p, q = \overline{1, s}$.

If $\det A_0 = 0$, then we consider, instead of $F(\lambda)$ and Λ_k, the matrix polynomial and region, respectively, as follows:

$$F_\alpha(\lambda) = F(\lambda + \alpha) = \sum_{i=0}^{s} \lambda^i A_{\alpha i}, \quad \Lambda_k^\alpha = \left\{ \lambda \in \mathbb{C}^1 : f(\lambda + \alpha, \overline{\lambda} + \overline{\alpha}) = \sum_{i,j=0}^{k} \gamma_{ij}^\alpha \lambda^i \overline{\lambda}^j > 0 \right\},$$

where $A_{\alpha 0} = F(\alpha)$, $\det A_{\alpha 0} \neq 0$. Since $F(\lambda)$ is a regular matrix polynomial, there is $\alpha \notin \sigma(F)$ with specified properties. Thus, $\sigma(F) \subset \Lambda_k \iff \sigma(F_\alpha) \subset \Lambda_k^\alpha$, and $\sigma(F_\alpha) = \sigma(F) - \alpha$.

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There are various methods of reduction of spectral problems for matrix polynomials to similar problems for linear pencils of matrices (see, for example, [6]–[9]). We use the method [9] based on applying the matrices of type $A^\perp$ satisfying (2.3). Using the block representation $A^{\perp*} = [B_0, \ldots, B_s]$, we construct the linear pencil of matrices

$$(3.10) \qquad D_1 - \lambda D_0 = [B_1, \ldots, B_s] - \lambda [B_0, \ldots, B_{s-1}]$$

and consider the following relations

$$(3.11) \qquad v^* F(\lambda) = 0, \quad v \neq 0;$$

$$(3.12) \qquad u^*(D_1 - \lambda D_0) = 0, \quad u \neq 0;$$

$$(3.13) \qquad u^* A^{\perp*} = v^* Z_s(\lambda), \quad v = B_0^* u \neq 0.$$

The relations (3.11) and (3.12) define the eigenvalues and the corresponding left eigenvectors of matrix polynomial (3.1) and pencil (3.10), respectively.

It is easy to establish equivalence of (3.12) and (3.13). On the other hand, from definition of $A^\perp$ and representations $F(\lambda) = Z_s(\lambda)A$, it follows that (3.11) is equivalent to (3.13) for certain vector $u \neq 0$. Hence, (3.11) and (3.12) are equivalent. Additionally, identical fulfilment of one of equalities (3.11) or (3.12) at every $\lambda \in \mathbb{C}^1$ implies identical fulfilment of other of them. This means that regularity properties of matrix polynomial (3.1) and pencil (3.10) are equivalent, too.

LEMMA 3.6. *Sets of all the various eigenvalues* λ_i *of matrix polynomial* (3.1) *and pencil* (3.10) *coincide, and their corresponding left eigenvectors are related by* $v_i^* = u_i^* B_0$ $(i = \overline{1, l})$.

Now we formulate criteria of inclusion $\sigma(F) \subset \Lambda_1$. In this case,

$$E = \begin{bmatrix} I_s \\ O_{1s} \end{bmatrix}, \quad \Delta = \begin{bmatrix} O_{1s} & 0 \\ I_s & O_{s1} \end{bmatrix},$$

$$(3.14) \qquad M(X) = \gamma_{00} D_0 X D_0^* + \gamma_{01} D_0 X D_1^* + \gamma_{10} D_1 X D_0^* + \gamma_{11} D_1 X D_1^*,$$

where D_0 and D_1 are matrices defined in (3.10). In particular, for matrix (3.8), we have

$$D_0 = \begin{bmatrix} \hat{A}_1 & I_n & \cdots & O_n \\ \vdots & \vdots & \ddots & \vdots \\ \hat{A}_{s-1} & O_n & \cdots & I_n \\ \hat{A}_s & O_n & \cdots & O_n \end{bmatrix}, \quad D_1 = I_{ns}.$$

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Considering Lemma 3.6 and general properties of positive invertible operators in the space of the Hermitian matrices, it is possible to formulate the following statement (see [2, 3]).

THEOREM 3.7. *The inclusion $\sigma(F) \subset \Lambda_1$ holds if and only if there exist Hermitian matrices X and Y satisfying the relations*

(a) $M(X) = Y \geq 0$, $D_0 X D_0^* \geq 0$, $\mathrm{rank}\,[D_1 - \lambda D_0, Y] \equiv ns\ (\lambda \in \hat{\Lambda}_1)$.

   If D_0 is nonsingular, then (a) *is equivalent to each of the following statements:*

(b) *the matrix inequality $M(X) > 0$ has a solution $X > 0$;*

(c) *given any $Y > 0$, the matrix equation $M(X) = Y$ has a solution $X > 0$;*

(d) *operator M is positive invertible with respect to a cone of nonnegative definite matrices.*

The proof of sufficiency of the criterion (a) consists in the following. If we assume that some eigenvalue $\lambda \in \hat{\Lambda}_1$, then, according to (3.12), for the corresponding left eigenvector u^*, the inconsistent relations should be held true:

$$f(\lambda, \overline{\lambda}) \leq 0, \quad u^* D_0 X D_0^* u \geq 0, \quad f(\lambda, \overline{\lambda})\, u^* D_0 X D_0^* u = u^* Y u > 0.$$

Provided that $\sigma(F) \subset \Lambda_1$, the matrices X and Y in criterion (a) always can be constructed in the form [2]

$$X = Z \hat{X} Z^*, \quad Y = D_0 Z \hat{Y} Z^* D_0^*, \quad \mathrm{rank}[D_1 Z, D_0 Z] = \mathrm{rank}(D_0 Z).$$

This fact is established using the Kronecker canonical form of a regular matrix pencil [11]. If D_0 is nonsingular, then the criteria (b), (c) and (d) follow from known theorems on localization of eigenvalues by means of the generalized Lyapunov equation.

EXAMPLE 3.8. Let $F(\lambda) = A_0 + \lambda A_1$ be a regular pencil of $n \times n$ matrices, and Λ_1 be a region of the form (3.2). The matrix inequality (3.6) in Theorem 3.1 may be expressed as

$$\left[\begin{array}{cc} A_0 H A_0^* + A_0 G_0^* + G_0 A_0^* + \gamma_{00} X & A_0 H A_1^* + A_0 G_1^* + G_0 A_1^* + \gamma_{01} X \\ A_1 H A_0^* + A_1 G_0^* + G_1 A_0^* + \gamma_{10} X & A_1 H A_1^* + A_1 G_1^* + G_1 A_1^* + \gamma_{11} X \end{array} \right] > 0.$$

The matrix inequality (3.7) constructed for the linear pencil $F_\alpha(\lambda) = F(\alpha) + \lambda A_1$ and the regions $\Lambda_1^\alpha = \{\lambda \in \mathbb{C}^1 : f(\lambda + \alpha, \overline{\lambda} + \overline{\alpha}) > 0\}$ in virtue of $A^{\perp *} = [B_0, B_1] = [-A_1 F^{-1}(\alpha), I_n]$ is reduced to the form

$$M(X) = \gamma_{00} A_1 Z A_1^* - \gamma_{10} A_0 Z A_1^* - \gamma_{01} A_1 Z A_0^* + \gamma_{11} A_0 Z A_0^* > 0,$$

where $X = F(\alpha) Z F^*(\alpha)$, $\alpha \notin \sigma(F)$. The given inequality due to equivalence of the inclusions $\sigma(F) \subset \Lambda_1$ and $\sigma(F_\alpha) \subset \Lambda_1^\alpha$, we use in Theorems 3.1 and 3.7 for initial

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matrix pencil $F(\lambda)$ and regions Λ_1. In particular, existence of solutions $Z = Z^* > 0$ of the matrix inequalities

$$A_0 Z A_1^* + A_1 Z A_0^* > 0, \quad A_1 Z A_1^* - A_0 Z A_0^* > 0$$

is equivalent to location of the spectrum $\sigma(F)$ inside the left half-plane and the unit disk, respectively.

EXAMPLE 3.9. Let $F(\lambda) = A_0 + \lambda A_1 + \lambda^2 A_2$ be a quadratic pencil of $n \times n$ matrices, and Λ_k be a region of the type (3.2) for $k \le 2$. Then the expression $AG^* + GA^* + AHA^*$ in (3.6) has the block form

$$\begin{bmatrix} A_0 G_0^* + G_0 A_0^* + A_0 H A_0^* & A_0 G_1^* + G_0 A_1^* + A_0 H A_1^* & A_0 G_2^* + G_0 A_2^* + A_0 H A_2^* \\ A_1 G_0^* + G_1 A_0^* + A_1 H A_0^* & A_1 G_1^* + G_1 A_1^* + A_1 H A_1^* & A_1 G_2^* + G_1 A_2^* + A_1 H A_2^* \\ A_2 G_0^* + G_2 A_0^* + A_2 H A_0^* & A_2 G_1^* + G_2 A_1^* + A_2 H A_1^* & A_2 G_2^* + G_2 A_2^* + A_2 H A_2^* \end{bmatrix}.$$

For the class of regions Λ_1, we have $1 = k < s = 2$, $m = 2$, $r = 1$,

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \Delta = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad R = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad X = \begin{bmatrix} X_{00} & X_{01} \\ X_{10} & X_{11} \end{bmatrix},$$

$$L(X) = \begin{bmatrix} \gamma_{00} X_{00} & \gamma_{00} X_{01} + \gamma_{01} X_{00} & \gamma_{01} X_{01} \\ \gamma_{00} X_{10} + \gamma_{10} X_{00} & \sum_{i,j=0}^{1} \gamma_{ij} X_{1-i1-j} & \gamma_{01} X_{11} + \gamma_{11} X_{01} \\ \gamma_{10} X_{10} & \gamma_{10} X_{11} + \gamma_{11} X_{10} & \gamma_{11} X_{11} \end{bmatrix}.$$

For the class of regions Λ_2, in (3.6) and (3.9), we use the operators

$$L(X) = \begin{bmatrix} \gamma_{00} X & \gamma_{01} X & \gamma_{02} X \\ \gamma_{10} X & \gamma_{11} X & \gamma_{12} X \\ \gamma_{20} X & \gamma_{21} X & \gamma_{22} X \end{bmatrix}, \quad M(X) = \begin{bmatrix} Y_{11}(Z) & Y_{12}(Z) \\ Y_{21}(Z) & Y_{22}(Z) \end{bmatrix},$$

where

$$Y_{11}(Z) = \gamma_{00} A_1 Z A_1^* - \gamma_{10} A_0 Z A_1^* - \gamma_{01} A_1 Z A_0^* + \gamma_{11} A_0 Z A_0^*,$$
$$Y_{12}(Z) = \gamma_{00} A_1 Z A_2^* - \gamma_{10} A_0 Z A_2^* - \gamma_{02} A_1 Z A_0^* + \gamma_{12} A_0 Z A_0^*,$$
$$Y_{22}(Z) = \gamma_{00} A_2 Z A_2^* - \gamma_{20} A_0 Z A_2^* - \gamma_{02} A_2 Z A_0^* + \gamma_{22} A_0 Z A_0^*.$$

EXAMPLE 3.10. Let $F(\lambda) = a_0 + \lambda a_1 + \cdots + \lambda^s a_s$ be a scalar polynomial of degree s. The matrix inequality (3.6) has the form

(3.15) $$ag^* + ga^* + haa^* + L(X) > 0,$$

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where $a = [a_0, \ldots, a_s]^*$, $h \in \mathbb{R}^1$, $g \in \mathbb{C}^{s+1}$. Specifying $L(X)$ for the class of regions Λ_1, we use the matrices

$$E = \begin{bmatrix} I_s \\ O_{1s} \end{bmatrix}, \quad \Delta = \begin{bmatrix} O_{1s} & 0 \\ I_s & O_{s1} \end{bmatrix}, \quad R = \left[\begin{array}{c|c} I_s & O_{1s} \\ O_{1s} & I_s \end{array} \right], \quad X = \|x_{ij}\|_{i,j=0}^{s-1}.$$

In particular, for the left half-plane and the unit disk defined by corresponding matrices (3.3), we have

$$L(X) = - \begin{bmatrix} 0 & x_{00} & \cdots & x_{0s-2} & x_{0s-1} \\ x_{00} & x_{01} + x_{10} & \cdots & x_{0s-1} + x_{1s-2} & x_{1s-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ x_{s-20} & x_{s-10} + x_{s-21} & \cdots & x_{s-1s-2} + x_{s-2s-1} & x_{s-1s-1} \\ x_{s-10} & x_{s-11} & \cdots & x_{s-1s-1} & 0 \end{bmatrix}$$

and

$$L(X) = \begin{bmatrix} x_{00} & x_{01} & \cdots & x_{0s-1} & 0 \\ x_{10} & x_{11} - x_{00} & \cdots & x_{1s-1} - x_{0s-2} & -x_{0s-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ x_{s-10} & x_{s-11} - x_{s-20} & \cdots & x_{s-1s-1} - x_{s-2s-2} & -x_{s-2s-1} \\ 0 & -x_{s-10} & \cdots & -x_{s-1s-2} & -x_{s-1s-1} \end{bmatrix}.$$

The operator $L(X) = x\Gamma$ is a matrix-valued function of scalar argument $X = x$ for any region Λ_k at $k \geq s$.

Note that in [10, Theorem 1], it is obtained a LMI characterization of the roots of polynomials inclusion into regions Λ_1, which is similar to (3.15) with other form of the operator $L(X)$.

4. Robust spectrum localization. In applications, the robust stability and the robust spectrum localization problems formulated for dynamic systems with parametric uncertainty are very important (see, for example, [12]–[14]). Solving these problems, there can be useful results formulated above for matrix functions.

As an example, we consider the parametric set of regular matrix polynomials:

$$(4.1) \quad F(\lambda, p) = A_0(p_0) + \lambda A_1(p_1) + \cdots + \lambda^s A_s(p_s), \quad \det F(\lambda, p) \not\equiv 0, \ \lambda \in \mathbb{C}^1,$$

The values of coefficient matrices $A_i(p_i)$ depending on vector parameters

$$p_i = [p_{i1}, \ldots, p_{i\nu_i}]^T \in \mathcal{P}_{\nu_i} \triangleq \left\{ q \in \mathbb{R}^{\nu_i} : q_1 \geq 0, \ldots, q_{\nu_i} \geq 0, \ \sum_{j=1}^{\nu_i} q_j = 1 \right\},$$

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constitute a set of the matrix polytopes

$$A_i = \left\{ A \in \mathbb{C}^{n \times n} : A = \sum_{j=1}^{\nu_i} p_{ij} A_{ij}, \ p_i \in \mathcal{P}_{\nu_i} \right\}, \quad i = \overline{0, s}.$$

The general vector of parameters $p = [p_0^T, \ldots, p_s^T]^T \in \mathcal{P} = \mathcal{P}_{\nu_0} \times \cdots \times \mathcal{P}_{\nu_s}$ has order $\nu = \nu_0 + \cdots + \nu_s$.

We specify all the $\nu_0 \cdots \nu_s$ polynomial matrix vertices as follows:

$$F_{t_0 \cdots t_s}(\lambda) = A_{0 t_0} + \lambda A_{1 t_1} + \cdots + \lambda^s A_{s t_s}, \quad t_i \in \{1, \ldots, \nu_i\}, \quad i = \overline{0, s}.$$

If $\nu_i = 1$ for some i, then $t_i = 1$, and A_i in (4.1) does not depend on p_i.

LEMMA 4.1. *If $p \in \mathcal{P}_\nu$, then $pp^T \leq P = \mathrm{diag}\{p_1, \ldots, p_\nu\}$.*

Proof. For any vector $x \in R^\nu$, we have

$$x^T(P - pp^T)x = \sum_{i=1}^{\nu} p_i x_i^2 - \left(\sum_{i=1}^{\nu} p_i x_i \right)^2 = \left(\sum_{i=1}^{\nu} p_i x_i^2 \right) \left(\sum_{j=1}^{\nu} p_j \right) - \left(\sum_{i=1}^{\nu} p_i x_i \right)^2 =$$

$$= \sum_{i \neq j} p_i p_j x_i^2 - \sum_{i \neq j} p_i p_j x_i x_j = \sum_{i < j} p_i p_j (x_i - x_j)^2 \geq 0.$$

Hence, $pp^T \leq P$. ☐

THEOREM 4.2. *Let G, $H = H^* \leq 0$ and $X_{t_0 \cdots t_s} = X_{t_0 \cdots t_s}^*$ satisfy (3.5) and the system of matrix inequalities*

$$(4.2) \qquad A_{t_0 \cdots t_s} G^* + G A_{t_0 \cdots t_s}^* + A_{t_0 \cdots t_s} H A_{t_0 \cdots t_s}^* + L(X_{t_0 \cdots t_s}) > 0,$$

where

$$A_{t_0 \cdots t_s} = \begin{bmatrix} A_0 \\ \vdots \\ A_m \end{bmatrix}, \quad A_i = \begin{cases} A_{it_i}, & i \leq s \\ O_n, & i > s \end{cases}, \quad t_i \in \{1, \ldots, \nu_i\}, \quad i = \overline{0, s}.$$

Then for any $p \in \mathcal{P}$, all the eigenvalues of $F(\lambda, p)$ belong to region (3.2).

Proof. We show that for any $p \in \mathcal{P}$,

$$(4.3) \qquad A(p) G^* + G A(p)^* + A(p) H A(p)^* + L(X(p)) > 0,$$

where

$$A(p) = \begin{bmatrix} A_0 \\ \vdots \\ A_m \end{bmatrix}, \quad A_i = \begin{cases} A_i(p_i), & i \leq s \\ O_n, & i > s \end{cases}, \quad t_i \in \{1, \ldots, \nu_i\}, \quad i = \overline{0, s},$$

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$X(p)$ is a nonnegative linear combination of the matrices $X_{t_0 \cdots t_s}$ satisfied (3.5).

The inequalities (4.2) are ordered by sets of indexes $\{t_0 \cdots t_s\}$. We specify the indexes $i = 0$ and $t_j \in \{1, \ldots, \nu_j\}$, $j \neq 0$, and multiply the ν_0 inequalities (4.2) corresponding to the sets of indexes $\{1, t_1 \cdots t_s\}$, ..., $\{\nu_0, t_1 \cdots t_s\}$ accordingly to p_{01}, ..., $p_{0\nu_0}$ and summarize them in consideration of $p_0 = [p_{01}, \ldots, p_{0\nu_0}]^T \in \mathcal{P}_{\nu_0}$. For various combinations of indexes $t_j \in \{1, \ldots, \nu_j\}$, $j \neq 0$, we will perform the same operations. We will use the received inequalities in (4.2) instead of already considered inequalities. Thus, all the inequalities of the given system can be ordered by sets of indexes $\{t_1 \cdots t_s\}$, where $t_j \in \{1, \ldots, \nu_j\}$, $j = \overline{1, s}$. We will perform similar procedure for every $i = \overline{1, s}$ using the vectors $p_i = [p_{i1}, \ldots, p_{i\nu_i}]^T \in \mathcal{P}_{\nu_i}$. Finally, we will receive one block inequality and put the corresponding expressions $A_i(p_i) H A_i(p_i)^*$ in the first $s + 1$ diagonal blocks. As a result, we will obtain

$$A(p)G^* + GA(p)^* + A(p)HA(p)^* + S(p) + L(X(p)) > 0,$$

where $S(p)$ is a block-diagonal matrix with the diagonal blocks

$$S_i = \begin{cases} \sum\limits_{j=1}^{\nu_i} p_{ij} A_{ij} H A_{ij}^* - A_i(p_i) H A_i(p_i)^*, & i \leq s \\ O_n, & i > s \end{cases}, \quad i = \overline{1, m}.$$

Note that for $i \leq s$, we have

$$S_i = W_i \big[(P_i - p_i p_i^T) \otimes H \big] W_i^*, \quad W_i = [A_{i1}, \ldots, A_{i\nu_i}], \quad P_i = \mathrm{diag}\{p_{i1}, \ldots, p_{i\nu_i}\}.$$

According to Lemma 4.1, $P_i \geq p_i p_i^T$ for $p_i \in \mathcal{P}_i$. Using the following property of the Kronecker product

$$P = P^* \geq 0, \quad Q = Q^* \leq 0 \implies P \otimes Q \leq 0,$$

we get that $S(p) \leq 0$ for $p \in \mathcal{P}$. Hence, the matrix inequality (4.3) holds, and according to Theorem 3.1, all the eigenvalues of the matrix polynomial (4.1) lie in the region (3.2) for any $p \in \mathcal{P}$. $\square$

REMARK 4.3. If (4.2) holds for $H = H^* \leq 0$, then it is true for $H = O_n$ also. Therefore, we can suppose that $H = O_n$ always when Theorem 4.2 is used. If $H = H^* < 0$, then (4.3) is equivalent to the block inequality

(4.4)
$$\begin{bmatrix} A(p)G^* + GA(p)^* + L(X(p)) & A(p) \\ A(p)^* & -H^{-1} \end{bmatrix} > 0.$$

The inequalities (4.2) are represented in similar form. Dependence on vector parameter p in (4.4) is linear. Therefore, in this case, it is possible to prove Theorem 4.2 without usage of Lemma 4.1.

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Theorem 4.2 can be used for parametrical and interval sets of regular matrix polynomials of the type

$$(4.5) \qquad F(\lambda, p) = \sum_{i=1}^{\nu} p_i (A_{0i} + \lambda A_{1i} + \cdots + \lambda^s A_{si}), \quad p \in \mathcal{P}_\nu, \quad \lambda \in \mathbb{C}^1,$$

$$(4.6) \quad F(\lambda) = A_0 + \lambda A_1 + \cdots + \lambda^s A_s, \quad \underline{A}_i \le A_i \le \overline{A}_i, \quad i = \overline{0, s}, \quad \lambda \in \mathbb{C}^1.$$

The set (4.1) is reduced to the form (4.5) when all the vectors $p_i \in \mathcal{P}_{\nu_i}$ have the same dimension ν and coincide. In this case, the system (4.2) consists from ν matrix inequalities. The interval set (4.6) is used most often in applications. It is described in the form (4.1) also. Indeed, for this purpose in (4.1), it is necessary to suppose that

$$A_i = A_i(p_i) = \sum_{j=1}^{\nu_i} p_{ij} A_{ij}, \quad A_{ij} = \|a_{t\tau}^{ij}\|_{t,\tau=1}^n, \ a_{t\tau}^{ij} \in \left\{ \underline{a}_{t\tau}^i, \overline{a}_{t\tau}^i \right\}, \ \nu_i = 2^{n^2},$$

$$\underline{A}_i = \|\underline{a}_{t\tau}^i\|_{t,\tau=1}^n, \quad \overline{A}_i = \|\overline{a}_{t\tau}^i\|_{t,\tau=1}^n, \quad p_i = [p_{i1}, \ldots, p_{i\nu_i}]^T \in \mathcal{P}_{\nu_i}, \quad i = \overline{0, s}.$$

Then the system (4.2) includes $2^{(s+1)n^2}$ matrix inequalities.

Note that, using Lemmas 2.1 and 2.3, we can obtain the generalizations and analogues of Theorem 4.2 for a parametric set of regular matrix functions of the type

$$F(\lambda, p) = \sum_{i=0}^{m} f_i(\lambda) A_i(p_i), \quad \det F(\lambda, p) \not\equiv 0, \ \lambda \in \mathbb{C}^1, \quad p = [p_0^T, \ldots, p_s^T]^T \in \mathcal{P}$$

with the corresponding class of regions Ω.

EXAMPLE 4.4. Consider the mechanical system in Figure 4.1 studied in [14]. The system is described by the following differential equations:

$$(4.7) \qquad \begin{cases} m_1 \ddot{x}_1 + d_1 \dot{x}_1 + (c_1 + c_{12}) x_1 - c_{12} x_2 = 0, \\ m_2 \ddot{x}_2 + d_2 \dot{x}_2 + (c_2 + c_{12}) x_2 - c_{12} x_1 = u. \end{cases}$$

In [5], the robust stability analysis of the system was carried out with the root-locus inclusion demands in the disk of radius 12 centered at (-12,0) for all admissible interval uncertainty of parameters $\underline{a} \le a = [c_1, c_2, d_1, d_2, m_1, m_2] \le \overline{a}$.

We choose the stability region located in the left half plane:

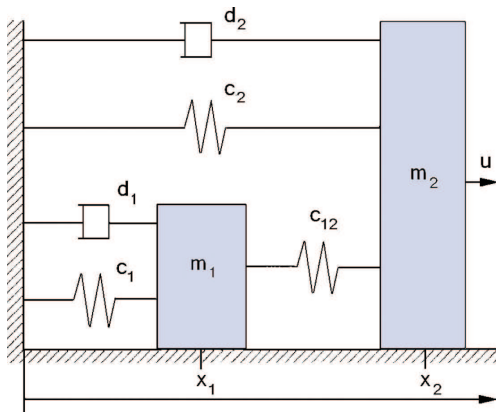$$(4.8) \qquad \Lambda_2 = \{\lambda \in \mathbb{C}^1 : z(\lambda) \Gamma z^*(\lambda) > 0\},$$
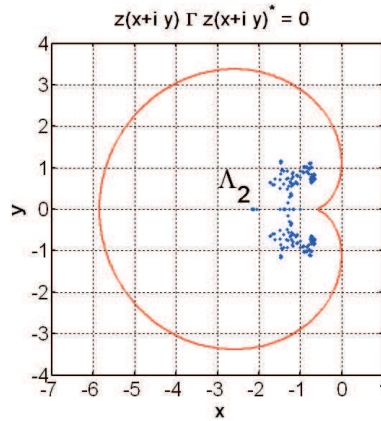
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FIG. 4.1. *The mechanical system.*

FIG. 4.2. *The stability region Λ_2.*

where

$$z(\lambda) = [1, \lambda, \lambda^2], \quad \Gamma = - \begin{bmatrix} \frac{9}{16}\alpha^4 & \frac{7}{4}\alpha^3 & \frac{9}{4}\alpha^2 \\ \frac{7}{4}\alpha^3 & 3\alpha^2 & 3\alpha \\ \frac{9}{4}\alpha^2 & 3\alpha & 1 \end{bmatrix}, \quad \alpha > 0.$$

A boundary of Λ_2 is the 4-th order curve called as the Pascal's limacon (see Figure 4.2) and defined by the equation

$$[(x+h)^2 + y^2 + 2\alpha(x+h)]^2 - \beta^2[(x+h)^2 + y^2] = 0,$$

where $\beta = 2\alpha$ and $h = \alpha/2$. We set

$$\alpha = 1.3, \quad \underline{a} = [5, 6, 6, 9, 2, 4], \quad \overline{a} = [6, 7, 7, 10, 4, 7], \quad c_{12} = 1.$$

Then the interval matrix polynomial (4.6) corresponding to the open-loop system is given by $F(\lambda) = A_0 + \lambda A_1 + \lambda^2 A_2$, where

$$\underline{A}_0 = \begin{bmatrix} 6 & -1 \\ -1 & 7 \end{bmatrix} \leq A_0 = \begin{bmatrix} c_1 + 1 & -1 \\ -1 & c_2 + 1 \end{bmatrix} \leq \overline{A}_0 = \begin{bmatrix} 7 & -1 \\ -1 & 8 \end{bmatrix},$$

$$\underline{A}_1 = \begin{bmatrix} 6 & 0 \\ 0 & 9 \end{bmatrix} \leq A_1 = \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix} \leq \overline{A}_1 = \begin{bmatrix} 7 & 0 \\ 0 & 10 \end{bmatrix},$$

$$\underline{A}_2 = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} \leq A_2 = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \leq \overline{A}_2 = \begin{bmatrix} 4 & 0 \\ 0 & 7 \end{bmatrix}.$$

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The system (4.2) to be solved with respect to $X_{t_0 t_1 t_2}$ ($t_0, t_1, t_2 \in \{1, \ldots, 4\}$), G and H consists of the 64 matrix inequalities. Assuming that $G = (\underline{A} + \overline{A})/2$ and $H = 0$, with the help of MATLAB, we find this system feasible. Hence, all the eigenvalues of the stable open-loop mechanical system remain in the domain (4.8) for the specified interval uncertainties. Figure 4.2 shows the roots location of the 64 matrix polynomials $F_{t_0 t_1 t_2}(\lambda)$ with respect to the Pascal's limacon.

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