The Electronic Journal of Linear Algebra.
A publication of the International Linear Algebra Society.
Volume 5, pp. 24-38, February 1999.
ISSN 1081-3810. http://math.technion.ac.il/iic/ela

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COMPARISON THEOREMS FOR WEAK NONNEGATIVE SPLITTINGS OF K-MONOTONE MATRICES*

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Abstract. The comparison of the asymptotic rates of convergence of two iteration matrices induced by two splittings of the same matrix has arisen in the works of many authors. In this paper new comparison theorems for weak nonnegative splittings of K-monotone matrices are derived which extend some results on regular splittings by Csordas and Varga (1984) for weak nonnegative splittings of the same or different types.

Key words. nonsingular matrix, iterative methods, spectral radius, proper cone, K-nonnegative matrix, K-monotone matrix, comparison conditions, weak nonnegative splitting.

AMS subject classification. 65J10

1. Introduction.

For the iterative solution of a linear system of equations

$$Ax = b \tag{1.1}$$

where A is a nonsingular matrix, it is customary to consider a splitting $A = M - N$ and the iterative scheme

$$x^{(k+1)} = M^{-1}Nx^{(k)} + M^{-1}b, \quad k \geq 0, \tag{1.2}$$

where $x^{(0)}$ is the initial guess.

It is well known that scheme (1.2) converges to the unique solution of system (1.1) for all $x^{(0)}$ if and only if $\rho(M^{-1}N) < 1$, where $\rho(M^{-1}N)$ denotes the spectral radius of the iteration matrix $M^{-1}N$. The rate of convergence of scheme (1.2) depends on $\rho(M^{-1}N)$; see, for example, Berman and Plemmons [2], Varga [18], or Young [20].

In 1960, Varga [18, Theorem 3.15] introduces results for the case where A is a monotone matrix which allow us to establish, which of two regular splittings converges faster. In 1973 Woźnicki (see for example [19, Theorem 5.1]) presents a more general result, also for regular splittings. With these, we have a series of comparison results which are easy to check from a practical point of view as they depend on matrices which are easy to compute in many of the iterative methods of the form (1.2). In recent years, some authors, such as Beauwens [1], Miller and Neumann [11], and Song [15, 16] introduce new general comparison conditions for weak splittings of the first type (see Definition 2.1 below) without requiring the matrix A to be monotone. These conditions, although quite complex to check numerically, are sometimes useful in proving convergence of other methods; see for example, Lanzkron, Rose, and Szyld [8]. Others authors, such as Elsner [6], Marek and Szyld [10], and Nabben [12] introduce some comparison theorems for different types of matrices and splittings.

In Section 3, we generalize the result introduced by Csordas and Varga [5, Theorem 2] for regular splittings of matrices to weak nonnegative splittings with respect

*Received by the editors on 26 September 1997. Accepted for publication on 10 February 1999. Handling Editor: Daniel B. Szyld.

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to a general proper cone K. We also present new comparison conditions for weak nonnegative splittings of the same or different type (also, with respect to K). These conditions use the transpose of a matrix, in a way similar to the results of Woźnicki [19]. Here we do not need the hypothesis that the matrix A be symmetric, cf. Climent and Perea [3]. Some of the results presented in Section 3 were introduced by Climent and Perea [4] for the case $K = \mathbb{R}^n_+$.

The main result of the paper is Theorem 3.5, where we generalize the partial reciprocal introduced by Csordas and Varga [5, Theorem 4] for regular splittings, for weak nonnegative splittings (with respect to a cone K) of the same or different type.

Finally, in Section 4 we establish some relations between the comparison conditions introduced by Climent and Perea [3] and the comparison conditions introduced in Section 3.

2. Preliminaries. The following notation will be used throughout the paper. Denote by K a proper cone in $\mathbb{R}^n$ (see Berman and Plemmons [2] and Kreĭn and Rutman [7] for the infinite dimensional case) and let $\mathrm{int}(K)$ be the interior of K. A vector x in $\mathbb{R}^n$ is called K-nonnegative (respectively, K-positive) if x belongs to K (respectively, x belongs to $\mathrm{int}(K)$), denoted $x \geq 0$ (respectively, $x > 0$). The $n \times n$ real matrix A is called K-nonnegative (respectively, K-positive) if $AK \subseteq K$ (respectively, $A(K \setminus \{0\}) \subseteq \mathrm{int}(K)$) and denoted $A \geq 0$ (respectively, $A > 0$). Similarly, for A and B $n \times n$ real matrices we denote $A - B \geq 0$ (respectively, $A - B > 0$) by $A \geq B$ (respectively, $A > B$). A is K-monotone if $A^{-1} \geq 0$, and $A \geq 0$ is K-irreducible if A has exactly one (up to scalar multiples) eigenvector in K, and this vector belongs to $\mathrm{int}(K)$ (see Berman and Plemmons [2]). Recall that when we consider the particular case $K = \mathbb{R}^n_+$, that is, the set of all vectors with nonnegative entries, then $A \geq 0$ (respectively, $A > 0$) denotes the matrices with nonnegative (respectively, positive) entries. We will consider this particular case in all the examples of the paper.

We use the following results without any explicit reference to them (see for example, Varga [18] and Young [20]): if A, B are $n \times n$ real matrices then $(AB)^T = B^T A^T$, $\rho(AB) = \rho(BA)$, and $\rho(A^T) = \rho(A)$ where A^T denotes the transpose matrix of A. Furthermore, we also will use the spectral properties of K-nonnegative matrices stated in Berman and Plemmons [2].

The different splittings of the second type in Definition 2.1 below, were introduced by Woźnicki [19] for the particular case $K = \mathbb{R}^n_+$. The concept of weak splitting of the first type was introduced by Marek [9] for positive operators on a general cone with the name of splitting of positive type. We note that not all authors use the same nomenclature; see, for example, Beawens [1], Berman and Plemmons [2], Elsner [6], Neumann and Plemmons [13], and Ortega and Rheinboldt [14].

Definition 2.1. *Let A be a square matrix. The representation*

$$A = M - N$$

is called a ***splitting*** *of A if M is nonsingular. In addition, the splitting is*
regular *if $M^{-1} \geq 0$, and $N \geq 0$,*
nonnegative *if $M^{-1} \geq 0$, $M^{-1}N \geq 0$, and $NM^{-1} \geq 0$.*
weak nonnegative of the first type *if $M^{-1} \geq 0$ and $M^{-1}N \geq 0$,* ***weak nonnegative of the second type*** *if $M^{-1} \geq 0$ and $NM^{-1} \geq 0$,*
weak of the first type *if $M^{-1}N \geq 0$,* ***weak of the second type*** *if $NM^{-1} \geq 0$.*

In order to simplify notation, we will say that a splitting is **weak nonnegative** (respectively, **weak**) if it is weak nonnegative (respectively, weak) of the first or of the second type.

As it is easy to see, the different types of splittings of Definition 2.1 are ordered from the most restrictive to the least one. This can be found in Corollaries 3.1 and 6.1 of Woźnicki [19] for $K = \mathbb{R}^n_+$, and in Theorem 1 of Climent and Perea [3] for a proper cone K. Consequently, even though in this paper we only present results for weak nonnegative splittings, they are also valid for regular and nonnegative splittings. Furthermore, all the results in Sections 3 and 4 are still valid if we change "weak nonnegative splitting" by "convergent and weak splitting"; see Climent and Perea [3, pages 96-97].

Using the above-mentioned classification, Climent and Perea [3] present convergence results in a general infinite-dimensional case, which include as a particular case those introduced by other authors such as Varga [18], Bermann and Plemmons [2], Song [15, 16], and Woźnicki [19], for finite–dimensional case. We reformulate some of these results, for matrices, in the following theorem for further reference.

Theorem 2.2 (Theorem 3 and Remark 4 of [3]). *Let A be a nonsingular matrix and let $A = M - N$ be a weak nonnegative splitting of the first (respectively, second) type. The following conditions are equivalent.*

1. $A^{-1} \geq 0$
2. $A^{-1} \geq M^{-1}$
3. $A^{-1}M \geq 0$ *(respectively,* $MA^{-1} \geq 0$*).*
4. $\rho(M^{-1}N) = \dfrac{\rho(A^{-1}M) - 1}{\rho(A^{-1}M)}$ *(respectively,* $\rho(NM^{-1}) = \dfrac{\rho(MA^{-1}) - 1}{\rho(MA^{-1})}$*).*
5. $\rho(M^{-1}N) = \rho(NM^{-1}) < 1$.
6. $I - M^{-1}N$ *is* K*-monotone (respectively,* $I - NM^{-1}$ *is* K*-monotone).*
7. $A^{-1}N \geq 0$ *(respectively,* $NA^{-1} \geq 0$*).*
8. $A^{-1}N \geq M^{-1}N$ *(respectively,* $NA^{-1} \geq NM^{-1}$*).*
9. $\rho(M^{-1}N) = \dfrac{\rho(A^{-1}N)}{1 + \rho(A^{-1}N)}$ *(respectively,* $\rho(NM^{-1}) = \dfrac{\rho(NA^{-1})}{1 + \rho(NA^{-1})}$*).*

Moreover, we state the following results of Marek and Szyld [10] for the finite-dimensional case. Similar results can be found in Berman and Plemmons [2].

Lemma 2.3.

1. *(Corollary 3.2 of [10]) Let $T \geq 0$, and let $x \geq 0$ be such that $Tx - \alpha x \geq 0$. Then $\alpha \leq \rho(T)$. Moreover if $Tx - \alpha x > 0$, then $\alpha < \rho(T)$.*
2. *(Lemma 3.3 of [10]) Let $T \geq 0$ and let $x > 0$ be such that $\alpha x - Tx \geq 0$. Then $\rho(T) \leq \alpha$. Moreover, if $\alpha x - Tx > 0$ then $\rho(T) < \alpha$.*

To finish this section we introduce the following lemma which we will use in the next sections.

Lemma 2.4 (Lemma 6.1 of [7]). *Let that $AK \subseteq K$ and for some $v > 0$ and $\rho > 0$, $Av = \rho v$. Then for each $x > 0$, the sequence of vectors $\left\{ \left(\dfrac{A}{\rho} \right)^j x \right\}_{j=1}^{\infty}$ lies at a positive distance from the frontier of K.*

Observe, that in case $K = \mathbb{R}^n_+$ if we denote by $y_j = \left(\dfrac{A}{\rho(A)} \right)^j x$, Lemma 2.4 says that if $A \geq 0$ then the sequence of vectors $\{y_j\}_{j=1}^{\infty}$ satisfies $y_j \geq \delta u$ with some $\delta > 0$ independent of j and $u = (1, \ldots, 1)^T$.

3. Weak nonnegative splittings. Consider two splittings $A = M_1 - N_1 = M_2 - N_2$. One of the main purposes of this section is to derive new comparison theorems for weak nonnegative splittings, so that we can establish the inequality $\rho(M_1^{-1}N_1) \leq \rho(M_2^{-1}N_2)$ for a larger variety of cases. Also, we generalize the partial reciprocal of Csordas and Varga [5] for weak nonnegative splittings.

We begin with the result introduced by Csordas and Varga [5] for regular splittings of matrices and that Woźnicki [19] extends for weak nonnegative splittings of different type; that is, one of the first type and the other of the second type, and that now we extend for a general proper cone K. The proof is a generalization of Woźnicki's proof, but with some differences for the cases with strict inequality.

THEOREM 3.1. *Let A be a nonsingular matrix and let $A = M_1 - N_1 = M_2 - N_2$ be two weak nonnegative splittings of different type.*

1. If $A^{-1} \geq 0$ and

$$(A^{-1}N_1)^j A^{-1} \leq (A^{-1}N_2)^j A^{-1} \quad \textit{for some} \quad j \geq 1, \tag{3.1}$$

then

$$\rho(M_1^{-1}N_1) \leq \rho(M_2^{-1}N_2) < 1. \tag{3.2}$$

2. If $A^{-1} > 0$ and

$$(A^{-1}N_1)^j A^{-1} < (A^{-1}N_2)^j A^{-1} \quad \textit{for some} \quad j \geq 1, \tag{3.3}$$

then

$$\rho(M_1^{-1}N_1) < \rho(M_2^{-1}N_2) < 1. \tag{3.4}$$

Proof. Since $A^{-1} \geq 0$ by Theorem 2.2 both splittings are convergent.

1. Suppose that $A = M_1 - N_1$ is of the first type, then also by Theorem 2.2 we have that $A^{-1}N_1 \geq 0$ and $N_2 A^{-1} \geq 0$. Now, for the eigenvalue $\rho(A^{-1}N_1)$ there exists an eigenvector $x_1 \geq 0$ such that

$$x_1^T A^{-1} N_1 = \rho(A^{-1}N_1) x_1^T. \tag{3.5}$$

Hence from inequality (3.1) we have that

$$\begin{aligned} \rho(A^{-1}N_1)^j x_1^T A^{-1} &= x_1^T (A^{-1}N_1)^j A^{-1} \\ &\leq x_1^T (A^{-1}N_2)^j A^{-1} \\ &= x_1^T A^{-1} (N_2 A^{-1})^j. \end{aligned}$$

Now considering $y_1^T = x_1^T A^{-1} \geq 0$, we can write the above inequality as

$$\rho(A^{-1}N_1)^j y_1^T \leq y_1^T (N_2 A^{-1})^j,$$

so by part 1 of Lemma 2.3 we have that

$$\rho(A^{-1}N_1)^j \leq \rho(N_2 A^{-1})^j,$$

and then

$$\rho(A^{-1}N_1) \leq \rho(A^{-1}N_2). \tag{3.6}$$

Since $f(\alpha) = \dfrac{\alpha}{1+\alpha}$ is an increasing function for $\alpha \geq 0$, from part 9 of Theorem 2.2, the inequality (3.2) follows.

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If $A = M_1 - N_1$ is of the second type, then by Theorem 2.2 we have that $A^{-1}N_1 \geq 0$ and for the eigenvalue $\rho(A^{-1}N_1)$ there exists and eigenvector $x_1 \geq 0$ such that

$$(N_1A^{-1})x_1 = \rho(N_1A^{-1})x_1. \tag{3.7}$$

Now, taking into account that $(A^{-1}N_1)^jA^{-1} = A^{-1}(N_1A^{-1})^j$ and using (3.7) instead of equality (3.5) by a similar argument inequality (3.2) follows.

2. If $A = M_1 - N_1$ is of the first type, by a similar argument that in part 1 we have that $A^{-1}N_1 \geq 0$ and $N_2A^{-1} \geq 0$. Now for the eigenvalue $\rho(N_2A^{-1})$ there exists an eigenvector $x_2 \geq 0$ such that

$$(N_2A^{-1})x_2 = \rho(N_2A^{-1})x_2. \tag{3.8}$$

Hence, from inequality (3.3) we have that

$$(A^{-1}N_1)^jA^{-1}x_2 < (A^{-1}N_2)^jA^{-1}x_2 = A^{-1}(N_2A^{-1})^jx_2 = \rho(N_2A^{-1})^jA^{-1}x_2.$$

Now considering $y_2 = A^{-1}x_2 > 0$ we can write the above inequality as

$$(A^{-1}N_1)^jy_2 < \rho(N_2A^{-1})^jy_2.$$

Then by part 2 of Lemma 2.3

$$\rho(A^{-1}N_1)^j < \rho(A^{-1}N_2)^j.$$

Therefore, inequality (3.6) follows with strict inequality, and then, inequality (3.4) follows.

If $A = M_1 - N_1$ is of the second type, the proof is analogous using (3.7) instead of equality (3.8) and part 1 of Lemma 2.3 instead of part 2 of Lemma 2.3. □

REMARK 3.2. Theorem 3.1 also holds if we replace $A^{-1}N_i$ by $A^{-1}M_i$ for $i = 1, 2$, in (3.1) and (3.3).

Theorem 3.1 does not hold if we replace different type with same type (see Example 8 of Climent and Perea [3]).

Csordas and Varga [5] also introduced the following partial reciprocal to part 2 of Theorem 3.1 considering the particular case $K = \mathbb{R}^n_+$.

THEOREM 3.3 (Theorem 4 of [5]). *Let A be a nonsingular matrix with $A^{-1} > 0$. Let $A = M_1 - N_1 = M_2 - N_2$ be two regular splittings. If inequality (3.4) holds, then there exists an integer $j_0 \geq 1$ such that*

$$(A^{-1}N_1)^jA^{-1} < (A^{-1}N_2)^jA^{-1} \quad \textit{for all} \quad j \geq j_0. \tag{3.9}$$

However, the above theorem does not hold for weak nonnegative splittings of different types nor of the same type, even carrying out the change $A^{-1}N_i$ for $A^{-1}M_i$ for $i = 1, 2$ in inequality (3.9) as in Remark 3.2, as the following example shows.

EXAMPLE 3.4. We consider the monotone matrix $A = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & -2 \\ -1 & 0 & 2 \end{bmatrix}$ and the splittings $A = P_i - Q_i$, $i = 1, 2, 3$, where

$$P_1 = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & -\frac{5}{2} \\ -1 & 0 & 3 \end{bmatrix}, \; P_2 = \begin{bmatrix} 2 & -3 & 1 \\ 0 & 2 & -2 \\ -\frac{3}{2} & 1 & 2 \end{bmatrix} \text{ and } P_3 = \begin{bmatrix} 2 & -2 & -\frac{1}{2} \\ 0 & 2 & -2 \\ -2 & 0 & 3 \end{bmatrix}.$$

For $i = 1, 3$ the splittings are weak nonnegative of the first type and for $i = 2$ the splitting is weak nonnegative of the second type.

Let $M_1 = P_1$ and $M_2 = P_2$, then $\rho(M_1^{-1}N_1) = \dfrac{1}{3} < \dfrac{1}{2} = \rho(M_2^{-1}N_2)$ but inequality (3.9) does not hold because for $j \geq 1$ matrices $(A^{-1}N_1)^j A^{-1}$ and $(A^{-1}N_2)^j A^{-1}$ have the same $(3, 3)$ entry. The same occurs with matrices $(A^{-1}M_1)^j A^{-1}$ and $(A^{-1}M_2)^j A^{-1}$.

Now let $M_1 = P_1$ and $M_2 = P_3$, then again we have that $\rho(M_1^{-1}N_1) = \dfrac{1}{3} < \dfrac{1}{2} = \rho(M_2^{-1}N_2)$ but inequality (3.9) does not hold because for $j \geq 1$ matrices $(A^{-1}N_1)^j A^{-1}$ and $(A^{-1}N_2)^j A^{-1}$ have the same third row. The same occurs with matrices $(A^{-1}M_1)^j A^{-1}$ and $(A^{-1}M_2)^j A^{-1}$.

Observe that in all cases the matrices N_2A^{-1}, M_2A^{-1}, $A^{-1}N_2$ and $A^{-1}M_2$ are not irreducible.

The next theorem, which is the main result of the paper, establishes that Theorem 3.3 is true for weak nonnegative splittings of the same or different types considering a general proper cone K, if we impose the additional condition that the matrix $A^{-1}N_2$ (respectively, N_2A^{-1}) is K-irreducible, when $A = M_2 - N_2$ is a weak nonnegative splitting of the first (respectively, second) type.

THEOREM 3.5. *Let A be a nonsingular matrix with $A^{-1} > 0$. Let $A = M_1 - N_1 = M_2 - N_2$ be two weak nonnegative splittings of the same or different type. Assume that the matrix $A^{-1}N_2$ (respectively, N_2A^{-1}) is K-irreducible when $A = M_2 - N_2$ is of the first (respectively, second) type. If inequality (3.4) holds, then there exists an integer $j_0 \geq 1$ such that inequality (3.9) holds.*

Proof. Let $\lambda_2 = \rho(A^{-1}N_2) = \rho(N_2A^{-1})$. Assume that both splittings are weak nonnegative of the first type. From part 9 of Theorem 2.2 we have that inequality (3.4) implies

$$\rho(A^{-1}N_1) < \lambda_2. \tag{3.10}$$

Since $A^{-1} > 0$, then for all $x \geq 0$ we have that $y = A^{-1}x > 0$. Now, by the K-irreducibility of $A^{-1}N_2$, we have that for the eigenvalue λ_2 there exists an eigenvector $x_2 > 0$ such that

$$A^{-1}N_2x_2 = \lambda_2x_2.$$

Then from Lemma 2.4 we obtain that the sequence of vectors $\left\{ \left(\frac{A^{-1}N_2}{\lambda_2} \right)^j y \right\}_{j=1}^{\infty}$ lies at positive distance of the frontier of K; and then

$$\left(\frac{A^{-1}N_2}{\lambda_2} \right)^j A^{-1} > 0 \quad \text{for all} \quad j \geq 1. \tag{3.11}$$

On the other hand, from (3.10) it follows that $\lim\limits_{j\to\infty} \left(\dfrac{A^{-1}N_1}{\lambda_2} \right)^j = 0$ so that

$$\lim_{j\to\infty} \left(\frac{A^{-1}N_1}{\lambda_2} \right)^j A^{-1} = 0. \tag{3.12}$$

Now, from (3.11) and (3.12) there exists a positive integer j_0 such that

$$\left(\frac{A^{-1}N_1}{\lambda_2} \right)^j A^{-1} < \left(\frac{A^{-1}N_2}{\lambda_2} \right)^j A^{-1}, \quad \text{for all} \quad j \geq j_0,$$

and inequality (3.9) follows.

If $A = M_1 - N_1$ is of the first type and $A = M_2 - N_2$ is of the second type, taking into account that $\lambda_2 = \rho(A^{-1}N_2) = \rho(N_2A^{-1})$, and that $\rho(M_2^{-1}N_2) = \rho(N_2M_2^{-1})$, and using N_2A^{-1} instead of $A^{-1}N_2$, by a similar argument, we obtain that

$$A^{-1}\left(\frac{N_2A^{-1}}{\lambda_2}\right)^j > 0 \text{ for all } j \geq 1, \quad \text{and} \quad \lim_{j\to\infty}\left(\frac{A^{-1}N_1}{\lambda_2}\right)^j A^{-1} = 0.$$

Therefore, there exists a positive integer j_0 such that

$$\left(\frac{A^{-1}N_1}{\lambda_2}\right)^j A^{-1} < A^{-1}\left(\frac{N_2A^{-1}}{\lambda_2}\right)^j = \left(\frac{A^{-1}N_2}{\lambda_2}\right)^j A^{-1}$$

for all $j \geq j_0$. Then, inequality (3.9) follows.

If $A = M_1 - N_1$ is weak nonnegative of the second type and $A = M_2 - N_2$ is weak nonnegative of the first or second type, by a similar argument, but using N_1A^{-1} instead of $A^{-1}N_1$ inequality (3.9) holds. □

REMARK 3.6. Theorem 3.5 is still valid if we replace $A^{-1}N_i$ by $A^{-1}M_i$ for $i = 1, 2$ and N_2A^{-1} by M_2A^{-1}.

Observe that if we consider in Theorem 3.5 the proper cone $K = \mathbb{R}^n_+$ the condition "$A^{-1}N_2$ (respectively, N_2A^{-1}) is K-irreducible" is equivalent to "$M_2^{-1}N_2$ (respectively, $N_2M_2^{-1}$) is K-irreducible" by Theorem 1 of Szyld [17].

As we have mentioned before, Theorem 3.1 does not hold for weak nonnegative splittings of the same type. However, if we introduce other conditions similar to those of Theorem 3.1, we obtain the following results. The proof is similar to that of part 1 of Theorem 3.1.

THEOREM 3.7. *Let A be a nonsingular and K-monotone matrix. Let $A = M_1 - N_1 = M_2 - N_2$ be two weak nonnegative splittings.*

1. *If both splittings are of the first type and*

$$A^{-1}(N_1A^{-1})^j \leq A^{-1}(A^{-1}N_2)^j, \quad \textit{for some} \quad j \geq 1, \tag{3.13}$$

or

$$A^{-1}(A^{-1}N_1)^j \leq A^{-1}(N_2A^{-1})^j, \quad \textit{for some} \quad j \geq 1, \tag{3.14}$$

then inequality (3.2) holds.

2. *If both splittings are of the second type and*

$$(N_1A^{-1})^jA^{-1} \leq (A^{-1}N_2)^jA^{-1}, \quad \textit{for some} \quad j \geq 1, \tag{3.15}$$

or

$$(A^{-1}N_1)^jA^{-1} \leq (N_2A^{-1})^jA^{-1}, \quad \textit{for some} \quad j \geq 1, \tag{3.16}$$

then inequality (3.2) holds.

REMARK 3.8. Theorem 3.7 is still valid if we replace $A^{-1}N_i$ by $A^{-1}M_i$ and N_iA^{-1} by M_iA^{-1} for $i = 1, 2$.

Note that if in Theorem 3.7 and in Remark 3.8 we replace the inequalities by strict inequalities, we obtain comparison conditions similar to those of part 2 of Theorem 3.1 for weak nonnegative splittings of the same type. In this case, the proof is similar to that of part 2 of Theorem 3.1.

Furthermore, Theorem 3.5 is still valid if we replace inequality (3.9) with one of the inequalities (3.13)–(3.16) of Theorem 3.7 or in Remark 3.8, but with strict inequality.

Finally, note that if in Theorem 3.7 and in Remark 3.8 we consider the particular case where $j = 1$ we obtain new comparison conditions similar to the comparison condition introduced by Csordas and Varga [5] for the particular case $K = \mathbb{R}^n_+$ and regular splitting extended by Woźnicki [19] for weak nonnegative splitting of different type. For example, for the splittings $M_1 = P_3$ and $M_2 = P_1$ of Example 4.8 in Section 4, the comparison conditions of Theorem 3.15 of Varga [18] are not satisfied, but we have that $A^{-1}A^{-1}N_1 \leq A^{-1}N_2A^{-1}$. Moreover, for the splittings $M_1 = P_{10}$ and $M_2 = P_9$, also of Example 4.8, the comparison conditions of Theorem 14 of Climent and Perea [3] are not satisfied, but we have that $N_1A^{-1}A^{-1} \leq A^{-1}N_2A^{-1}$. Therefore, with these new conditions we can affirm that $\rho(M_1^{-1}N_1) \leq \rho(M_2^{-1}N_2)$ in both cases.

With the idea introduced by Woźnicki [19] to use the transpose matrix in one of the matrices of the splittings we can obtain a series of new comparison conditions similar to those seen so far in this section.

THEOREM 3.9. *Let A be a nonsingular and K-monotone matrix. Let $A = M_1 - N_1 = M_2 - N_2$ be two weak nonnegative splittings.*

1. *If both splittings are of the first type and*

$$A^{-1}(N_1A^{-1})^j \leq A^{-1}\left((A^{-1}N_2)^T\right)^j, \quad \textit{for some} \quad j \geq 1, \tag{3.17}$$

or

$$(A^{-1}N_1)^j(A^{-1})^T \leq \left((N_2A^{-1})^T\right)^j(A^{-1})^T, \quad \textit{for some} \quad j \geq 1, \tag{3.18}$$

then inequality (3.2) holds.

2. *If both splittings are of the second type and*

$$(A^{-1})^T(N_1A^{-1})^j \leq (A^{-1})^T\left((A^{-1}N_2^T)\right)^j, \quad \textit{for some} \quad j \geq 1, \tag{3.19}$$

or

$$(A^{-1}N_1)^jA^{-1} \leq \left((N_2A^{-1})^T\right)^jA^{-1}, \quad \textit{for some} \quad j \geq 1, \tag{3.20}$$

then inequality (3.2) holds.

3. *If $A = M_1 - N_1$ is of the first type, $A = M_2 - N_2$ is of the second type and*

$$A^{-1}(N_1A^{-1})^j \leq A^{-1}\left((N_2A^{-1})^T\right)^j, \quad \textit{for some} \quad j \geq 1, \tag{3.21}$$

or

$$(A^{-1})^T(A^{-1}N_1)^j \leq (A^{-1})^T\left((A^{-1}N_2)^T\right)^j, \quad \textit{for some} \quad j \geq 1, \tag{3.22}$$

then inequality (3.2) holds.

4. *If $A = M_1 - N_1$ is of the second type, $A = M_2 - N_2$ is of the first type and*

$$(A^{-1}N_1)^jA^{-1} \leq \left((A^{-1}N_2)^T\right)^jA^{-1}, \quad \textit{for some} \quad j \geq 1, \tag{3.23}$$

or

$$(N_1A^{-1})^j(A^{-1})^T \leq \left((N_2A^{-1})^T\right)^j(A^{-1})^T, \quad \textit{for some} \quad j \geq 1, \tag{3.24}$$

then inequality (3.2) holds.

Proof. The proof is also similar to that of part 1 of Theorem 3.1 taking into account that $\rho\left((A^{-1}N_2)^T\right) = \rho(A^{-1}N_2)$. □

REMARK 3.10. Theorem 3.9 is still valid if we replace $A^{-1}N_i$ by $A^{-1}M_i$ and N_iA^{-1} by M_iA^{-1} for $i = 1, 2$.

Note that Theorem 3.9 and Remark 3.10 are still valid if we replace inequalities by strict inequalities. Furthermore, Theorem 3.5 is still valid if we replace inequality (3.9) by one of the inequalities (3.17)–(3.24) in Theorem 3.9 or in Remark 3.10, but with strict inequality.

On the other hand, when both splittings are of the first type, inequalities (3.17) and (3.18) of Theorem 3.9 may be written, for $j = 1$, as

$$A^{-1}N_1A^{-1} \leq A^{-1}N_2^T(A^{-1})^T \quad \text{and} \quad A^{-1}N_1(A^{-1})^T \leq (A^{-1})^T N_2^T (A^{-1})^T$$

respectively. Therefore, we obtain new comparison conditions for weak nonnegative splittings of the first type. Note that for the splittings $M_1 = P_8$ and $M_2 = P_1$ of Example 4.8, we have that the inequality (3.18) of Theorem 3.9 for $j = 1$ holds, and none of the comparison conditions introduced before by Varga [18], or Csordas and Varga [5], are satisfied; that is, these conditions are not redundant.

Furthermore, if we take $j = 1$ in inequalities (3.19)–(3.24) of Theorem 3.9 we also obtain new comparison conditions for weak nonnegative splittings of the second type and for weak nonnegative splittings of different types. For example, if we consider the splittings $M_1 = P_1$ and $M_2 = P_{21}$ of Example 4.8, the inequality (3.21) holds; but none of the other similar comparison conditions we have had up to now are satisfied.

In the particular case where the matrix A is symmetric, the inequalities (3.17)–(3.20) of Theorem 3.9, for $j = 1$, can be written as

$$A^{-1}N_1A^{-1} \leq A^{-1}N_2^TA^{-1}$$

therefore obtaining Theorem 12 of Climent and Perea [3] for the finite dimensional case as a particular case.

Moreover, if we take $j = 1$ in inequalities (3.21)–(3.24), we obtain the following corollary.

COROLLARY 3.11. *Let A be a symmetric, nonsingular and K-monotone matrix. Let $A = M_1 - N_1 = M_2 - N_2$ be two weak nonnegative splittings of different types.*

1. *If $A = M_1 - N_1$ is of the first type and*

$$A^{-1}N_1A^{-1} \leq A^{-1}A^{-1}N_2^T \quad \textit{or} \quad A^{-1}A^{-1}N_1 \leq A^{-1}N_2^TA^{-1},$$

   then inequality (3.2) holds.

2. *If $A = M_1 - N_1$ is of the second type and*

$$A^{-1}N_1A^{-1} \leq N_2^TA^{-1}A^{-1} \quad \textit{or} \quad N_1A^{-1}A^{-1} \leq A^{-1}N_2^TA^{-1},$$

   then inequality (3.2) holds.

4. Relations between comparison conditions. The aim of this section is to complete the chain of implications introduced in Section 4 of Climent and Perea [3] for the finite dimensional case, with the new comparison conditions presented in this paper. In this chain, we relate the comparison conditions introduced in this paper in such a way that we establish a series of implications which are ordered from the most restrictive (usually easier to check) to the weakest (usually the most difficult to check).

THEOREM 4.1. *Let A be a nonsingular and K-monotone matrix. Let $A = M_1 - N_1 = M_2 - N_2$ be two weak nonnegative splittings of different types.*

1. If $N_1 \leq N_2$*, then* $M_1^{-1} \geq M_2^{-1}$.
2. If $M_1^{-1} \geq M_2^{-1}$*, then* $A^{-1}N_1A^{-1} \leq A^{-1}N_2A^{-1}$.
3. If $A^{-1}N_1A^{-1} \leq A^{-1}N_2A^{-1}$*, then the following inequalities hold for all* $j \geq 1$
  a. $(A^{-1}N_1)^jA^{-1} \leq (A^{-1}N_2)^jA^{-1}$.
  b. $(A^{-1}M_1)^jA^{-1} \leq (A^{-1}M_2)^jA^{-1}$.
4. If either 3a or 3b hold for some $j \geq 1$*, then inequality (3.2) holds.*

Proof. For parts 1 and 2 see Climent and Perea [3, Theorem 15].

3. Assume that $A = M_1 - N_1$ is weak nonnegative of the first (respectively, second) type. By hypothesis, we have that the inequality of part 3a holds for $j = 1$. Now, taking into account that, from Theorem 2.2, $A^{-1}N_1 \geq 0$ (respectively, $N_1A^{-1} \geq 0$), the proof follows by induction over j.

Next, since $A^{-1}N_1A^{-1} \leq A^{-1}N_2A^{-1}$ if and only if $A^{-1}M_1A^{-1} \leq A^{-1}M_2A^{-1}$, and once again, by induction over j the inequality of part 3b follows.

4. By Theorem 2.2 and Remark 3.2. □

Parts 1 and 3 of Theorem 4.1 are also valid for weak nonnegative splittings of the same type. However, for parts 2 and 4 this is not true, as we can see in Example 8 of Climent and Perea [3]. This example also show that the converses of parts 1 and 2 of Theorem 4.1 do not hold either. In Examples 4.8 and 4.9 below, we also show that the converses of parts 3 and 4 of Theorem 4.1 do not hold.

THEOREM 4.2. *Let* A *be a nonsingular and* K*-monotone matrix. Let* $A = M_1 - N_1 = M_2 - N_2$ *be two weak nonnegative splittings of the first type.*
1. If $0 \leq N_1A^{-1} \leq A^{-1}N_2$*, then the following inequalities hold for all* $j \geq 1$
  a. $A^{-1}(N_1A^{-1})^j \leq A^{-1}(A^{-1}N_2)^j$.
  b. $A^{-1}(M_1A^{-1})^j \leq A^{-1}(A^{-1}M_2)^j$.
2. If $0 \leq A^{-1}N_1 \leq N_2A^{-1}$*, then the following inequalities hold for all* $j \geq 1$
  a. $A^{-1}(A^{-1}N_1)^j \leq A^{-1}(N_2A^{-1})^j$.
  b. $A^{-1}(A^{-1}M_1)^j \leq A^{-1}(M_2A^{-1})^j$.
3. If some of the conditions 1a, 1b, 2a, or 2b hold for some $j \geq 1$*, then inequality (3.2) holds.*

Proof. 1. Clearly $0 \leq (N_1A^{-1})^j \leq (A^{-1}N_2)^j$ and multiplying both sides, on the left, by $A^{-1} \geq 0$ we obtain the inequality in part 1a.

On the other hand, taking into account that $N_i = M_i - A$ for $i = 1, 2$, we have that $0 \leq M_1A^{-1} \leq A^{-1}M_2$. Now, by a similar argument as above, the inequality of part 1b follows.

2. Similar to part 1.

3. From Theorem 3.7 and Remark 3.10. □

REMARK 4.3. Theorem 4.2 is still valid if we make the following changes.

- "first type" by "second type", $A^{-1}N_i$ and $M_i^{-1}N_i$ by N_iA^{-1} and $N_iM_i^{-1}$, respectively,
- $A^{-1}(N_1A^{-1})^j \leq A^{-1}(A^{-1}N_2)^j$ by $(N_1A^{-1})^jA^{-1} \leq (A^{-1}N_2)^jA^{-1}$,
- $A^{-1}(M_1A^{-1})^j \leq A^{-1}(A^{-1}M_2)^j$ by $(M_1A^{-1})^jA^{-1} \leq (A^{-1}M_2)^jA^{-1}$,
- $A^{-1}(A^{-1}N_1)^j \leq A^{-1}(N_2A^{-1})^j$ by $(A^{-1}N_1)^jA^{-1} \leq (N_2A^{-1})^jA^{-1}$,
- $A^{-1}(A^{-1}M_1)^j \leq A^{-1}(M_2A^{-1})^j$ by $(A^{-1}M_1)^jA^{-1} \leq (M_2A^{-1})^jA^{-1}$.

Observe that parts 1 and 2 of Theorem 4.2 and parts 1 and 2 of Remark 4.3 (really, parts 1 and 2 of Theorem 4.2 together with the changes proposed in Remark 4.3) still hold regardless of the type of splitting. This is not true for part 3 as we can see in Example 4.8 below. Examples 4.8 and 4.9 below also show that the converses of Theorem 4.2 are not true.

THEOREM 4.4. *Let* A *be a nonsingular and* K*-monotone matrix. Let* $A = M_1 -$

$N_1 = M_2 - N_2$ be two weak nonnegative splittings of the first type.

1. *If $0 \leq N_1 A^{-1} \leq (A^{-1}N_2)^T$ then the following inequalities hold for all $j \geq 1$*
   a. $A^{-1}(N_1A^{-1})^j \leq A^{-1}\left((A^{-1}N_2)^T\right)^j$.
   b. $A^{-1}(M_1A^{-1})^j \leq A^{-1}\left((A^{-1}M_2)^T\right)^j$.
2. *If $0 \leq A^{-1}N_1 \leq (N_2A^{-1})^T$, then the following inequalities hold for all $j \geq 1$*
   a. $(A^{-1}N_1)^j(A^{-1})^T \leq \left((N_2A^{-1})^T\right)^j(A^{-1})^T$.
   b. $(A^{-1}M_1)^j(A^{-1})^T \leq \left((M_2A^{-1})^T\right)^j(A^{-1})^T$.
3. *If some of the conditions 1a, 1b, 2a, or 2b hold for some $j \geq 1$, then inequality (3.2) holds.*

Proof. The proof of parts 1 and 2 is similar to the proofs of the same parts in Theorem 4.2. The proof of part 3 follows from Theorem 3.9 and Remark 3.10. □

Remark 4.5. Theorem 4.4 is still valid if we make the following changes.

- "first type" by "second type", $A^{-1}N_i$ and $M_i^{-1}N_i$ by N_iA^{-1} and $N_iM_i^{-1}$, respectively,
- $A^{-1}(N_1A^{-1})^j \leq A^{-1}\left((A^{-1}N_2)^T\right)^j$ by
  $(A^{-1})^T(N_1A^{-1})^j \leq (A^{-1})^T\left((A^{-1}N_2)^T\right)^j$,
- $A^{-1}(M_1A^{-1})^j \leq A^{-1}\left((A^{-1}M_2)^T\right)^j$ by
  $(A^{-1})^T(M_1A^{-1})^j \leq (A^{-1})^T\left((A^{-1}M_2)^T\right)^j$,
- $(A^{-1}N_1)^j(A^{-1})^T \leq \left((N_2A^{-1})^T\right)^j(A^{-1})^T$ by
  $(A^{-1}N_1)^jA^{-1} \leq \left((N_2A^{-1})^T\right)^jA^{-1}$,
- $(A^{-1}M_1)^j(A^{-1})^T \leq \left((M_2A^{-1})^T\right)^j(A^{-1})^T$ by
  $(A^{-1}M_1)^jA^{-1} \leq \left((M_2A^{-1})^T\right)^jA^{-1}$.

Observe that parts 1 and 2 of Theorem 4.4 and parts 1 and 2 of Remark 4.5 (really, parts 1 and 2 of Theorem 4.4 with the changes proposed in Remark 4.5) still hold regardless of the type of the splittings. This is not true for part 3 as we can see in Example 4.8 below. Examples 4.8 and 4.9 below also show that the converses of Theorem 4.4 do not hold.

Similar to Theorem 4.4 and Remark 4.5 we have the following results.

Theorem 4.6. *Let A be a nonsingular and K-monotone matrix. Let $A = M_1 - N_1$ be a weak nonnegative splitting of the first type and let $A = M_2 - N_2$ be a weak nonnegative splitting of the second type.*

1. *If $0 \leq N_1A^{-1} \leq (N_2A^{-1})^T$ then the following inequalities hold for all $j \geq 1$*
   a. $A^{-1}(N_1A^{-1})^j \leq A^{-1}\left((N_2A^{-1})^T\right)^j$.
   b. $A^{-1}(M_1A^{-1})^j \leq A^{-1}\left((M_2A^{-1})^T\right)^j$.
2. *If $0 \leq A^{-1}N_1 \leq (A^{-1}N_2)^T$ then the following inequalities hold for all $j \geq 1$*
   a. $(A^{-1})^T(A^{-1}N_1)^j \leq (A^{-1})^T\left(A^{-1}(N_2)^T\right)^j$.
   b. $(A^{-1})^T(A^{-1}M_1)^j \leq (A^{-1})^T\left(A^{-1}(M_2)^T\right)^j$.
3. *If some of the conditions 1a, 1b, 2a, or 2b hold for some $j \geq 1$, then inequality (3.2) holds.*

Remark 4.7. Theorem 4.6 is still valid if we make the following changes.

- "$A = M_1 - N_1$ first type" by "$A = M_1 - N_1$ second type",
- "$A = M_2 - N_2$ second type" by "$A = M_2 - N_2$ first type",
- $A^{-1}(N_1A^{-1})^j \leq A^{-1}\left((N_2A^{-1})^T\right)^j$ by
  $(N_1A^{-1})^j(A^{-1})^T \leq \left((N_2A^{-1})^T\right)^j(A^{-1})^T$,
- $A^{-1}(M_1A^{-1})^j \leq A^{-1}\left((M_2A^{-1})^T\right)^j$ by

$(M_1A^{-1})^j(A^{-1})^T \leq \left((M_2A^{-1})^T\right)^j (A^{-1})^T$,

- $(A^{-1})^T(A^{-1}N_1)^j \leq (A^{-1})^T \left(A^{-1}(N_2)^T\right)^j$ by $(A^{-1}N_1)^jA^{-1} \leq \left((A^{-1}N_2)^T\right)^j A^{-1}$,
- $(A^{-1})^T(A^{-1}M_1)^j \leq (A^{-1})^T \left(A^{-1}(M_2)^T\right)^j$ by $(A^{-1}M_1)^jA^{-1} \leq \left((A^{-1}M_2)^T\right)^j A^{-1}$.

Observe that parts 1 and 2 of Theorem 4.6 and parts 1 and 2 of Remark 4.7 (really, parts 1 and 2 of Theorem 4.6 together with the changes proposed in Remark 4.7) still hold regardless of the type of the splittings. This is not true for part 3 as we can see in Example 4.8 below. Examples 4.8 and 4.9 below also show that the converses of Theorem 4.6 do not hold.

Example 4.8. We consider the monotone matrix of Example 3.4 and the splittings $A = P_i - Q_i$ with $i = 1, 2, \ldots, 15$, where

$$P_1 = \begin{bmatrix} 3 & -2 & 1 \\ 0 & 2 & -2 \\ -2 & 0 & 2 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 2 & -2 & 2 \\ 0 & 2 & -2 \\ -\frac{3}{2} & 0 & \frac{3}{2} \end{bmatrix},$$

$$P_3 = \begin{bmatrix} 1 & -2 & 2 \\ 0 & 2 & -4 \\ -1 & 0 & 4 \end{bmatrix}, \quad P_4 = \begin{bmatrix} \frac{3}{2} & -\frac{1}{2} & 1 \\ -1 & 1 & -2 \\ 0 & 0 & 2 \end{bmatrix},$$

$$P_5 = \begin{bmatrix} 1 & -2 & 1 \\ -\frac{1}{3} & 4 & -2 \\ -\frac{1}{2} & \frac{1}{4} & 2 \end{bmatrix}, \quad P_6 = \begin{bmatrix} \frac{8}{9} & -\frac{20}{9} & \frac{10}{9} \\ \frac{2}{9} & \frac{40}{9} & -\frac{20}{9} \\ -\frac{4}{9} & \frac{10}{9} & \frac{40}{9} \end{bmatrix},$$

$$P_7 = \begin{bmatrix} 1 & -1 & \frac{1}{2} \\ 0 & 2 & -1 \\ 0 & 0 & 5 \end{bmatrix}, \quad P_8 = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 4 & -3 \\ -\frac{1}{2} & 0 & 2 \end{bmatrix},$$

$$P_9 = \begin{bmatrix} 3 & -6 & 3 \\ 1 & 2 & -3 \\ -3 & 0 & 6 \end{bmatrix}, \quad P_{10} = \begin{bmatrix} 1 & -4 & 4 \\ 0 & 2 & -2 \\ -1 & 0 & 2 \end{bmatrix},$$

$$P_{11} = \begin{bmatrix} \frac{5}{2} & -4 & 3 \\ -1 & 4 & -4 \\ -1 & 0 & 3 \end{bmatrix}, \quad P_{12} = \begin{bmatrix} \frac{5}{2} & -5 & \frac{5}{2} \\ -\frac{1}{2} & 3 & -\frac{3}{2} \\ -2 & 0 & 4 \end{bmatrix},$$

$$P_{13} = \begin{bmatrix} \frac{3}{2} & -2 & \frac{1}{2} \\ 0 & 2 & -2 \\ -\frac{1}{2} & 0 & \frac{3}{2} \end{bmatrix}, \quad P_{14} = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -2 \\ -1 & 2 & 2 \end{bmatrix},$$

$$P_{15} = \begin{bmatrix} 2 & -4 & 2 \\ 0 & 4 & -4 \\ -3 & 0 & 6 \end{bmatrix}.$$

For $i = 1, 2, 3, 4, 6, 11, 14$, the splittings are weak nonnegative of the first type. For $i = 7$, the splitting is nonnegative and for $i = 5, 8, 9, 10, 12, 13, 14, 15$, the splittings are weak nonnegative of the second type.

Let $M_1 = P_1$ and $M_2 = P_5$ it follows then that $(A^{-1}N_1)^4A^{-1} \leq (A^{-1}N_2)^4A^{-1}$ and $(A^{-1}M_1)^5A^{-1} \leq (A^{-1}M_2)^5A^{-1}$, but $A^{-1}N_1A^{-1} \not\leq A^{-1}N_2A^{-1}$. Therefore the converse of part 3 of Theorem 4.1 does not hold.

Let $M_1 = P_{15}$ and $M_2 = P_{11}$ it follows then that $A^{-1}N_1A^{-1} \leq A^{-1}A^{-1}N_2$ and $A^{-1}M_1A^{-1} \leq A^{-1}A^{-1}M_2$ but $\rho(M_1^{-1}N_1) \not\leq \rho(M_2^{-1}N_2)$. Now, for $M_1 = P_{11}$ and $M_2 = P_{12}$ it follows that $A^{-1}A^{-1}N_1 \leq A^{-1}N_2A^{-1}$ and $A^{-1}A^{-1}N_1 \leq A^{-1}N_2A^{-1}$ but $\rho(M_1^{-1}N_1) \not\leq \rho(M_2^{-1}N_2)$. Therefore, part 3 of Theorem 4.2 does not hold if both splittings are not weak nonnegative of the first type.

Let $M_1 = P_7$ and $M_2 = P_6$ it follows then that $A^{-1}(N_1A^{-1})^6 \leq A^{-1}(A^{-1}N_2)^6$ and $A^{-1}(M_1A^{-1})^7 \leq A^{-1}(A^{-1}M_2)^7$ but $N_1A^{-1} \not\leq A^{-1}N_2$. Now, for $M_1 = P_2$ and $M_2 = P_1$ it follows then that $A^{-1}(A^{-1}N_1)^4 \leq A^{-1}(N_2A^{-1})^4$ and $A^{-1}(A^{-1}M_1)^5 \leq A^{-1}(M_2A^{-1})^5$ but $A^{-1}N_1 \not\leq N_2A^{-1}$. Therefore, the converses of parts 1 and 2 of Theorem 4.2 do not hold.

Let $M_1 = P_{10}$ and $M_2 = P_4$ it follows then that $A^{-1}N_1A^{-1} \leq A^{-1}(A^{-1}N_2)^T$ and $A^{-1}M_1A^{-1} \leq A^{-1}(A^{-1}M_2)^T$ but $\rho(M_1^{-1}N_1) \not\leq \rho(M_2^{-1}N_2)$. Now, for $M_1 = P_1$ and $M_2 = P_9$ it follows that $A^{-1}N_1(A^{-1})^T \leq (N_2A^{-1})^T(A^{-1})^T$ and $A^{-1}M_1(A^{-1})^T \leq (M_2A^{-1})^T(A^{-1})^T$ but $\rho(M_1^{-1}N_1) \not\leq \rho(M_2^{-1}N_2)$. Therefore, part 3 of Theorem 4.4 does not hold if both splittings are not weak nonnegative of the first type.

Let $M_1 = P_2$ and $M_2 = P_1$ it follows then that $A^{-1}(N_1A^{-1})^3 \leq A^{-1}\left((A^{-1}N_2)^T\right)^3$ and $A^{-1}(M_1A^{-1})^5 \leq A^{-1}\left((A^{-1}M_2)^T\right)^5$ but $N_1A^{-1} \not\leq (A^{-1}N_2)^T$. Also we have that

$$(A^{-1}N_1)^2(A^{-1})^T \leq \left((N_2A^{-1})^T\right)^2(A^{-1})^T$$

and

$$(A^{-1}M_1)^3(A^{-1})^T \leq \left((M_2A^{-1})^T\right)^3(A^{-1})^T,$$

but $A^{-1}N_1 \not\leq (N_2A^{-1})^T$. Therefore, the converses of parts 1 and 2 of Theorem 4.4 do not hold.

Let $M_1 = P_{10}$ and $M_2 = P_{13}$ it follows that

$$A^{-1}N_1A^{-1} \leq A^{-1}(N_2A^{-1})^T \text{ and } A^{-1}M_1A^{-1} \leq A^{-1}(M_2A^{-1})^T,$$

but $\rho(M_1^{-1}N_1) \not\leq \rho(M_2^{-1}N_2)$. Now, for $M_1 = P_3$ and $M_2 = P_{14}$ it follows that $(A^{-1})^TA^{-1}N_1 \leq (A^{-1})^T(A^{-1}N_2)^T$ and $(A^{-1})^TA^{-1}M_1 \leq (A^{-1})^T(A^{-1}M_2)^T$, but

$$\rho(M_1^{-1}N_1) \not\leq \rho(M_2^{-1}N_2).$$

Therefore, part 3 of Theorem 4.6 does not hold when $A = M_1 - N_1$ is not weak nonnegative of the first type and $A = M_2 - N_2$ is not weak nonnegative of the second type.

Let $M_1 = P_1$ and $M_2 = P_7$ it follows then that $A^{-1}(N_1A^{-1})^2 \leq A^{-1}\left((A^{-1}N_2)^T\right)^2$ and $A^{-1}(M_1A^{-1})^3 \leq A^{-1}\left((A^{-1}M_2)^T\right)^3$ but $N_1A^{-1} \not\leq (N_2A^{-1})^T$. Now, for $M_1 = P_3$ and $M_2 = P_8$ it follows that $(A^{-1})^TA^{-1}N_1 \leq (A^{-1})^T(A^{-1}N_2)^T$ and $(A^{-1})^TA^{-1}N_1 \leq (A^{-1})^T(A^{-1}N_2)^T$ but $A^{-1}N_1 \not\leq (A^{-1}N_2)^T$. Therefore, the converses of parts 1 and 2 of Theorem 4.6 do not hold.

EXAMPLE 4.9. For the monotone matrix $A = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, consider the splittings $A = P_i - Q_i$, $i = 1, 2, 3$, where

$$P_1 = \begin{bmatrix} 0 & 1 & -1 \\ 2 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 0 & 3 & -3 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad P_3 = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

For $i = 1, 3$ the splittings are regular and for $i = 2$ the splitting is weak nonnegative of the second type.

Let $M_1 = P_1$ and $M_2 = P_2$ it follows then that $\rho(M_1^{-1}N_1) \leq \rho(M_2^{-1}N_2)$, but for all $j \geq 1$, $(A^{-1}N_1)^j A^{-1} \not\leq (A^{-1}N_2)^j A^{-1}$ and $(A^{-1}M_1)^j A^{-1} \not\leq (A^{-1}M_2)^j A^{-1}$. Therefore, the converse of part 4 of Theorem 4.1 does not hold.

Now, let $M_1 = P_1$ and $M_2 = P_3$. It is easy to prove that the converses of parts 3 of Theorems 4.2, 4.4 and 4.6 do not hold.

It is immediate the equivalence between the inequalities

$$A^{-1}N_1A^{-1} \leq A^{-1}N_2A^{-1} \quad \text{and} \quad A^{-1}M_1A^{-1} \leq A^{-1}M_2A^{-1};$$

as well as the comparison conditions which appear in Theorems 3.7 and 3.9 and those which appear in the corresponding remarks in which the N_i with $i = 1, 2$, and the similar conditions in which the M_i with $i = 1, 2$, appear for $j = 1$ are equivalent. However, if we consider the matrix A of Example 4.8 and the splittings $M_1 = P_7$ and $M_2 = P_6$ of the same example, it follows that $A^{-1}(N_1A^{-1})^6 \leq A^{-1}(A^{-1}N_2)^6$ but $A^{-1}(M_1A^{-1})^6 \not\leq A^{-1}(A^{-1}M_2)^6$. Therefore, the equivalence is not true for $j > 1$. On the other hand, the examples obtained suggest that if the inequality with M_i, $i = 1, 2$ for a certain $j > 1$ holds, then the same condition with N_i, $i = 1, 2$, for the same j holds. However, we have not been able to prove this result or to find a counterexample.

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