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STRONGLY STABLE GYROSCOPIC SYSTEMS*

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Abstract. Here, gyroscopic systems are time-invariant systems for which motions can be characterized by properties of a matrix pencil $L(\lambda) = \lambda^2 I + \lambda G - C$, where $G^T = -G$ and $C > 0$. A strong stability condition is known which depends only on $|G|$ $(= (G^T G)^{1/2} \geq 0)$ and C. If a system with coefficients G_0 and C satisfies this condition then all systems with the same C and with a G satisfying $|G| \geq |G_0|$ are also strongly stable. In order to develop a sense of those variations in G_0 which are admissible (preserve strong stability), the class of real skew-symmetric matrices G for which this inequality holds is investigated, and also those G for which $|G| = |G_0|$.

Key words. Gyroscopic systems. Stability. Skew-symmetric matrices.

AMS subject classifications. 47A56, 15A22

1. Introduction. For the purposes of this paper a gyroscopic system is defined to be an $n \times n$ matrix-valued function

$$L(\lambda) = \lambda^2 A + \lambda G + K,$$

where $A, G, K \in \mathcal{R}^{n \times n}$, $A^T = A, G^T = -G$, and $K^T = K$ (the index T denotes transposition). There is, of course, an underlying system of homogeneous differential equations:

$$A\ddot{x}(t) + G\dot{x}(t) + Kx(t) = 0 \tag{1.1}$$

(where dots denote t-derivatives). We are mainly interested in the case that A, K are positive and negative definite, respectively, written $A > 0, K < 0$. There is no loss of generality if it is assumed from the beginning that $A = I$ and K is diagonal. This can be achieved by employing modal coordinates determined by A and K. We write $K = -C = -\text{diag}[c_1^2, c_2^2, \ldots, c_n^2]$ and assume $c_j > 0$ for j = 1,2,...,n. Thus, we consider the reduced form:

$$L(\lambda) = \lambda^2 I + \lambda G - C. \tag{1.2}$$

The eigenvalues of a system are the zeros of $\det(L(\lambda))$ and the multiplicity of an eigenvalue is the order of the corresponding zero in $\det(L(\lambda))$. If λ_0 is an eigenvalue, the nonzero vectors in the nullspace of $L(\lambda_0)$ are the eigenvectors associated with λ_0. As a set of numbers in the complex plane, the eigenvalues have "Hamiltonian" symmetry, i.e., they are symmetrically placed (with associated multiplicities) with respect to both the real and imaginary axes. These facts follow from the relations

$$L(\lambda)^* = L(-\bar{\lambda}), \quad L(\lambda)^T = L(-\lambda),$$

and * denotes the conjugate transpose.

The system is said to be *stable* if all solutions of (1.1) are bounded for all non-negative t. Algebraically, this is equivalent to having all eigenvalues of $L(\lambda)$ on the

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imaginary axis and, in addition, all eigenvalues semi-simple, i.e., if the eigenvalue has multiplicity k, there are k linearly independent associated eigenvectors. A less familiar concept which plays a major role in this work is that of *strong stability.* A system has this property when it is stable and, in addition, all neighbouring systems are stable. By a neighbouring system we mean one with coefficients A, G, K, which are arbitrarily close to the given system but have the same symmetries. Thus, strong stability is a form of "robustness", and it means that, for neighbouring systems, the Hamiltonian symmetry of the set of eigenvalues (the spectrum) is preserved. Of course, strong stability requires that all eigenvalues are on the imaginary axis and are semi-simple, but in addition they must have a *definiteness* property (see [BLM], for example). In particular, all simple eigenvalues (with multiplicity one) are necessarily definite, so strong stability depends on the nature of the multiple eigenvalues on the imaginary axis, if any.

The hypothesis $C > 0$, (or $K < 0$), implies that when $G = 0$ the system is unstable. The eigenvalues are just $\pm c_1, \pm c_2, \ldots, \pm c_n$ and the positive eigenvalues account for the instability. We are to investigate a class of matrices G (resulting from gyroscopic effects) for which the system is strongly stable which means, in particular, that all eigenvalues are on the imaginary axis.

The objective of this paper is easily explained. It is known that if the inequality

$$|G| > kI + k^{-1}C \tag{1.3}$$

holds for some $k > 0$, then the system (1.2) is strongly stable (see [BL] and [BLM]). Here, $|G|$ denotes the positive semi-definite square root of G^TG. (However if (1.3) holds, then $|G|$ is necessarily positive definite.) Thus, if a gyroscopic term G_0 satisfies such an inequality then so does every matrix in the class of gyroscopic (real skew-symmetric) terms

$$\mathcal{G} = \{G \in \mathcal{R}^{n\times n} : G^T = -G \text{ and } |G| \geq |G_0|\}.$$

Naturally, there is special interest in the subset

$$\mathcal{G}_0 = \{G \in \mathcal{R}^{n\times n} : G^T = -G \text{ and } |G| = |G_0|\}.$$

The strategy is to take a fixed easily recognized G_0 satisfying (1.3) and then examine $\mathcal{G}$ and $\mathcal{G}_0$ with a view to identifying variations in the gyroscopic term (with C fixed) which retain strong stability of the system. In Section 2 we consider a useful class of initial stabilizing matrices G_0. In Sections 3, 4, and 5 we examine $\mathcal{G}_0$ and include a complete decription in Section 4, and a constructive procedure in Section 5. In Section 6 we discuss the more problematic class $\mathcal{G}$.

There are two ways in which this analysis may be useful. Firstly, in the sense of inverse problems, for a given diagonal matrix C it may be possible to choose a strongly stabilizing G from the classes described here. Secondly, for a given system, it may be possible to identify a stabilizing sub-matrix G_0 within G and, knowing something about the classes described here, deduce a stabilizing property for the whole matrix G.

The idea of strong stability has a long history in the mathematics literature but, with the possible exception of Duffin's work on over-damped systems, has not been widely recognized in the engineering literature. Works of M.G.Krein [K] and Gelfand and Lidskii [GL] of 1955 concern differential equations with periodic coefficients which, by the Floquet theory, are transformed to time-invariant systems. In their work the phrase systems with "strongly bounded solutions" is used. When put into our context this turns out to be essentially the same notion as "strong stability". For more recent developments and further references see, for example, [GLR] (especially Theorem III.2.2) and [LMM] (especially Theorem 8).

2. Perfectly matched systems. We are to build up a class of simple gyroscopic terms, G_0, beginning with the (generic) 2×2 example. It will be useful to examine it in detail. Let

$$G = \begin{bmatrix} 0 & g \\ -g & 0 \end{bmatrix}, \; C = \begin{bmatrix} c_1^2 & 0 \\ 0 & c_2^2 \end{bmatrix}.$$

The eigenvalues are the zeros of

$$\det(L(\lambda)) = \lambda^4 + (g^2 - c_1^2 - c_2^2)\lambda^2 + c_1^2 c_2^2 = 0.$$

We follow the eigenvalues as functions of g as g increases from zero, keeping the Hamiltonian symmetry in mind. Supposing $c_2 > c_1$, the behaviour is illustrated in Figure 1. At $g = 0$ there are four distinct real eigenvalues.

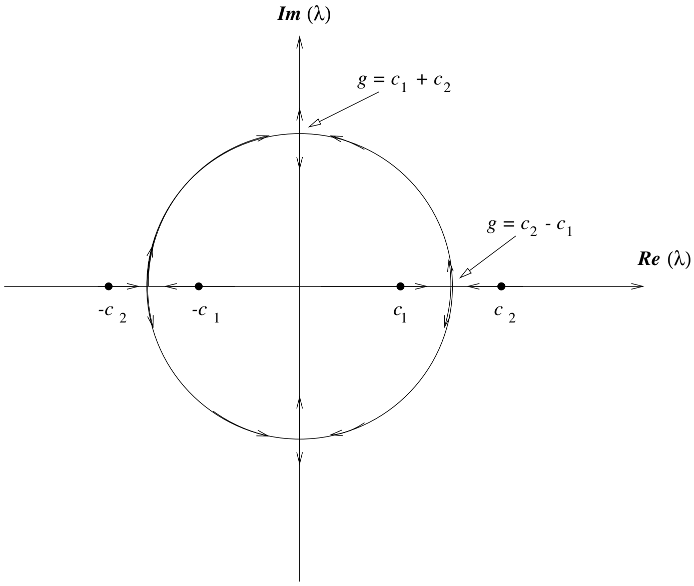

FIGURE 1. Locus of eigenvalues for 2×2 systems.

At $g = c_2 - c_1$ there are two double real eigenvalues at $\lambda = \pm\sqrt{c_1 c_2}$. As g increases further the eigenvalues lie on the circle $|\lambda| = \sqrt{c_1 c_2}$ and at $g = c_1 + c_2$ there are two double eigenvalues at $\lambda = \pm i\sqrt{c_1 c_2}$. Thereafter there are four distinct purely imaginary eigenvalues. The system is unstable for $0 \leq g \leq c_1 + c_2$, and strongly stable for $g > c_1 + c_2$.

For gyroscopic systems of order $2n$ the idea of (pairwise) "perfect matching" of gyroscopic forces has been introduced by Seyranian et al. [SSK]. The matrix G is said to couple the coordinates in a pefect matching if G contains one and only one non-zero element in each row and column. It is not difficult to see that such a matrix can be expressed in the form $G = PJ_0P^T$ where P is a permutation matrix,

$$J_0 = \operatorname{diag}\left\{ \begin{bmatrix} 0 & g_1 \\ -g_1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & g_2 \\ -g_2 & 0 \end{bmatrix}, \ldots \begin{bmatrix} 0 & g_n \\ -g_n & 0 \end{bmatrix} \right\},$$

and $g_1, g_2, \ldots, g_n$ are non-zero. Thus, J_0 couples coordinates 1 and 2, 3 and 4, and so on. Such matrices will appear frequently in the sequel so let us introduce the abbreviations $\hat{J} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and

$$J_0 = \operatorname{diag}[g_1\hat{J}, g_2\hat{J}, \ldots, g_n\hat{J}]. \tag{2.1}$$

It will also be convenient to assume that $g_j > 0$ for $j = 1, 2, \ldots, n$. We say that a system of the form (1.2) is perfectly matched if G is defined by a perfect matching.

For a perfectly matched system we may now write

$$L(\lambda) = \lambda^2 I + \lambda G - C = P(\lambda^2 I + \lambda J_0 - C_0)P^T,$$

where C_0 retains the diagonal property. Thus, to within a permutation of coordinates (which does not affect the eigenvalues of $L(\lambda)$), a perfectly matched system is just a direct sum of 2×2 gyroscopic systems. If we now write $C_0 = \operatorname{diag}[c_1^2, c_2^2, \ldots, c_n^2]$ then each component 2×2 system is strongly stable if

$$g_j > c_{2j-1} + c_{2j}, \quad j = 1, 2, \ldots, n$$

(see the discussion above). However, their direct sum need not be stable because eigenvalues of two or more components may coincide to form a multiple eigenvalue of $L(\lambda)$. We make two formal statements about strong stability.

Theorem 2.1. *A perfectly matched system for which each component system is stable and for which all $2n$ eigenvalues are distinct is strongly stable.*

Proof. The fact that the n component systems are stable means that all eigenvalues of $L(\lambda)$ are purely imaginary. Now the eigenvalues are assumed to be distinct and they depend continuously on G and C. Furthermore, the admissible perturbations retain the Hamiltonian symmetry of the eigenvalue distribution. Consequently, for sufficiently small perturbations the eigenvalues remain purely imaginary and distinct so the system remains stable. $\square$

Theorem 2.1 concerns small perturbations. The next result admits finite variations above a certain limit. To that end, let us define

$$c_+ = \max_{1 \le k \le 2n}(c_k), \quad c_- = \min_{1 \le k \le 2n}(c_k),$$

$$g_+ = \max_{1 \le k \le n}(g_k), \quad g_- = \min_{1 \le k \le n}(g_k).$$

Proposition 2.2. *Let $L(\lambda)$ be a perfectly matched system for which*

$$g_- > 2c_+. \tag{2.2}$$

Then $L(\lambda)$ is strongly stable.

Proof. Choose $k = c_+$. Then

$$k + k^{-1}c_j^2 = c_+ + c_j^2/c_+ \le 2c_+,$$

for $1 \le j \le 2n$. Now, using (2.2), we have

$$|J_0| > 2c_+ I \ge kI + k^{-1}C,$$

and it follows from Theorem 3.2 of [BLM] (see also (1.3) above) that $L(\lambda)$ is strongly stable.

The last statement involves some easy technicalities. The theorem in question is stated in terms of

$$M(\mu) := \mu^2 I + \mu(-iG_0) + C$$

with Hermitian coefficients, and obtained from $L(\lambda)$ by setting $\lambda = i\mu$. The strong stability notion used here corresponds to the fact that all eigenvalues of $M(\mu)$ have "definite type" as defined in [BLM]. □

There is an interesting parallel between perfectly matched gyroscopic systems and damped (non-gyroscopic) systems with proportional damping. Systems of the latter type are frequently used models in the engineering literature, and more complicated systems are discussed as perturbations or variations about a system with proportional damping. An underlying reason for this is that, in general, three real symmetric matrix coefficients cannot be diagonalized simultaneously by congruence, so two are diagonalized and the third (as a first approximation) is taken as a linear combination of them. Likewise, for gyroscopic systems, the three matrix coefficients cannot be simultaneously diagonalized; indeed, with one of them skew-symmetric this would mean reduction to the zero matrix. However, two may be diagonalized (A and K in our case), and the simplest form for G is associated with the perfect matching concept. Here, more general problems are then generated by perturbations (not necessarily small) about the perfectly matched system.

It will also be useful to note that matrices of the form (2.1) arise naturally as a canonical form for real skew-symmetric matrices under similarity. Indeed, we have the following result.

THEOREM 2.3. *A matrix $G \in \mathcal{R}^{n\times n}$ is skew-symmetric if and only if there is a real matrix J_0 of the form* (2.1) *and a real orthogonal matrix U such that $G = UJ_0U^T$. In this case, $|G| = U|J_0|U^T$ and*

$$|J_0| = \text{diag}[g_1, g_1, g_2, \dots, g_n, g_n].$$

In this context, J_0 is recognized as the real Jordan form for G (see [LT], for example). Thus, perfect matching also plays a fundamental algebraic role.

Proof. In one direction the proof of this theorem is immediate. For the converse, the proof is a special case of a more general theorem from a famous work of Ostrowski and Schneider (Theorem 2 of [OS]). For the last statement observe that

$$|G|^2 = G^TG = U(J_0^TJ_0)U^T = (U|J_0|U^T)^2.$$

The expression for $|J_0|$ is easily verified. □

It is apparent from this theorem that the eigenvalues of G are just those of J_0, namely $\pm ig_1, \pm ig_2, \dots, \pm ig_n$. With this result in hand, we easily see that Proposition 2.2 is just a special case of Theorem 6 of [BL], as follows.

THEOREM 2.4. *Let $L(\lambda)$ be a gyroscopic system, and let the matrix G have J_0 for its real Jordan form. Then the system is strongly stable if $g_- > 2c_+$.*

Proof. Clearly, using the diagonal form of $|J_0|$, the hypothesis implies that $|J_0| > 2c_+I$. But the preceding theorem now gives

$$|G| - 2c_+I = U(|J_0| - 2c_+I)U^T > 0$$

where U is real orthogonal. Thus, as in the proof of Proposition 2.2,

$$|G| > 2c_+ I \geq kI + k^{-1}C,$$

where $k = c_+$, and the result follows. $\square$

3. The class $\mathcal{G}_0$: the Riccati approach. In this section we make some preliminary investigations of the set $\mathcal{G}_0$. In the next section it will be seen how they fit into a complete description of $\mathcal{G}_0$. A constructive way of finding members of this class is described in Section 5. Observe first that $G = G_0 + X \in \mathcal{G}_0$ if and only if $X^T = -X$ and X satisfies the Riccati-like equation

$$G_0^T X + X^T G_0 + X^T X = 0. \tag{3.1}$$

Indeed, if we define $A = iG_0$ and $Y = iX$ then A is Hermitian and we seek Hermitian solutions Y of the Riccati equation

$$Y^2 + YA + AY = 0.$$

Information about the solution set can be obtained from the general analysis of [LR], for example (see especially their Corollary 7.3.3), via properties of the asociated Hamiltonian matrix

$$\begin{bmatrix} A & I \\ 0 & -A \end{bmatrix}.$$

In particular, it is found that when $g_1, g_2, \ldots, g_n$ are all distinct there are only finitely many solutions of (3.1) (including the trivial solutions $X = 0$ and $X = -2G_0$). However, when there is at least one repetition among the g_j there is a continuum of Hermitian solutions Y of the Riccati equation. Guided by this qualitative information we make a case study of 4×4 systems in which $g_1 = g_2 = g > 0$. Thus,

$$G_0 = g \begin{bmatrix} \hat{J} & 0 \\ 0 & \hat{J} \end{bmatrix}, \tag{3.2}$$

and we anticipate a continuum of solutions.

Let us write

$$X = \begin{bmatrix} x_1 \hat{J} & \Gamma \\ -\Gamma^T & x_2 \hat{J} \end{bmatrix}, \tag{3.3}$$

where $\Gamma,\ x_1,\ x_2$ represent six scalar parameters. Then the upper-right block of equation (3.1) gives

$$\hat{g_1} \hat{J} \Gamma + \hat{g_2} \Gamma \hat{J} = 0,$$

where $\hat{g_1} = g + x_1$ and $\hat{g_2} = g + x_2$. Then this is equivalent to the four scalar equations

$$\begin{aligned} \hat{g_2}\gamma_{11} + \hat{g_1}\gamma_{22} = 0, &\quad \hat{g_1}\gamma_{12} - \hat{g_2}\gamma_{21} = 0, \\ \hat{g_1}\gamma_{11} + \hat{g_2}\gamma_{22} = 0, &\quad -\hat{g_2}\gamma_{12} + \hat{g_1}\gamma_{21} = 0. \end{aligned}$$

For the existence of solutions with $\Gamma \neq 0$ we must have $\hat{g_2}^2 = \hat{g_1}^2$ and when this holds Γ has one of two forms:

$$\Gamma = \Gamma_1 = \begin{bmatrix} \gamma_1 & \gamma_2 \\ -\gamma_2 & \gamma_1 \end{bmatrix}, \quad \Gamma = \Gamma_2 = \begin{bmatrix} \gamma_1 & \gamma_2 \\ \gamma_2 & -\gamma_1 \end{bmatrix}, \tag{3.4}$$

and in either case det$\Gamma \neq 0$. In the first case $\hat{g_2} = -\hat{g_1}$ (so that $x_1 + x_2 = 2g$), and in the second $\hat{g_2} = \hat{g_1}$ (so that $x_1 = x_2$).

The upper-left and lower-right blocks of equation (3.1) give

$$x_j(2g + x_j)I + \Gamma\Gamma^T = 0$$

for $j = 1, 2$, and since $\Gamma\Gamma^T = (\gamma_1^2 + \gamma_2^2)I$, we find that $x_1,\ x_2$ are roots of

$$x^2 + 2gx + (\gamma_1^2 + \gamma_2^2) = 0. \tag{3.5}$$

The results can be summarized as follows.

PROPOSITION 3.1. *Let G_0 and X have the forms* (3.2) *and* (3.3), *respectively, and assume $\Gamma \neq 0$. Then $G_0 + X \in \mathcal{G}_0$ if and only if Γ has one of the forms* (3.4) *and $\gamma_1^2 + \gamma_2^2 \leq g^2$. In this case x_1 and x_2 are roots of equation* (3.5). *Furthermore, if $\Gamma = \Gamma_1$ then x_1 and x_2 are the two roots of* (3.5), *and if $\Gamma = \Gamma_2$ then $x_1 = x_2$.*

We observe that there are just two free parameters defining the class $\mathcal{G}_0$. The next examples are simple one parameter families.

EXAMPLE 3.2. Put $g = 1$, $\gamma_1 = t$, $\gamma_2 = 0$ and take the first choice in (3.4). It is found that the following family is in $\mathcal{G}_0$:

$$G(t) = \begin{bmatrix} 0 & \sqrt{1-t^2} & t & 0 \\ -\sqrt{1-t^2} & 0 & 0 & t \\ -t & 0 & 0 & -\sqrt{1-t^2} \\ 0 & -t & \sqrt{1-t^2} & 0 \end{bmatrix}, \quad t \in [-1,\ 1].$$

EXAMPLE 3.3. Put $g = 1$, $\gamma_1 = \gamma_2 = s/\sqrt{2}$ and take the second choice of Γ in (3.4). We obtain the following family $G(s)$ in $\mathcal{G}_0$:

$$G(s) = \begin{bmatrix} 0 & \sqrt{1-s^2} & s/\sqrt{2} & s/\sqrt{2} \\ -\sqrt{1-s^2} & 0 & s/\sqrt{2} & -s/\sqrt{2} \\ -s/\sqrt{2} & -s/\sqrt{2} & 0 & \sqrt{1-s^2} \\ -s/\sqrt{2} & s/\sqrt{2} & -\sqrt{1-s^2} & 0 \end{bmatrix}, \quad s \in [-1,\ 1].$$

Observe that $G(0) = G_0$ and

$$G(1) = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \\ -1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \end{bmatrix}.$$

4. Characterization of $\mathcal{G}_0$. In this section we give a complete description of the class $\mathcal{G}_0$ and show how the results of the preceding section fit into this description. We first need an important lemma. In particular, it shows that matrices of $\mathcal{G}_0$ are isospectral and, since the spectral norm is unitarily invariant, it implies (as is otherwise clear) that all members of $\mathcal{G}_0$ have the same spectral norm.

LEMMA 4.1. *If $G \in \mathcal{G}_0$ then G and G_0 are orthogonally similar.*

Proof. Let $G \in \mathcal{G}_0$. Using Theorem 2.3 write

$$G = WJW^T, \quad G_0 = W_0 J_0 W_0^T,$$

where J and J_0 are real Jordan forms and W, W_0 are orthogonal. Also, without loss of generality, it may be assumed that the diagonal entries of the diagonal matrices $|J|$

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and $|J_0|$ are in non-decreasing order. Then the second part of Theorem 2.3 implies that

$$|J| = (W^T W_0)|J_0|(W^T W_0)^T \tag{4.1}$$

and $W^T W_0$ is real orthogonal. But this is a similarity and implies that $|J|$ and$|J_0|$ have the same eigenvalues. Because the diagonal entries are ordered it follows that $|J| = |J_0|$. Using our convention that $g_j > 0$ we obtain $J = J_0$ and

$$G = WJW^T = WJ_0W^T = (WW_0^T)G_0(WW_0^T)^T. \quad \square$$

THEOREM 4.2. *Let $G_0 = W_0 J_0 W_0^T$ be the real Jordan representation of G_0. Then $G \in \mathcal{G}_0$ if and only if*

$$G = (W_0 U)J_0(W_0 U)^T$$

for a real orthogonal matrix U which commutes with $|J_0|$, i.e., $U|J_0| = |J_0|U$.

Proof. If $|G| = |G_0|$ then, as in the lemma, J_0 is a real Jordan form for both G and G_0. If we define $U = W_0^T W$, (4.1) gives $|J_0|U = U|J_0|$. Furthermore, $G = WJ_0W^T = (W_0U)J_0(W_0U)^T$, as required. For the converse, the last equation implies

$$|G| = (W_0U)|J_0|(W_0U)^T = W_0|J_0|W_0^T = |G_0|. \quad \square$$

It remains to describe those real orthogonal matrices U which commute with $|J_0|$. For this purpose re-define the symbols $g_1, \ldots, g_r$ (all positive) as the *distinct* eigenvalues of $|G_0|$ $(r \leq n)$ and then

$$J_0 = \mathrm{diag}[g_1 J_1, g_2 J_2, \ldots, g_r J_r],$$

where (as in (3.2)) J_j is a direct sum of m_j copies of $\hat{J}$, for $j = 1, 2, \ldots, r$. Now we have

$$|J_0| = \mathrm{diag}[g_1 I_{2m_1}, \ldots, g_r I_{2m_r}].$$

It is easily seen that if U commutes with $|J_0|$ and is orthogonal, then

$$U = \mathrm{diag}[V_1, V_2, \ldots, V_r], \tag{4.2}$$

where V_j is a real orthogonal matrix of size $2m_j$.

COROLLARY 4.3. *The matrix $G \in \mathcal{G}_0$ if and only if there are real orthogonal matrices $V_1, V_2, \ldots, V_r$ such that*

$$G = W_0\, \mathrm{diag}[g_1 V_1 J_1 V_1^T,\ g_2 V_2 J_2 V_2^T, \ldots, g_r V_r J_r V_r^T] W_0^T. \tag{4.3}$$

Now consider some special cases. First, when G_0 is associated with a perfect matching, then $G_0 = J_0$ and $W_0 = I$ in the above discussion. Retain this hypothesis and assume also that $r = n$ (i.e., $g_1, \ldots, g_n$ are positive and distinct), then there are n 2×2 blocks in (4.3). It is easily verified (using the general form of 2×2 orthogonal matrices) that

$$g_j V_j J_j V_j^T = \pm g_j \hat{J}.$$

Thus, in this case, and as predicted in Section 3, $\mathcal{G}_0$ consists of 2^n distinct real skew-symmetric matrices; not an interesting case.

When $r < n$ there is a continuum of matrices in $\mathcal{G}_0$ representing more interesting possibilities for variation about G_0 without change in absolute value. These results are consistent with those of Section 3. In particular, when $r = 1$ and $n = 2$ Proposition 3.1 applies and gives a more detailed description of $\mathcal{G}_0$.

To interpret Example 3.2 in the present context, observe that $G_0 = J_0$, and the matrix function $G(t)$ of that example would be represented here in the form

$$G(t) = V(t)J_0V(t)^T$$

for a function $V(t)$ taking values in the real orthogonal matrices for $t \in [-1,\ 1]$. Similarly for Example 3.3. In these examples J_0 is orthogonal and so $G(t)$, $G(s)$ are also orthogonal families.

These results can be interpreted in a rather different way concerning the set of square roots of positive definite matrices, as follows.

THEOREM 4.4. *Let a real $2n \times 2n$ matrix $A > 0$ be given. Then there exists a real skew-symmetric matrix G for which $A^{\frac{1}{2}} = |G|$ if and only if the eigenvalues of A occur in equal (positive) pairs.*

If, in Theorem 4.2, we put $A = |G_0|^2$, then Corollary 4.3 describes all real skew-symmetric matrices G for which $|G| = A^{\frac{1}{2}}$.

5. Constructing matrices in $\mathcal{G}_0$. Here, we give another construction for matrices $G \in \mathcal{G}_0$. It is assumed that the real Jordan canonical form J_0 of G_0 is known. Indeed, this approach will be most useful when G_0 already has real Jordan form, i.e., in the context of gyroscopic systems, G_0 corresponds to a perfect matching.

In general, suppose that we have a real Jordan representation $G_0 = W_0J_0W_0^T$ and define $\Lambda = \mathrm{diag}[g_1,\ g_2, \ldots, g_n]$. We also introduce the permutation matrix P defined by unit coordinate vectors as follows:

$$P = [e_1\ \ e_3\ \ e_5\ \ \ldots\ \ e_{2n-1}\ \ e_2\ \ e_4\ \ \ldots\ \ e_{2n}].$$

Guided by Corollary 4.3 and equation (4.2), determine a real orthogonal matrix U such that $U|J_0| = |J_0|U$.

Now define $2n \times n$ matrices X and Y by writing

$$[X\ \ Y] = W_0UP,$$

and it is claimed that the real skew-symmetric matrix

$$G := X\Lambda Y^T - Y\Lambda X^T$$

is in $\mathcal{G}_0$. Furthermore, it can be shown that the map from matrices U to $\mathcal{G}_0$ defined in this way is surjective.

Let us verify the first claim. Note first that the matrix $[X\ \ Y] = W_0UP$ is real and orthogonal and this implies that $X^TY = Y^TX = 0$. Then a simple computation gives

$$G^TG = [X\ \ Y] \begin{bmatrix} \Lambda^2 & 0 \\ 0 & \Lambda^2 \end{bmatrix} \begin{bmatrix} X^T \\ Y^T \end{bmatrix},$$

and hence

$$|G| = W_0UP \begin{bmatrix} \Lambda & 0 \\ 0 & \Lambda \end{bmatrix} P^TU^TW_0^T.$$

But we also have $P\begin{bmatrix} \Lambda & 0 \\ 0 & \Lambda \end{bmatrix}P^T = |J_0|$. Thus, by definition of U,

$$|G| = W_0U|J_0|U^TW_0^T = W_0|J_0|W_0^T = |G_0|.$$

The matrices Λ, X and Y defined in this way also contain complete information on the spectral properties of G over the complex field. The following statements are easily verified:

1. (Eigenvalue-eigenvector decomposition)

$$G(X+iY) = (X+iY)i\Lambda, \quad G(X-iY) = (X-iY)(-i\Lambda)$$

and the matrix $Z := \frac{1}{\sqrt{2}}[X+iY \;\; X-iY]$ is unitary.

2. (Spectral decomposition)

$$G = Z\begin{bmatrix} i\Lambda & 0 \\ 0 & -i\Lambda \end{bmatrix}Z^*.$$

3. $$|G| = X\Lambda X^T + Y\Lambda Y^T.$$

6. The class $\mathcal{G}$. Our concern now is with a fixed real, skew-symmetric matrix G_0 with a known real Jordan form, J_0, as described above, and with properties of matrices in the associated class $\mathcal{G}$. Our results here are fragmentary and a full description of $\mathcal{G}$ in spectral terms is not known. We begin with a strong sufficient condition for $G \in \mathcal{G}$.

THEOREM 6.1. *Let G have eigenvalues $\pm i\gamma_j$, for $j = 1, 2, \ldots, n$ and $\gamma_- = min_{1\le j\le n}|\gamma_j|$. Then $\gamma_- \ge g_+$ implies that $G \in \mathcal{G}$.*

Proof. Let G and G_0 have real Jordan forms J and J_0, respectively. Then, by Theorem 2.3, there are real orthogonal matrices U and V such that $|G| = U\,|J|\,U^T$ and $|G_0| = V\,|J_0|\,V^T$. For any vector x with $\|x\| = 1$ we also have $\|U^Tx\| = \|V^Tx\| = 1$ and so

$$x^TV|J_0|V^Tx \le g_+ \le \gamma_- \le x^TU|J|U^Tx.$$

It follows that $V|J_0|V^T \le U|J|U^T$, i.e., $|G_0| \le |G|$, as required. □

Now we can give a complete description of $\mathcal{G}$ in a special case.

THEOREM 6.2. *Let G_0 have only two distinct eigenvalues, $\pm ig_1$, $g_1 > 0$. Then $G \in \mathcal{G}$ if and only if $G = UJU^T$ where U is real orthogonal, and*

$$J = \mathrm{diag}[\gamma_1\hat{J}, \ldots, \gamma_n\hat{J}]$$

with $\min_j|\gamma_j| \ge g_1$.

Proof. If G has the given form it follows from the preceding theorem that $G \in \mathcal{G}$. Conversely, we have $|G| = U|J|U^T$ and Theorem 2.3 gives $|G_0| = g_1I$. Thus, $|G| \ge |G_0|$ implies $U|J_0|U^T \ge g_1I$, or $|J_0| \ge \gamma I$. Hence $\min_j|\gamma_j| \ge g_1$. □

To start a different line of attack, let $G \in \mathcal{G}$ and (as in Section 3) write $G = G_0 + X$, where $X^T = -X$. Define $\triangle$ by writing $G^TG = G_0^TG_0 + \triangle$. Thus,

$$\triangle = G_0^TX + X^TG_0 + X^TX, \tag{6.1}$$

and observe that $G \in \mathcal{G}$ if $\triangle \ge 0$ (see Appendix A). As in Theorem (2.3) write $G_0 = UJ_0U^T$, and if we define $Y = U^TXU$ then

$$U^T\triangle U = J_0^TY + Y^TJ_0 + Y^TY.$$

If G_0 corresponds to a perfect matching then, of course, we may take $U = I$ and $Y = X$.

We generalize the basic asumption of the last theorem and suppose now that G_0 has size $2n$ and four eigenvalues; namely $\pm ig_1$ and $\pm ig_2$. Then a real Jordan form for G_0 is

$$J_0 = \mathrm{diag}[g_1 J_1,\ g_2 J_2]. \tag{6.2}$$

where J_1 and J_2 have the form $\mathrm{diag}[\hat{J}, \hat{J}, \dots, \hat{J}]$ with, say, m_1 and m_2 blocks, respectively, and we assume that $m_1 \le m_2$. Thus, when $g_1 \neq g_2$, ig_1 and $-ig_1$ each have mutilplicity m_1, ig_2 and $-ig_2$ each have multiplicity m_2, and $m_1 + m_2 = n$. Note also that $J_1^2 = -I_{2m_1}$ and $J_2^2 = -I_{2m_2}$.

Now partition Y in the form

$$Y = \begin{bmatrix} Y_1 & \Gamma \\ -\Gamma^T & Y_2 \end{bmatrix}, \tag{6.3}$$

where $Y_1^T = -Y_1$, $Y_2^T = -Y_2$ and Γ has size $2m_1 \times 2m_2$. Then it is found that

$$U^T \triangle U = \begin{bmatrix} \triangle_1 & F \\ F^T & \triangle_2 \end{bmatrix},$$

where

$$\triangle_1 = g_1(J_1^T Y_1 + Y_1^T J_1) + Y_1^T Y_1 + \Gamma\Gamma^T, \tag{6.4}$$

$$\triangle_2 = g_2(J_2^T Y_2 + Y_2^T J_2) + Y_2^T Y_2 + \Gamma^T\Gamma, \quad \text{and} \tag{6.5}$$

$$F = (g_1 J_1^T + Y_1^T)\Gamma - \Gamma(g_2 J_2 + Y_2). \tag{6.6}$$

To characterize the condition $\triangle \ge 0$ (i.e., $G \in \mathcal{G}$) we now apply a well-known criterion for this 2×2 block matrix to be positive-semidefinite (see [A], for example). Thus, we have $\triangle \ge 0$ if and only if

$$\triangle_1 \ge 0, \quad \triangle_2 \ge F^T \triangle_1^+ F, \quad \triangle_1 \triangle_1^+ F = F, \tag{6.7}$$

where the index + denotes the Moore-Penrose generalized inverse. (The reason for assuming only two pairs of eigenvalues for G_0 is now clear. If there are more blocks in Y we do not have a nice extension of the criteria (6.7).)

THEOREM 6.3. *If G_0 has the real Jordan form* (6.2) *then, with $Y = U^T X U$ and the definitions* (6.4), (6.5), *and* (6.6), *$G = G_0 + X \in \mathcal{G}$ if conditions* (6.7) *are satisfied.*

Now we introduce some hypotheses that make this result more tractable.

HYPOTHESIS 6.4. G_0 is defined by a perfect matching. (Then $U = I$ and $Y = X$.)

HYPOTHESIS 6.5. $\triangle_1 \ge 0$ and $\triangle_2 \ge 0$. (For example, this is the case when $U = I$ and $Y_1 = Y_2 = 0$ and, with Hypothesis 6.4, the variation about G_0 is confined to the linking term Γ.) When this holds write

$$\triangle_1 = \Gamma_1 \Gamma_1^T, \quad \triangle_2 = \Gamma_2^T \Gamma_2 \tag{6.8}$$

(so that $\Gamma_j = \Gamma$ if $Y_j = 0$, for $j = 1, 2$).

HYPOTHESIS 6.6. $\triangle_1 > 0$. (Since $m_1 \le m_2$ this is equivalent to $\mathrm{rank}(\Gamma_1) = 2m_1$. Now $\triangle^+ = \triangle^{-1}$ and the last condition of (6.7) always holds.)

Corollary 6.7. *If Hypotheses 6.5 and 6.6 hold, and* $\triangle_2 \geq F^T\triangle_1^{-1}F$ *then* $G \in \mathcal{G}$.

Hypothesis 6.8. $m_1 = m_2$ and $\triangle_2 > 0$.

Corollary 6.9. *If Hypotheses 6.6 and 6.8 hold, then* $G \in \mathcal{G}$ *if*

$$\|\Gamma_1^{-1}\{(g_1J_1 - Y_1^T)\Gamma + \Gamma(g_2J_1 + Y_2)\}\Gamma_2^{-1}\|_s \leq 1. \tag{6.9}$$

Proof. (Here, $\|.\|_s$ denotes the spectral matrix norm, i.e., the norm induced by the euclidean vector norm.) These hypotheses imply that Γ_1, F, and Γ_2 are square matrices of the same size and Γ_1, Γ_2 are nonsingular. The preceding Corollary holds and, using (6.8), gives $G \in \mathcal{G}$ if

$$\Gamma_2^T\Gamma_2 \geq F^T(\Gamma_1\Gamma_1^T)^{-1}F.$$

This is equivalent to $(\Gamma_1^{-1}F\Gamma_2^{-1})^T(\Gamma_1^{-1}F\Gamma_2^{-1}) \leq I$ which, in turn (and using (6.6)), is equivalent to (6.9). Note also that now $J_1 = J_2$. □

An interesting special case arises when the four hypotheses hold and $Y_1 = Y_2 = 0$. We suppose that (2.2) is satisfied when $G = G_0$ so that the unperturbed system is perfectly matched and strongly stable. The condition $Y_1 = Y_2 = 0$ means that variations about G_0 are confined to the linking term Γ (see (6.3)). Now (6.9) reduces to

$$\|g_1\Gamma^{-1}J_1 + g_2J_1\Gamma^{-1}\|_s \leq 1. \tag{6.10}$$

Here, g_1, g_2 and J_1 are determined by G_0 and there is a suggestion that stability is ensured if only Γ is "large" in such a way that Γ^{-1} is "small". On the other hand, because the unperturbed system is strongly stable, we have stability for Γ sufficiently small. Thus, as Γ increases the stabilizing role is transferred from the entries g_1J_1, g_2J_1 to those of Γ. However there may be an interregnum of instability. This is illustrated in Examples 6.11 and 6.12 below.

Notice also that this argument depends on the existence of Γ^{-1}. Indeed, examples suggest that if Γ is singular (Hypothesis 6.8 does not hold) then increasing Γ will generally be destabilizing. This is supported by Examples 6.13 and 6.14 below and the next result.

Theorem 6.10. *Let Hypothesis 6.4 hold and let* $Y_1 = Y_2 = 0$. *If the nullspace of* Γ *has dimension one then* $\triangle$ *is not positive semi-definite (or definite), i.e.,* $G = G_0 + X \notin \mathcal{G}$.

Proof. We now have

$$U^T\triangle U = \begin{bmatrix} \Gamma\Gamma^T & F \\ F^T & \Gamma^T\Gamma \end{bmatrix},$$

where $F = g_1J_1^T\Gamma + g_2\Gamma J_1^T$. Choose an $x_0 \neq 0$ such that $\Gamma^Tx_0 = 0$. Then J_1x_0 is not a multiple of x_0, so that $\Gamma^T(J_1x_0) \neq 0$. If also $\Gamma y_0 = 0$, $y_0 \neq 0$, then define $y = y_0 - \epsilon\Gamma^TJ_1x_0$ and a computation gives

$$[x_0^T \;\; y^T]U^T\triangle U \begin{bmatrix} x_0 \\ y \end{bmatrix} = -2\epsilon g_1\|\Gamma^TJ_1x_0\|^2 + \epsilon^2\|\Gamma\Gamma^TJ_1x_0\|^2.$$

Since $\Gamma^TJ_1x_0 \neq 0$ there is an ϵ such that the right-hand side is negative. Hence the result. □

Notice that if $n=2$ the conditions of this theorem are satisfied when $\det\Gamma = 0$ and $\Gamma \neq 0$.

EXAMPLE 6.11. Suppose $m_1 = m_2 = 1$ (so that $n = 2$) and $Y_1 = Y_2 = 0$. Let Hypothesis 6.4 hold and $\Gamma = \gamma I_2 > 0$. Then in (6.10),

$$g_1\Gamma^{-1}J_1 + g_2J_1\Gamma^{-1} = \frac{g_1+g_2}{\gamma}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

and $G \in \mathcal{G}$ provided $\gamma \geq g_1 + g_2$. Thus, if G_0 satisfies (1.3) so does

$$G = \begin{bmatrix} 0 & g_1 & \gamma & 0 \\ -g_1 & 0 & 0 & \gamma \\ -\gamma & 0 & 0 & g_2 \\ 0 & -\gamma & -g_2 & 0 \end{bmatrix}$$

provided $\gamma \geq g_1 + g_2$, by Corollary 6.9. Also, condition (1.3) ensures that the system defined by G is strongly stable when γ is sufficiently small.

EXAMPLE 6.12. This is a special case of Example 6.11 obtained by setting $C = \mathrm{diag}[0.1,\ 0.2,\ 0.15,\ 0.24]$, $g_1 = g_2 = 1$, and $\Gamma = \gamma I_2 > 0$. Thus

$$G = \begin{bmatrix} 0 & 1 & \gamma & 0 \\ -1 & 0 & 0 & \gamma \\ -\gamma & 0 & 0 & 1 \\ 0 & -\gamma & -1 & 0 \end{bmatrix}.$$

It is easily verified that, when $\gamma = 0$, the system is perfectly matched and strongly stable. As in Example 6.11 the system is strongly stable when $\gamma > g_1 + g_2 = 2$. Numerical experiments show that the system is, in fact, unstable for $0.168 \leq \gamma \leq 1.836$.

EXAMPLE 6.13. Here, we use the same data as in Example 6.12 except that $\Gamma = \begin{bmatrix} \gamma & 0 \\ 0 & 0 \end{bmatrix}$, a singular matrix. Now Hypothesis 6.6 fails and our Corollaries do not apply. Numerical experiments show instability for all $\gamma \geq 0.326$.

EXAMPLE 6.14. Let

$$G_0 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}, \quad G_\alpha = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & \alpha & 0 \\ 0 & -\alpha & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}.$$

Here, Γ is singular and Theorem 6.10 holds. Direct calculations confirm that $|G_\alpha| \geq |G_0|$ does not hold for any nonzero α. In fact, $|G_0|$ has all eigenvalues equal to one, and the eigenvalues of $|G_\alpha|$ are the zeros of $\mu^2 - (2+\alpha^2)\mu + 1$, each with multiplicity two. Two of these have magnitudes less than one and the other two exceed one in magnitude.

Appendix A. A note on the partial order. It is well-known that, if matrices A and B satisfy $A \geq B \geq 0$ then it is not necessarily the case that $A^2 \geq B^2 \geq 0$ (see Chapter 5 of [B], for example). In contrast, the square-root function is better behaved (is "operator monotone"). In our context the following question arises naturally: If G and H are real, nonsingular, skew-symmetric matrices and $|G| \geq |H|$, does it follow that $|G|^2 \geq |H|^2$? It is trivially true that the answer is "yes" if the matrices are of size 2×2. However, this property does not extend to matrices of size four as the following example shows.

EXAMPLE A.1. Let

$$G = \frac{1}{2}\begin{bmatrix} 0 & 5 & 0 & 1 \\ -5 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \\ -1 & 0 & -1 & 0 \end{bmatrix}, \quad H = \begin{bmatrix} 0 & 2 & 0 & 0 \\ -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

It is found that $|G| \geq |H|$ but $|G|^2 - |H|^2$ is not positive semi-definite.

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