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STRUCTURED JORDAN CANONICAL FORMS FOR STRUCTURED MATRICES THAT ARE HERMITIAN, SKEW HERMITIAN OR UNITARY WITH RESPECT TO INDEFINITE INNER PRODUCTS*

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Abstract. For inner products defined by a symmetric indefinite matrix $\Sigma_{p,q}$, canonical forms for real or complex $\Sigma_{p,q}$-Hermitian matrices, $\Sigma_{p,q}$-skew Hermitian matrices and $\Sigma_{p,q}$-unitary matrices are studied under equivalence transformations which keep the class invariant.

Key words. structured eigenvalue problems, Lie group, Lie algebra, Jordan algebra

AMS subject classifications. 15A21, 65F15

1. Introduction. In several recent papers [1, 8, 10, 9, 6] the topic of canonical forms for structured matrices and pencils associated with classical Lie groups, Lie algebras and Jordan algebras has been studied. The motivation for these analyses is the development of new structure preserving numerical methods for the solution of the eigenvalue problem for matrices in these classes. The main motivation is to use equivalence transformations that preserve the algebraic structures, i.e., for example the symmetry in the spectrum in finite arithmetic. This means that the transformation matrices are restricted to be from the associated Lie groups only. If such structure preserving methods can be constructed, then this usually leads to a reduction in complexity and at the same time it avoids that in finite arithmetic physically meaningless results are obtained. Often one also has a better perturbation and error analysis, see for example [2, 3, 4]. The latter is obtained in particular if one uses unitary transformations which are at the same time in the associated Lie group, since then the methods, usually, are also numerically backwards stable. However, for numerical computations we need to know the proper condensed forms within the given structures, usually called structured Schur like forms, that the numerical methods can possibly generate, and from which the eigenvalues and eigenstructures can be easily read off. The structured Jordan like canonical forms that we describe here are the simplest versions of such condensed forms, although they need nonunitary transformations. Hence these Jordan like forms will be the fundamental theory for studying the proper structured Schur like forms and therefore for developing numerical methods.

The invariants under similarity transformations have been classified already for quite a while [5, 11]. There, for physical applications the canonical forms are restricted to be classical Jordan forms. So the transformation matrices are not in the associated Lie groups. Such canonical forms are not what we are interested in. Hence it is necessary to convert these forms to the desired forms.

A complete analysis for the case of Hamiltonian, skew Hamiltonian and symplectic matrices, i.e., matrices that are Hermitian, skew Hermitian and unitary with respect to an indefinite scalar product given by a skew symmetric matrix, has recently been given in [8]. In this paper we now derive analogous results for the matrices that are Hermitian, skew Hermitian and unitary with respect to an inner product defined via

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the indefinite symmetric matrix $\Sigma_{p,q} := \begin{bmatrix} I_p & 0 \\ 0 & -I_q \end{bmatrix}$, where I_k is the $k \times k$ identity matrix. We consider the following classes of matrices.

DEFINITION 1.1. *Let $\mathbb{R}$ and $\mathbb{C}$ denote the real and complex field, respectively.*
A matrix $\mathcal{C} \in \mathbb{C}^{(p+q)\times(p+q)}$ is called $\Sigma_{p,q}$-Hermitian if $\mathcal{C}\Sigma_{p,q} = (\mathcal{C}\Sigma_{p,q})^H$. $\mathcal{C}$ is called $\Sigma_{p,q}$-symmetric if it is $\Sigma_{p,q}$-Hermitian and real.
A matrix $\mathcal{C} \in \mathbb{C}^{(p+q)\times(p+q)}$ is called $\Sigma_{p,q}$-skew Hermitian if $\mathcal{C}\Sigma_{p,q} = -(\mathcal{C}\Sigma_{p,q})^H$. $\mathcal{C}$ is called $\Sigma_{p,q}$-skew symmetric if it is $\Sigma_{p,q}$-skew Hermitian and real.
A matrix $\mathcal{G} \in \mathbb{C}^{(p+q)\times(p+q)}$ is called $\Sigma_{p,q}$-unitary if $\mathcal{G}^H\Sigma_{p,q}\mathcal{G} = \Sigma_{p,q}$. It is called $\Sigma_{p,q}$-orthogonal if it is $\Sigma_{p,q}$-unitary and real. Note that the $\Sigma_{p,q}$-Hermitian matrices form a Jordan algebra, the $\Sigma_{p,q}$-skew Hermitian matrices from a Lie algebra, and the $\Sigma_{p,q}$-unitary matrices form a Lie group. The algebras and group are invariant under similarity transformations with $\Sigma_{p,q}$-unitary matrices.

PROPOSITION 1.2.
1. If $\mathcal{C}$ is $\Sigma_{p,q}$-Hermitian and $\mathcal{G}$ is $\Sigma_{p,q}$-unitary then $\mathcal{G}^{-1}\mathcal{C}\mathcal{G}$ is $\Sigma_{p,q}$-Hermitian.
2. If $\mathcal{C}$ is $\Sigma_{p,q}$-skew Hermitian and $\mathcal{G}$ is $\Sigma_{p,q}$-unitary then $\mathcal{G}^{-1}\mathcal{C}\mathcal{G}$ is $\Sigma_{p,q}$-skew Hermitian.
3. If $\mathcal{G}_1$ and $\mathcal{G}_2$ are $\Sigma_{p,q}$-unitary then $\mathcal{G}_1\mathcal{G}_2$ is also $\Sigma_{p,q}$-unitary.

Similar to the approach for Hamiltonian and symplectic matrices in [8] we derive structured Jordan canonical forms for these classes of matrices. But different from the case of Hamiltonian and symplectic matrices and pencils, for matrices that are $\Sigma_{p,q}$-Hermitian, skew Hermitian or unitary, it is difficult to derive the appropriate structured Schur like forms with similarity transformations that are both unitary and $\Sigma_{p,q}$-unitary, since this class has only a very small dimension. Currently the best that one can do in this respect are the fishbone like forms of [1]. As mentioned above the approach that we present here will be taken as the first step for the structured Schur-like forms. To make the idea more clear let us consider the case of $\Sigma_{p,q}$-Hermitian matrices. The discussion for the other cases is similar. There are many different approaches that one can take to derive canonical and condensed forms for such matrices. A very simple approach to obtain a canonical form is the idea to express the $\Sigma_{p,q}$-Hermitian matrix $\mathcal{C}$ as an Hermitian pencil $\lambda\Sigma_{p,q} - \Sigma_{p,q}\mathcal{C}$. Using congruence transformations $U^H(\lambda\Sigma_{p,q} - \Sigma_{p,q}\mathcal{C})U$, we obtain a canonical form via classical results, see e.g., [7, 12, 13, 5]. In view of our goals, however, this is not quite what we want, since in general these forms do not give that $U^H\Sigma_{p,q}U = \Sigma_{p,q}$, hence they do not lead directly to the desired structured form. Clearly, however, the characteristic quantities that we obtain from this canonical form will have to appear in our canonical form, too.

The outline of the paper is as follows: We will present some basic preliminary results and some notations in Section 2 and then present structured canonical forms for $\Sigma_{p,q}$-Hermitian matrices and $\Sigma_{p,q}$-skew Hermitian matrices under $\Sigma_{p,q}$-unitary similarity transformations in Section 3 and Section 4, respectively. By combining the Cayley transformation and the structured canonical forms for $\Sigma_{p,q}$-skew Hermitian matrices we will then derive the structured canonical forms for $\Sigma_{p,q}$-unitary matrices in Section 5. All canonical forms are represented both for real and complex matrices. For comparisons and derivations we also list the already known classical canonical forms.

The theorems for the main results are listed in Table 1.1. Here $\mathbb{C}$ and $\mathbb{R}$ represent the complex and real case respectively, and J and U represent the classical structured Jordan canonical forms and the structured Jordan forms under $\Sigma_{p,q}$-unitary similarity

	$\Sigma_{p,q}$-Hermitian	$\Sigma_{p,q}$-skew Hermitian	$\Sigma_{p,q}$-unitary
C& J	Theorem 3.1	Theorem 4.1	Theorem 5.9
R& J	Theorem 3.2	Theorem 4.2	Theorem 5.10
C& U	Theorem 3.3	Theorem 4.3	Theorem 5.11
R& U	Theorem 3.6	Theorem 4.4	Theorem 5.12

TABLE 1.1
Main results

transformations, respectively.

2. Preliminaries. In this section we introduce the notation and give some preliminary results that are needed for the canonical forms. Our construction of structured Jordan froms will be based on the combination of different blocks of the classical, unstructured Jordan form. Let us recall some facts from the classical theory.

Let $\Lambda(A)$ denote the spectrum of a matrix A. We begin with a well-known fact on the relationship between left and right invariant subspaces, which follows clearly from the Jordan canonical form.

PROPOSITION 2.1. *Let the columns of U span the left invariant subspace of a square matrix A corresponding to $\lambda_1 \in \Lambda(A)$ and let the columns of V span the right invariant subspace corresponding to $\lambda_2 \in \Lambda(A)$. If $\lambda_1 \neq \lambda_2$ then $U^H V = 0$ and if $\lambda_1 = \lambda_2$ then* $\det(U^H V) \neq 0$.

Let

$$N_r := \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix}$$

be an $r \times r$ nilpotent Jordan block. Define accordingly

$$P_r := \begin{bmatrix} & & -1 \\ & (-1)^2 & \\ \cdot & & \\ (-1)^r & & \end{bmatrix}, \qquad \hat{P}_r := \begin{bmatrix} & & 1 \\ & \cdot & \\ \cdot & & \\ 1 & & \end{bmatrix}_{r\times r}.$$

For any given nilpotent matrix

$$N = \operatorname{diag}(N_{r_1}, \ldots, N_{r_s})$$

we set

$$P_N := \operatorname{diag}(P_{r_1}, \ldots, P_{r_s}), \qquad \hat{P}_N := \operatorname{diag}(\hat{P}_{r_1}, \ldots, \hat{P}_{r_s}).$$

Then these matrices have the following easily verified properties.

PROPOSITION 2.2.
i) $P_r^H = P_r^{-1} = (-1)^{r-1} P_r$;
ii) $P_N^{-1} N^H P_N = -N$;
iii) $\hat{P}_N = \hat{P}_N^H = \hat{P}_N^{-1}$;
iv) $\hat{P}_N^{-1} N^H \hat{P}_N = N$.

For matrices A and B, $A \otimes B = [a_{ij}B]$ denotes the Kronecker product and for a $t \times t$ matrix Z with $t = 1$, or 2 we set

$$N_r(Z) := I_r \otimes Z + N_r \otimes I_t, \quad N(Z) := I \otimes Z + N \otimes I_t.$$

Also set

$$\Sigma_{1,1} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad J_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

From [5], or from a direct derivation as done in [8] we have the following properties.

PROPOSITION 2.3. *Let $\mathcal{C}$ be a complex square matrix and let λ be an eigenvalue of $\mathcal{C}$ with associated Jordan structure $N(\lambda)$. If* $\Lambda := \begin{bmatrix} \operatorname{Re}\lambda & \operatorname{Im}\lambda \\ -\operatorname{Im}\lambda & \operatorname{Re}\lambda \end{bmatrix}$, *then we have the following results.*

I.a *If $\mathcal{C}$ is complex $\Sigma_{p,q}$-Hermitian, then there exists a nonsingular matrix U such that*

i) if $\operatorname{Im}\lambda \neq 0$, *then*

$$U^H \Sigma_{p,q} U = \begin{bmatrix} 0 & \hat{P}_N \\ \hat{P}_N^H & 0 \end{bmatrix}, \qquad \mathcal{C}U = U \begin{bmatrix} N(\lambda) & 0 \\ 0 & N(\bar{\lambda}) \end{bmatrix},$$

ii) if λ is real, then

$$U^H \Sigma_{p,q} U = \operatorname{diag}(\pi_1 \hat{P}_{r_1} \dots, \pi_s \hat{P}_{r_s}), \qquad \mathcal{C}U = UN(\lambda),$$

where all $\pi_i = \pm 1$.

I.b *If $\mathcal{C}$ is real $\Sigma_{p,q}$-symmetric, then there exists a real nonsingular matrix U such that*

i) if $\operatorname{Im}\lambda \neq 0$, *then*

$$U^T \Sigma_{p,q} U = \hat{P}_N \otimes \Sigma_{1,1}, \qquad \mathcal{C}U = UN(\Lambda),$$

ii) if λ is real, then

$$U^T \Sigma_{p,q} U = \operatorname{diag}(\pi_1 \hat{P}_{r_1} \dots, \pi_s \hat{P}_{r_s}), \qquad \mathcal{C}U = UN(\lambda),$$

where all $\pi_i = \pm 1$.

II.a *If $\mathcal{C}$ is complex $\Sigma_{p,q}$-skew Hermitian, then there exists a nonsingular matrix U such that*

i) if $\operatorname{Re}\lambda \neq 0$, *then*

$$U^H \Sigma_{p,q} U = \begin{bmatrix} 0 & P_N \\ P_N^H & 0 \end{bmatrix}, \qquad \mathcal{C}U = U \begin{bmatrix} N(\lambda) & 0 \\ 0 & N(-\bar{\lambda}) \end{bmatrix},$$

ii) if $\operatorname{Re}\lambda = 0$, *then*

$$U^H \Sigma_{p,q} U = \operatorname{diag}(\pi_1 P_{r_1} \dots, \pi_s P_{r_s}), \qquad \mathcal{C}U = UN(\lambda),$$

where $\pi_k = \pm i$ for even r_k and $\pi_k = \pm 1$ if odd r_k.

II.b *If $\mathcal{C}$ is real $\Sigma_{p,q}$-skew symmetric, then there exists a real nonsingular matrix U such that* *i) if* $\operatorname{Re}\lambda \operatorname{Im}\lambda \neq 0$, *then*

$$U^T\Sigma_{p,q}U = \begin{bmatrix} 0 & P_N \otimes \Sigma_{1,1} \\ P_N^T \otimes \Sigma_{1,1} & 0 \end{bmatrix}, \ \mathcal{C}U = U \begin{bmatrix} N(\Lambda) & 0 \\ 0 & N(-\Lambda) \end{bmatrix},$$

ii) if $\operatorname{Re}\lambda \neq 0$, $\operatorname{Im}\lambda = 0$ *then*

$$U^T\Sigma_{p,q}U = \begin{bmatrix} 0 & P_N \\ P_N^T & 0 \end{bmatrix}, \qquad \mathcal{C}U = U \begin{bmatrix} N(\lambda) & 0 \\ 0 & N(-\lambda) \end{bmatrix},$$

iii) if $\lambda \neq 0$, $\operatorname{Re}\lambda = 0$, *then*

$$U^T\Sigma_{p,q}U = \operatorname{diag}(P_{r_1} \otimes \Xi_1, \ldots, P_{r_s} \otimes \Xi_s), \qquad \mathcal{C}U = UN((\operatorname{Im}\lambda)J_1),$$

where $\Xi_k = \pi_k I_2$ *for odd* r_k *and* $\Xi_k = (\operatorname{Im}\pi_k)J_1$ *for even* r_k, *and* π_k *is as in the complex case.*
iv) if $\lambda = 0$, *then*

$$U^T\Sigma_{p,q}U = \operatorname{diag}\left(\pi_1 P_{2u_1+1}, \ldots, \pi_a P_{2u_a+1}, \begin{bmatrix} 0 & P_{2v_1} \\ P_{2v_1}^T & 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 & P_{2v_b} \\ P_{2v_b}^T & 0 \end{bmatrix}\right),$$
$$\mathcal{C}U = UN := U \operatorname{diag}(N_{2u_1+1}, \ldots, N_{2u_a+1}, N_{2v_1}, N_{2v_1}, \ldots, N_{2v_b}, N_{2v_b}).$$

Note that the parameters π_k are invariant in the sense that for each group of Jordan blocks with the same size corresponding to λ the numbers of 1, -1, or i, $-i$, of the corresponding π_k are uniquely determined. For this reason we denote the complete set of parameters by $\operatorname{Ind}(\lambda) = \{\pi_1, \ldots, \pi_s\}$ and call this the *structure inertia index.* Note that for each Jordan block there is a unique corresponding structure inertia index.

Some obvious facts on the symmetry of the eigenvalues follow directly from Proposition 2.3.

PROPOSITION 2.4. *Let* λ *be an eigenvalue of a square matrix* $\mathcal{C}$. *Then we have the following properties.*
I. *If* $\mathcal{C}$ *is* $\Sigma_{p,q}$*-Hermitian (both complex and real) and* $\operatorname{Im}\lambda \neq 0$ *then* $\bar{\lambda}$ *is also an eigenvalue of* $\mathcal{C}$ *with the same Jordan structure as* λ.
II.a *If* $\mathcal{C}$ *is complex* $\Sigma_{p,q}$*-skew Hermitian and* λ *is not purely imaginary, then* $-\bar{\lambda}$ *is also an eigenvalue of* $\mathcal{C}$ *with the same Jordan structure as* λ.
II.b *If* $\mathcal{C}$ *is real* $\Sigma_{p,q}$*-skew symmetric and* λ *is not purely imaginary, then* $\bar{\lambda}$, $-\bar{\lambda}$, $-\lambda$ *are eigenvalues of* $\mathcal{C}$ *with the same Jordan structures as* λ. *If* $\lambda = 0$ *is an eigenvalue, then the number of each even sized corresponding Jordan blocks must be even.*

We will frequently use transformations with

$$\Upsilon_r = \frac{\sqrt{2}}{2} \begin{bmatrix} I_r & -I_r \\ I_r & I_r \end{bmatrix}, \tag{2.1}$$

for which we have

$$\Upsilon_r^H \begin{bmatrix} 0 & I_r \\ I_r & 0 \end{bmatrix} \Upsilon_r = \begin{bmatrix} I_r & 0 \\ 0 & -I_r \end{bmatrix}. \tag{2.2}$$

For $A \in \mathbf{C}^{n\times n}$ a simple calculation yields

$$\Upsilon_n^{-1} \begin{bmatrix} A & 0 \\ 0 & \pm A^H \end{bmatrix} \Upsilon_n = \begin{bmatrix} \frac{A\pm A^H}{2} & -\frac{A\mp A^H}{2} \\ -\frac{A\mp A^H}{2} & \frac{A\pm A^H}{2} \end{bmatrix}. \tag{2.3}$$

A variation of Υ_r is

$$\hat{\Upsilon}_r = \frac{\sqrt{2}}{2}\begin{bmatrix} I_r & 0 & -I_r \\ 0 & \sqrt{2} & 0 \\ I_r & 0 & I_r \end{bmatrix}. \tag{2.4}$$

We will also need, in the following, the symmetric and skew symmetric part of Jordan blocks

$$N_r^+ = \frac{1}{2}(N_r + N_r^T), \quad N_r^- = \frac{1}{2}(N_r - N_r^T),$$

and for a $t \times t$ matrix Z with $t = 1, 2$, we set

$$N_r^+(Z) = I_r \otimes Z + N_r^+ \otimes I_t, \quad N_r^-(Z) = I_r \otimes Z + N_r^- \otimes I_t.$$

Similarly we denote

$$N^+ = \frac{1}{2}(N + N^T), \quad N^- = \frac{1}{2}(N - N^T),$$

and for Z of $t \times t$ with $t = 1, 2$,

$$N^+(Z) = I \otimes Z + N^+ \otimes I_t, \quad N^-(Z) = I \otimes Z + N^- \otimes I_t.$$

Finally the symbol e_k represents the k-th unit vector.

With these notations and results in hand in the next two sections we will give the canonical forms for $\Sigma_{p,q}$-Hermitian and $\Sigma_{p,q}$-skew Hermitian matrices.

3. $\Sigma_{p,q}$-Hermitian matrices. In this section we derive structured Jordan canonical forms for $\Sigma_{p,q}$-Hermitian matrices. We will always consider two forms, a structured canonical form where the transformation matrices are not necessarily $\Sigma_{p,q}$-unitary and a structured canonical form under $\Sigma_{p,q}$-unitary matrices. Also we will always consider two cases, the complex $\Sigma_{p,q}$-Hermitian and the real $\Sigma_{p,q}$-symmetric matrices.

THEOREM 3.1. *Let $\mathcal{C}$ be a complex $\Sigma_{p,q}$-Hermitian matrix with pairwise different real eigenvalues $\alpha_1, \ldots, \alpha_\nu$ and pairwise different eigenvalues $\lambda_1, \ldots, \lambda_\mu$, with positive imaginary parts. Then there exists a nonsingular matrix $\mathcal{U}$ such that*

$$\mathcal{U}^{-1}\mathcal{C}\mathcal{U} = \mathrm{diag}(R_c^+, R_c^-, R_r),$$

where the blocks are

$$\begin{aligned} R_c^+ &= \mathrm{diag}(H_1(\lambda_1), \ldots, H_\mu(\lambda_\mu)), \quad R_c^- = \mathrm{diag}(H_1(\overline{\lambda_1}), \ldots, H_\mu(\overline{\lambda_\mu})), \\ R_r &= \mathrm{diag}(M_1(\alpha_1), \ldots, M_\nu(\alpha_\nu)), \end{aligned}$$

with substructures

$$H_k(\lambda_k) = \lambda_k I + H_k, \quad H_k(\overline{\lambda_k}) = \overline{\lambda_k} I + H_k, \quad H_k = \mathrm{diag}(N_{p_{k,1}}, \ldots, N_{p_{k,s_k}}),$$

for $k = 1, \ldots, \mu$, and

$$M_k(\alpha_k) = \alpha_k I + M_k, \quad M_k = \mathrm{diag}(N_{q_{k,1}}, \ldots N_{q_{k,t_k}}),$$

for $k = 1, \ldots, \nu$.

The matrix $\mathcal{U}$ satisfies

$$\mathcal{U}^H \Sigma_{p,q} \mathcal{U} = \begin{bmatrix} 0 & W_c & 0 \\ W_c^H & 0 & 0 \\ 0 & 0 & W_r \end{bmatrix}, \tag{3.1}$$

with $W_c = \operatorname{diag}(\hat{P}_{H_1}, \ldots, \hat{P}_{H_\mu})$ *and* $W_r = \operatorname{diag}(W_1^r, \ldots, W_\nu^r)$*, where for* $k = 1, \ldots, \mu$ *we have* $\hat{P}_{H_k} = \operatorname{diag}(\hat{P}_{p_{k,1}}, \ldots, \hat{P}_{p_{k,s_k}})$ *and for* $k = 1, \ldots, \nu$ *and* $\operatorname{Ind}(\alpha_k) = \{\pi_{k,1}, \ldots, \pi_{k,t_k}\}$ *we have* $W_k^r = \operatorname{diag}(\pi_{k,1}\hat{P}_{q_{k,1}}, \ldots, \pi_{k,t_k}\hat{P}_{q_{k,t_k}})$.

Proof. For each nonreal eigenvalue λ_k with the corresponding Jordan structure $H_k(\lambda_k)$, by I.a, i) of Proposition 2.3, we can choose a matrix U_k such that

$$U_k^H \Sigma_{p,q} U_k = \begin{bmatrix} 0 & \hat{P}_{H_k} \\ \hat{P}_{H_k}^H & 0 \end{bmatrix}, \quad \mathcal{C} U_k = U_k \operatorname{diag}(H_k(\lambda_k), H_k(\overline{\lambda_k})). \tag{3.2}$$

Partition $U_k = [U_{k,1}, U_{k,2}]$, where $U_{k,1}$, $U_{k,2}$ have the same size and set

$$\mathcal{U}_c = [U_{1,1}, \ldots, U_{\mu,1}; U_{1,2}, \ldots, U_{\mu,2}] = [\mathcal{U}_1^c; \mathcal{U}_2^c].$$

Note that by the symmetry of $\mathcal{C}$, the columns of $\mathcal{U}_1^c$, $\Sigma_{p,q}\mathcal{U}_2^c$ and $\mathcal{U}_2^c$, $\Sigma_{p,q}\mathcal{U}_1^c$ form bases of the right and left invariant subspaces corresponding to the two disjoint sets of eigenvalues $\{\lambda_1, \ldots, \lambda_\mu\}$ and $\{\overline{\lambda_1}, \ldots, \overline{\lambda_\mu}\}$, respectively. By Proposition 2.1 and (3.2) we have

$$\mathcal{U}_c^H \Sigma_{p,q} \mathcal{U}_c = \begin{bmatrix} 0 & W_c \\ W_c^H & 0 \end{bmatrix}, \quad \mathcal{C}\mathcal{U} = \mathcal{U} \operatorname{diag}(R_c^+, R_c^-),$$

with W_c, R_c^+ and R_c^- as asserted.

For each real eigenvalue α_k with the corresponding Jordan structure $M_k(\alpha_k)$, by I.a, ii) of Proposition 2.3 we can choose a matrix V_k such that

$$V_k^H \Sigma_{p,q} V_k = \operatorname{diag}(\pi_{k,1}\hat{P}_{q_{k,1}}, \ldots, \pi_{k,t_k}\hat{P}_{q_{k,t_k}}),$$

where $\operatorname{Ind}(\alpha_k) = \{\pi_{k,1}, \ldots, \pi_{k,t_k}\}$ and $\mathcal{C}V_k = V_k M_k(\alpha_k)$. Set $\mathcal{U}_r = [V_1, \ldots, V_\nu]$, then by Proposition 2.1 we have

$$\mathcal{U}_r^H \Sigma_{p,q} \mathcal{U}_r = W_r, \quad \mathcal{C}\mathcal{U}_r = \mathcal{U}_r R_r,$$

where W_r and R_r are of the asserted forms and with $\mathcal{U} = [\mathcal{U}_c, \mathcal{U}_r]$ the result follows from Proposition 2.1. □

Similarly for real $\Sigma_{p,q}$-symmetric matrices by employing I.b of Proposition 2.3 we have the following forms.

THEOREM 3.2. *Let $\mathcal{C}$ be a real $\Sigma_{p,q}$-symmetric matrix with pairwise different real eigenvalues $\alpha_1, \ldots, \alpha_\nu$ and pairwise different eigenvalues $\lambda_1, \ldots, \lambda_\mu$, with positive imaginary parts.*

Then there exists a real full rank matrix $\mathcal{U}$ such that

$$\mathcal{U}^{-1}\mathcal{C}\mathcal{U} = \operatorname{diag}(R_c, R_r),$$

where

$$R_c = \operatorname{diag}(H_1(\Lambda_1), \ldots, H_\mu(\Lambda_\mu))$$

and for $k=1,\ldots,\mu$ *the subblocks are* $H_k(\Lambda_k)=\operatorname{diag}(N_{p_{k,1}}(\Lambda_k),\ldots,N_{p_{k,s_k}}(\Lambda_k))$ *with* $\Lambda_k=\begin{bmatrix}\operatorname{Re}\lambda_k & \operatorname{Im}\lambda_k\\ -\operatorname{Im}\lambda_k & \operatorname{Re}\lambda_k\end{bmatrix}$. *The other diagonal block is*

$$R_r=\operatorname{diag}(M_1(\alpha_1),\ldots,M_\nu(\alpha_\nu)),$$

where for $k=1,\ldots,\nu$ *the subblocks are* $M_k(\alpha_k)=\alpha_k I+M_k$ *with* $M_k=\operatorname{diag}(N_{q_{k,1}},\ldots,N_{q_{k,t_k}})$. *The matrix* $\mathcal{U}$ *has the form*

$$\mathcal{U}^T\Sigma_{p,q}\mathcal{U}=\begin{bmatrix}W_c & 0\\ 0 & W_r\end{bmatrix}, \tag{3.3}$$

where $W_c=\operatorname{diag}(\hat{P}_{H_1}\otimes\Sigma_{1,1},\ldots,\hat{P}_{H_\mu}\otimes\Sigma_{1,1})$, $W_r=\operatorname{diag}(W_1^r,\ldots,W_\nu^r)$, *and where for* $k=1,\ldots,\mu$ *we have* $\hat{P}_{H_k}=\operatorname{diag}(\hat{P}_{p_{k,1}},\ldots,\hat{P}_{p_{k,s_k}})$ *and for* $\operatorname{Ind}(\alpha_k)=\{\pi_{k,1},\ldots,\pi_{k,t_k}\}$ *and* $k=1,\ldots,\nu$ *we have* $W_k^r=\operatorname{diag}(\pi_{k,1}\hat{P}_{q_{k,1}},\ldots,\pi_{k,t_k}\hat{P}_{q_{k,t_k}})$.

The canonical forms in Theorems 3.1 and 3.2 are just the results of [5] in matrix form. They are just the classical Jordan canonical forms, but the transformation matrices are constructed in such a way that they satisfy the relationship (3.1) and (3.3), respectively, associated with $\Sigma_{p,q}$. This is not quite what we want, since we wish to have that the transformation matrix is $\Sigma_{p,q}$-unitary. The following results give the structured canonical forms under $\Sigma_{p,q}$-unitary transformations.

THEOREM 3.3. *Let* $\mathcal{C}$ *be a* $\Sigma_{p,q}$*-Hermitian matrix with pairwise different real eigenvalues* $\alpha_1,\ldots,\alpha_\nu$ *and pairwise different eigenvalues* $\lambda_1,\ldots,\lambda_\mu$, *with positive imaginary parts. Then there exists a* $\Sigma_{p,q}$*-unitary matrix* $\mathcal{U}$ *such that*

$$\mathcal{U}^{-1}\mathcal{C}\mathcal{U}=\begin{bmatrix}R_c & & T_c & \\ & R_r^+ & & T_r\\ -T_c^H & & R_c & \\ & -T_r^H & & R_r^-\end{bmatrix}. \tag{3.4}$$

For the blocks we have the following substructures.
i) The blocks with index c, associated with the nonreal eigenvalues, are

$$R_c=\operatorname{diag}(R_1^c,\ldots,R_\mu^c),\quad T_c=\operatorname{diag}(T_1^c,\ldots,T_\mu^c),$$

where for $k=1,\ldots,\mu$ *we have*

$$R_k^c=\operatorname{diag}(N_{p_{k,1}}^+(\operatorname{Re}\lambda_k),\ldots,N_{p_{k,s_k}}^+(\operatorname{Re}\lambda_k)),$$
$$T_k^c=-\operatorname{diag}(N_{p_{k,1}}^-(i\operatorname{Im}\lambda_k),\ldots,N_{p_{k,s_k}}^-(i\operatorname{Im}\lambda_k)).$$

ii) The blocks with index r, associated with the real eigenvalues are

$$R_r^+=\operatorname{diag}(C_1,\ldots,C_\nu),\quad R_r^-=\operatorname{diag}(D_1,\ldots,D_\nu),\quad T_r=\operatorname{diag}(F_1,\ldots,F_\nu).$$

For $k=1,\ldots,\nu$ *these have the substructures*

$$C_k=\operatorname{diag}(C_k^e,C_k^+,C_k^-),\quad D_k=\operatorname{diag}(D_k^e,D_k^+,D_k^-),\quad F_k=\operatorname{diag}(F_k^e,F_k^+,F_k^-),$$

where

$$C_k^e=\operatorname{diag}(N_{q_{k,1}}^+(\alpha_k)+\frac{1}{2}\pi_{k,1}e_{q_{k,1}}e_{q_{k,1}}^H,\ldots,N_{q_{k,t_k}}^+(\alpha_k)+\frac{1}{2}\pi_{k,t_k}e_{q_{k,t_k}}e_{q_{k,t_k}}^H),$$

$$
\begin{aligned}
D_k^e &= \mathrm{diag}(N^+_{q_{k,1}}(\alpha_k) - \frac{1}{2}\pi_{k,1}e_{q_{k,1}}e^H_{q_{k,1}}, \ldots, N^+_{q_{k,t_k}}(\alpha_k) - \frac{1}{2}\pi_{k,t_k}e_{q_{k,t_k}}e^H_{q_{k,t_k}}),\\
F_k^e &= \mathrm{diag}(-N^-_{q_{k,1}} + \frac{1}{2}\pi_{k,1}e_{q_{k,1}}e^H_{q_{k,1}}, \ldots, -N^-_{q_{k,t_k}} + \frac{1}{2}\pi_{k,t_k}e_{q_{k,t_k}}e^H_{q_{k,t_k}}),\\
C_k^+ &= \mathrm{diag}\left(\begin{bmatrix} N^+_{u_{k,1}}(\alpha_k) & \frac{\sqrt{2}}{2}e_{u_{k,1}} \\ \frac{\sqrt{2}}{2}e^H_{u_{k,1}} & \alpha_k \end{bmatrix}, \ldots, \begin{bmatrix} N^+_{u_{k,w_k}}(\alpha_k) & \frac{\sqrt{2}}{2}e_{u_{k,w_k}} \\ \frac{\sqrt{2}}{2}e^H_{u_{k,w_k}} & \alpha_k \end{bmatrix}\right),\\
D_k^+ &= \mathrm{diag}(N^+_{u_{k,1}}(\alpha_k), \ldots, N^+_{u_{k,w_k}}(\alpha_k)),\\
F_k^+ &= \mathrm{diag}\left(\begin{bmatrix} -N^-_{u_{k,1}} \\ \frac{\sqrt{2}}{2}e^H_{u_{k,1}} \end{bmatrix}, \ldots, \begin{bmatrix} -N^-_{u_{k,w_k}} \\ \frac{\sqrt{2}}{2}e^H_{u_{k,w_k}} \end{bmatrix}\right),\\
C_k^- &= \mathrm{diag}(N^+_{v_{k,1}}(\alpha_k), \ldots, N^+_{v_{k,z_k}}(\alpha_k)),\\
D_k^- &= \mathrm{diag}\left(\begin{bmatrix} \alpha_k & -\frac{\sqrt{2}}{2}e^H_{v_{k,1}} \\ -\frac{\sqrt{2}}{2}e_{v_{k,1}} & N^+_{v_{k,1}}(\alpha_k) \end{bmatrix}, \ldots, \begin{bmatrix} \alpha_k & -\frac{\sqrt{2}}{2}e^H_{v_{k,z_k}} \\ -\frac{\sqrt{2}}{2}e_{v_{k,z_k}} & N^+_{v_{k,z_k}}(\alpha_k) \end{bmatrix}\right),\\
F_k^- &= \mathrm{diag}([\frac{\sqrt{2}}{2}e_{v_{k,1}}, -N^-_{v_{k,1}}], \ldots, [\frac{\sqrt{2}}{2}e_{v_{k,z_k}}, -N^-_{v_{k,z_k}}]).
\end{aligned}
$$

Here each nonreal λ_k $(\overline{\lambda_k})$ has s_k Jordan blocks of sizes $p_{k,1}, \ldots, p_{k,s_k}$ and each real eigenvalue α_k has
a) t_k even sized Jordan blocks of sizes $2q_{k,1}, \ldots, 2q_{k,t_k}$ and the corresponding structure inertia indices $\pi_{k,1}, \ldots, \pi_{k,t_k}$;
b) w_k odd sized Jordan blocks of sizes $2u_{k,1}+1, \ldots, 2u_{k,w_k}+1$, corresponding to the structure inertia index 1;
c) z_k odd sized Jordan blocks of sizes $2v_{k,1}+1, \ldots, 2v_{k,z_k}+1$, corresponding to the structure inertia index -1.

Proof. Let $m := \sum_{i=1}^{\mu}\sum_{j=1}^{s_i} p_{ij}$. For $\lambda_1, \ldots, \lambda_\mu$, by Theorem 3.1 there exists a matrix $\mathcal{U}_c$ such that

$$
\hat{\mathcal{U}}_c^H \Sigma_{p,q} \hat{\mathcal{U}}_c = \begin{bmatrix} 0 & W_c \\ W_c^H & 0 \end{bmatrix}, \qquad \mathcal{C}\hat{\mathcal{U}}_c = \hat{\mathcal{U}}_c \begin{bmatrix} R_c^+ & 0 \\ 0 & R_c^- \end{bmatrix}.
$$

Note $W_c^H W_c = W_c^2 = I_m$. Setting $\tilde{\mathcal{U}}_c := \hat{\mathcal{U}}_c \,\mathrm{diag}(I_m, W_c)$ then using the form of R_c^- and Proposition 2.2 we have

$$
\tilde{\mathcal{U}}_c^H \Sigma_{p,q} \tilde{\mathcal{U}}_c = \begin{bmatrix} 0 & I_m \\ I_m & 0 \end{bmatrix}, \qquad \mathcal{C}\tilde{\mathcal{U}}_c = \tilde{\mathcal{U}}_c \begin{bmatrix} R_c^+ & 0 \\ 0 & (R_c^+)^H \end{bmatrix}.
$$

Now let $\mathcal{U}_c := \tilde{\mathcal{U}}_c \Upsilon_m$, where Υ_m is defined in (2.1). By (2.2) and (2.3) and the special form of R_c^+ we then get

$$
\mathcal{U}_c^H \Sigma_{p,q} \mathcal{U}_c = \Sigma_{m,m}, \qquad \mathcal{C}U = U \begin{bmatrix} R_c & T_c \\ -T_c^H & R_c \end{bmatrix}, \tag{3.5}
$$

where R_c and T_c are in the asserted forms.

For real eigenvalues $\alpha_1, \ldots, \alpha_\nu$ the situation is relatively complicated. In this case we have to transform the Jordan blocks one by one. Let α be a real eigenvalue of $\mathcal{C}$ and $N_r(\alpha)$ be a Jordan block. Following from Proposition 2.3 there exists a matrix $\hat{U}$ such that

$$
\hat{U}^H \Sigma_{p,q} \hat{U} = \pi \hat{P}_r, \qquad \mathcal{C}\hat{U} = \hat{U} N_r(\alpha),
$$

where $\pi = \pm 1$. When r is even let $U := \hat{U}\,\mathrm{diag}(I_{\frac{r}{2}}, \pi \hat{P}_{\frac{r}{2}})\Upsilon_{\frac{r}{2}}$ then one can verify that

$$(3.6)\quad U^H \Sigma_{p,q} U = \Sigma_{\frac{r}{2},\frac{r}{2}}, \quad \mathcal{C}U = U \begin{bmatrix} N^+_{\frac{r}{2}}(\alpha) + \frac{1}{2}\pi e_{\frac{r}{2}} e^H_{\frac{r}{2}} & -N^-_{\frac{r}{2}} + \frac{1}{2}\pi e_{\frac{r}{2}} e^H_{\frac{r}{2}} \\ -N^-_{\frac{r}{2}} - \frac{1}{2}\pi e_{\frac{r}{2}} e^H_{\frac{r}{2}} & N^+_{\frac{r}{2}}(\alpha) - \frac{1}{2}\pi e_{\frac{r}{2}} e^H_{\frac{r}{2}} \end{bmatrix}.$$

Similarly when r is odd setting $U := \hat{U}\,\mathrm{diag}(I_{\frac{r+1}{2}}, \pi \hat{P}_{\frac{r-1}{2}})\hat{\Upsilon}_{\frac{r-1}{2}}$, where $\hat{\Upsilon}_{\frac{r-1}{2}}$ is defined in (2.4), then if $\pi = 1$,

$$U^H \Sigma_{p,q} U = \begin{bmatrix} I_{\frac{r+1}{2}} & 0 \\ 0 & -I_{\frac{r-1}{2}} \end{bmatrix},$$

$$(3.7)\quad \mathcal{C}U = U \left[\begin{array}{cc|c} N^+_{\frac{r-1}{2}}(\alpha) & \frac{\sqrt{2}}{2} e_{\frac{r-1}{2}} & -N^-_{\frac{r-1}{2}} \\ \frac{\sqrt{2}}{2} e^H_{\frac{r-1}{2}} & \alpha & \frac{\sqrt{2}}{2} e^H_{\frac{r-1}{2}} \\ \hline -N^-_{\frac{r-1}{2}} & -\frac{\sqrt{2}}{2} e_{\frac{r-1}{2}} & N^+_{\frac{r-1}{2}}(\alpha) \end{array}\right];$$

and if $\pi = -1$

$$U^H \Sigma_{p,q} U = \begin{bmatrix} I_{\frac{r-1}{2}} & 0 \\ 0 & -I_{\frac{r+1}{2}} \end{bmatrix},$$

$$(3.8)\quad \mathcal{C}U = U \left[\begin{array}{c|cc} N^+_{\frac{r-1}{2}}(\alpha) & \frac{\sqrt{2}}{2} e_{\frac{r-1}{2}} & -N^-_{\frac{r-1}{2}} \\ \hline -\frac{\sqrt{2}}{2} e^H_{\frac{r-1}{2}} & \alpha & -\frac{\sqrt{2}}{2} e^H_{\frac{r-1}{2}} \\ -N^-_{\frac{r-1}{2}} & -\frac{\sqrt{2}}{2} e_{\frac{r-1}{2}} & N^+_{\frac{r-1}{2}}(\alpha) \end{array}\right].$$

Let α_k be a real eigenvalue with associated even sized Jordan blocks of sizes $2q_{k,1}$, $\ldots$, $2q_{k,t_k}$, and associated odd sized Jordan blocks of sizes $2u_{k,1}+1, \ldots, 2u_{k,w_k}+1$ and $2v_{k,1}+1$, $\ldots$, $2v_{k,z_k}+1$ corresponding to the structure inertia indices 1 and -1 respectively. By Proposition 2.3 there exists a matrix $\hat{U}_k$ such that

$$\begin{aligned} \hat{U}_k^H \Sigma_{p,q} \hat{U}_k &= \mathrm{diag}(\pi_{k,1}\hat{P}_{2q_{k,1}}, \ldots, \pi_{k,t_k}\hat{P}_{2q_{k,t_k}}; \hat{P}_{2u_{k,1}+1}, \ldots, \hat{P}_{2u_{k,w_k}+1}; \\ &\qquad -\hat{P}_{2v_{k,1}+1}, \ldots, -\hat{P}_{2v_{k,z_k}+1}), \\ \mathcal{C}\hat{U}_k &= \hat{U}_k\,\mathrm{diag}(N_{2q_{k,1}}(\alpha_k), \ldots, N_{2q_{k,t_k}}(\alpha_k); N_{2u_{k,1}+1}(\alpha_k), \\ &\qquad \ldots, N_{2u_{k,w_k}+1}(\alpha_k); N_{2v_{k,1}+1}(\alpha_k), \ldots, N_{2u_{k,z_k}+1}(\alpha_k)). \end{aligned}$$

Set

$$\tilde{U}_k := \hat{U}_k\,\mathrm{diag}(Z^e_{k,1}, \ldots, Z^e_{k,t_k}; Z^+_{k,1}, \ldots, Z^+_{k,w_k}; Z^-_{k,1}, \ldots, Z^-_{k,z_k}),$$

where

$$\begin{aligned} Z^e_{k,j} &= \mathrm{diag}(I_{q_{k,j}}, \pi_{k,j}\hat{P}_{q_{k,j}})\Upsilon_{q_{k,j}}, \quad Z^+_{k,j} = \mathrm{diag}(I_{u_{k,j}+1}, \hat{P}_{u_{k,j}})\hat{\Upsilon}_{u_{k,j}}, \\ Z^-_{k,j} &= \mathrm{diag}(I_{v_{k,j}+1}, -\hat{P}_{v_{k,j}})\hat{\Upsilon}_{v_{k,j}}. \end{aligned}$$

Partition

$$\begin{aligned} \tilde{U}_k = [&V^e_{k,1}, W^e_{k,1}, \ldots, V^e_{k,t_k}, W^e_{k,t_k};\ V^+_{k,1}, W^+_{k,1}, \ldots, V^+_{k,w_k}, W^+_{k,w_k}; \\ &V^-_{k,1}, W^-_{k,1}, \ldots, V^-_{k,z_k}, W^-_{k,z_k}], \end{aligned}$$

where the columns of $V^e_{k,j}, W^e_{k,j}$ are $q_{k,j}$, the columns of $V^+_{k,j}$, $W^+_{k,j}$, $V^-_{k,j}$, $W^-_{k,j}$ are $u_{k,j}+1$, $u_{k,j}$, $v_{k,j}$, $v_{k+1}+1$, respectively. Set

$$\begin{aligned} V_k &= [V^e_{k,1},\dots,V^e_{k,t_k};\ V^+_{k,1},\dots,V^+_{k,w_k};\ V^-_{k,1},\dots,V^-_{k,z_k}],\\ W_k &= [W^e_{k,1},\dots,W^e_{k,t_k};\ W^+_{k,1},\dots,W^+_{k,w_k};\ W^-_{k,1},\dots,W^-_{k,z_k}] \end{aligned}$$

and $U_k = [V_k, W_k]$. Then by employing (3.6) – (3.8) we have

$$U_k^H \Sigma_{p,q} U_k = \begin{bmatrix} I_{n_{k,1}} & 0 \\ 0 & -I_{n_{k,2}} \end{bmatrix}, \quad \mathcal{C} U_k = U_k \begin{bmatrix} C_k & F_k \\ -F_k^H & D_k \end{bmatrix},$$

where C_k, F_k, D_k are as asserted, $n_{k,1} = w_k + \sum_{l=1}^{t_k} q_{k,l} + \sum_{l=1}^{w_k} u_{k,l} + \sum_{l=1}^{z_k} v_{k,l}$ and $n_{k,2} = z_k + \sum_{l=1}^{t_k} q_{k,l} + \sum_{l=1}^{w_k} u_{k,l} + \sum_{l=1}^{z_k} v_{k,l}$. Set $n_1 = \sum_{k=1}^{\nu} n_{k,1}$, $n_2 = \sum_{k=1}^{\nu} n_{k,2}$. Then with

$$V_r = [V_1,\dots,V_\nu], \qquad W_r = [W_1,\dots,W_\nu],$$

and $\mathcal{U}_r = [V_r, W_r]$ we have

$$\mathcal{U}_r^H \Sigma_{p,q} \mathcal{U}_r = \begin{bmatrix} I_{n_1} & 0 \\ 0 & -I_{n_2} \end{bmatrix}, \quad \mathcal{C}\mathcal{U}_r = \mathcal{U}_r \begin{bmatrix} R_r^+ & T_r \\ -T_r^H & R_r^- \end{bmatrix}, \tag{3.9}$$

Finally set $\mathcal{U} = [V_c, V_r;\ W_c, W_r]$, then by Proposition 2.1 and by above construction we have

$$\mathcal{U}^H \Sigma_{p,q} \mathcal{U} = \begin{bmatrix} I_{m+n_1} & 0 \\ 0 & -I_{m+n_2} \end{bmatrix}.$$

Since $\mathcal{U}$ is nonsingular it follows that $\mathcal{U}^H \Sigma_{p,q} \mathcal{U}$ is congruent to $\Sigma_{p,q}$ and hence $m+n_1 = p$, $m+n_2 = q$ and $\mathcal{U}^H \Sigma_{p,q} \mathcal{U} = \Sigma_{p,q}$. Equation (3.4) then follows from (3.5) and (3.9). □

Remark 3.4. The difference between the structured canonical forms of Theorems 3.1 and 3.3 is that in order to get a $\Sigma_{p,q}$-unitary transformation matrix we need to refine further and combine different blocks together. This leads to a loss in structure in the Jordan canonical form, which becomes more complicated, but shows that the classical Jordan canonical form somehow obscures the extra structure in the chains of root vectors.

Remark 3.5. By the structured Jordan form we immediately obtain the following relationships

$$\begin{aligned} p &= \sum_{k=1}^{\mu}\sum_{j=1}^{s_k} p_{k,j} + \sum_{k=1}^{\nu}(w_k + \sum_{j=1}^{t_k} q_{k,j} + \sum_{j=1}^{w_k} u_{k,j} + \sum_{j=1}^{z_k} v_{k,j}),\\ q &= \sum_{k=1}^{\mu}\sum_{j=1}^{s_k} p_{k,j} + \sum_{k=1}^{\nu}(z_k + \sum_{j=1}^{t_k} q_{k,j} + \sum_{j=1}^{w_k} u_{k,j} + \sum_{j=1}^{z_k} v_{k,j}), \end{aligned}$$

$$|p-q| = |\sum_{k=1}^{\nu}(w_k - z_k)|. \tag{3.10}$$

These relationship show that the parameters p, q will affect the eigenstructure of $\mathcal{C}$. For example, we get in the case $p = 0$ (or $q = 0$) that $\mathcal{C}$, which is Hermitian now, is unitarily similar to a real diagonal matrix. Another direct consequence is that for a real eigenvalue the largest size of the associated Jordan block is not larger than $2\min\{p,q\}+1$, and for a nonreal eigenvalue the largest size of the associated Jordan block is not larger than $\min\{p,q\}$. Furthermore, it is clear that if $|p-q|\neq 0$, then $\mathcal{C}$ must have real eigenvalues with at least $|p-q|$ odd sized Jordan blocks.

The real structured Jordan canonical form for a real $\Sigma_{p,q}$-symmetric matrix, under real $\Sigma_{p,q}$-orthogonal transformations can be obtained analogously.

THEOREM 3.6. *Let $\mathcal{C}$ be a real $\Sigma_{p,q}$-symmetric matrix with pairwise different real eigenvalues $\alpha_1,\ldots,\alpha_\nu$ and pairwise different eigenvalues $\lambda_1,\ldots,\lambda_\mu$ with positive imaginary parts. Then there exists a real $\Sigma_{p,q}$-orthogonal matrix $\mathcal{U}$, such that*

$$\mathcal{U}^{-1}\mathcal{C}\mathcal{U}=\begin{bmatrix} R_c^+ & & T_c & \\ & R_r^+ & & T_r \\ -T_c^T & & R_c^- & \\ & -T_r^T & & R_r^- \end{bmatrix}. \tag{3.11}$$

i) The blocks with index c, associated with nonreal eigenvalues, are

$$R_c^+=\operatorname{diag}(A_1,\ldots,A_\mu),\quad R_c^-=\operatorname{diag}(B_1,\ldots,B_\mu),\quad T_c=\operatorname{diag}(T_1^c,\ldots,T_\mu^c),$$

where for $k=1,\ldots,\mu$ we have

$$\begin{aligned}
A_k &= \operatorname{diag}(A_k^e,A_k^o),\quad B_k=\operatorname{diag}(B_k^e,B_k^o),\quad T_k^c=\operatorname{diag}(T_k^e,T_k^o),\\
A_k^e &= \operatorname{diag}(N_{p_{k,1}}^+((\operatorname{Re}\lambda_k)I_2)+E_{k,1},\ldots,N_{p_{k,s_k}}^+((\operatorname{Re}\lambda_k)I_2)+E_{k,s_k}),\\
B_k^e &= \operatorname{diag}(N_{p_{k,1}}^+((\operatorname{Re}\lambda_k)I_2)-E_{k,1},\ldots,N_{p_{k,s_k}}^+((\operatorname{Re}\lambda_k)I_2)-E_{k,s_k}),\\
T_k^e &= -\operatorname{diag}(N_{p_{k,1}}^-((\operatorname{Im}\lambda_k)J_1)-E_{k,1},\ldots,N_{p_{k,s_k}}^-((\operatorname{Im}\lambda_k)J_1)-E_{k,s_k}),\\
A_k^o &= \operatorname{diag}\left(\begin{bmatrix} N_{l_{k,1}}^+((\operatorname{Re}\lambda_k)I_2) & \frac{\sqrt{2}}{2}e_{2l_{k,1}-1}\\ \frac{\sqrt{2}}{2}e_{2l_{k,1}-1}^T & \operatorname{Re}\lambda_k\end{bmatrix},\right.\\
&\qquad\qquad \left.\ldots,\begin{bmatrix} N_{l_{k,x_k}}^+((\operatorname{Re}\lambda_k)I_2) & \frac{\sqrt{2}}{2}e_{2l_{k,x_k}-1}\\ \frac{\sqrt{2}}{2}e_{2l_{k,x_k}-1}^T & \operatorname{Re}\lambda_k\end{bmatrix}\right),\\
B_k^o &= \operatorname{diag}\left(\begin{bmatrix} N_{l_{k,1}}^+((\operatorname{Re}\lambda_k)I_2) & \frac{\sqrt{2}}{2}e_{2l_{k,1}}\\ \frac{\sqrt{2}}{2}e_{2l_{k,1}}^T & \operatorname{Re}\lambda_k\end{bmatrix},\right.\\
&\qquad\qquad \left.\ldots,\begin{bmatrix} N_{l_{k,x_k}}^+((\operatorname{Re}\lambda_k)I_2) & \frac{\sqrt{2}}{2}e_{2l_{k,x_k}}\\ \frac{\sqrt{2}}{2}e_{2l_{k,x_k}}^T & \operatorname{Re}\lambda_k\end{bmatrix}\right),\\
T_k^o &= \operatorname{diag}\left(\begin{bmatrix} -N_{l_{k,1}}^-((\operatorname{Im}\lambda_k)J_1) & -\frac{\sqrt{2}}{2}e_{2l_{k,1}}\\ \frac{\sqrt{2}}{2}e_{2l_{k,1}-1}^T & -\operatorname{Im}\lambda_k\end{bmatrix},\right.\\
&\qquad\qquad \left.\ldots,\begin{bmatrix} -N_{l_{k,x_k}}^-((\operatorname{Im}\lambda_k)J_1) & -\frac{\sqrt{2}}{2}e_{2l_{k,x_k}}\\ \frac{\sqrt{2}}{2}e_{2l_{k,x_k}-1}^T & -\operatorname{Im}\lambda_k\end{bmatrix}\right),
\end{aligned}$$

with $E_{k,j}=\frac{1}{2}\begin{bmatrix} 0 & 0\\ 0 & \Sigma_{1,1}\end{bmatrix}$.

ii) The blocks with index r, associated with real eigenvalues, are

$$R_r^+=\operatorname{diag}(C_1,\ldots,C_\nu),\quad R_r^-=\operatorname{diag}(D_1,\ldots,D_\nu),\quad T_r=\operatorname{diag}(F_1,\ldots,F_\nu).$$

These have for $k=1,\ldots,\nu$ the substructures

$$C_k=\operatorname{diag}(C_k^e,C_k^+,C_k^-),\quad D_k=\operatorname{diag}(D_k^e,D_k^+,D_k^-),\quad F_k=\operatorname{diag}(F_k^e,F_k^+,F_k^-),$$

where

$$\begin{aligned}
C_k^e &= \operatorname{diag}(N^+_{q_{k,1}}(\alpha_k)+\frac{1}{2}\pi_{k,1}e_{q_{k,1}}e^T_{q_{k,1}},\ldots,N^+_{q_{k,t_k}}(\alpha_k)+\frac{1}{2}\pi_{k,t_k}e_{q_{k,t_k}}e^T_{q_{k,t_k}}),\\
D_k^e &= \operatorname{diag}(N^+_{q_{k,1}}(\alpha_k)-\frac{1}{2}\pi_{k,1}e_{q_{k,1}}e^T_{q_{k,1}},\ldots,N^+_{q_{k,t_k}}(\alpha_k)-\frac{1}{2}\pi_{k,t_k}e_{q_{k,t_k}}e^T_{q_{k,t_k}}),\\
F_k^e &= \operatorname{diag}(-N^-_{q_{k,1}}+\frac{1}{2}\pi_{k,1}e_{q_{k,1}}e^T_{q_{k,1}},\ldots,-N^-_{q_{k,t_k}}+\frac{1}{2}\pi_{k,t_k}e_{q_{k,t_k}}e^T_{q_{k,t_k}}),\\
C_k^+ &= \operatorname{diag}\left(\left[\begin{array}{cc} N^+_{u_{k,1}}(\alpha_k) & \frac{\sqrt{2}}{2}e_{u_{k,1}}\\ \frac{\sqrt{2}}{2}e^T_{u_{k,1}} & \alpha_k\end{array}\right],\ldots,\left[\begin{array}{cc} N^+_{u_{k,w_k}}(\alpha_k) & \frac{\sqrt{2}}{2}e_{u_{k,w_k}}\\ \frac{\sqrt{2}}{2}e^T_{u_{k,w_k}} & \alpha_k\end{array}\right]\right),\\
D_k^+ &= \operatorname{diag}(N^+_{u_{k,1}}(\alpha_k),\ldots,N^+_{u_{k,w_k}}(\alpha_k)),\\
F_k^+ &= \operatorname{diag}\left(\left[\begin{array}{c} -N^-_{u_{k,1}}\\ \frac{\sqrt{2}}{2}e^T_{u_{k,1}}\end{array}\right],\ldots,\left[\begin{array}{c} -N^-_{u_{k,w_k}}\\ \frac{\sqrt{2}}{2}e^T_{u_{k,w_k}}\end{array}\right]\right),\\
C_k^- &= \operatorname{diag}(N^+_{v_{k,1}}(\alpha_k),\ldots,N^+_{v_{k,z_k}}(\alpha_k)),\\
D_k^- &= \operatorname{diag}\left(\left[\begin{array}{cc} \alpha_k & -\frac{\sqrt{2}}{2}e^T_{v_{k,1}}\\ -\frac{\sqrt{2}}{2}e_{v_{k,1}} & N^+_{v_{k,1}}(\alpha_k)\end{array}\right],\ldots,\left[\begin{array}{cc} \alpha_k & -\frac{\sqrt{2}}{2}e^T_{v_{k,z_k}}\\ -\frac{\sqrt{2}}{2}e_{v_{k,z_k}} & N^+_{v_{k,z_k}}(\alpha_k)\end{array}\right]\right),\\
F_k^- &= \operatorname{diag}([\frac{\sqrt{2}}{2}e_{v_{k,1}},-N^-_{v_{k,1}}],\ldots[\frac{\sqrt{2}}{2}e_{v_{k,z_k}},-N^-_{v_{k,z_k}}]).
\end{aligned}$$

Each λ_k $(\overline{\lambda_k})$ has s_k even sized Jordan blocks of sizes $2p_{k,1},\ldots,2p_{k,s_k}$, and x_k odd sized Jordan blocks of sizes $2l_{k,1}+1,\ldots,2l_{k,x_k}+1$.

For each real eigenvalue α_k there are
a) t_k even sized Jordan blocks of sizes $2q_{k,1},\ldots,2q_{k,t_k}$ corresponding to the structure inertia indices $\pi_{k,1},\ldots,\pi_{k,t_k}$;
b) w_k odd sized Jordan blocks of sizes $2u_{k,1}+1,\ldots,2u_{k,w_k}+1$ corresponding to the structure inertia index 1;
c) z_k odd sized Jordan blocks of sizes $2v_{k,1}+1,\ldots,2v_{k,z_k}+1$ corresponding to the structure inertia index -1.

Proof. For real eigenvalues $\alpha_1,\ldots,\alpha_\nu$, using I.b, ii) of Proposition 2.3, as in the proof of Theorem 3.3, there exists a real matrix $\mathcal{U}_r:=[V_r,W_r]$ such that

$$\mathcal{U}_r^T\Sigma_{p,q}\mathcal{U}_r=\left[\begin{array}{cc} I_{n_1} & 0\\ 0 & -I_{n_2}\end{array}\right],\quad \mathcal{C}\mathcal{U}_r=\mathcal{U}_r\left[\begin{array}{cc} R_r^+ & T_r\\ -T_r^T & R_r^-\end{array}\right],$$

which is the real version of (3.9).

For a nonreal eigenvalue λ of $\mathcal{C}$ with a Jordan block $N_r(\lambda)$, by Proposition 2.3 there is a real matrix $\hat{U}$ such that

$$\hat{U}^T\Sigma_{p,q}\hat{U}=\hat{P}_r\otimes\Sigma_{1,1},\qquad \mathcal{C}\hat{U}=\hat{U}N(\Lambda),$$

where $\Lambda=\left[\begin{array}{cc} \operatorname{Re}\lambda & \operatorname{Im}\lambda\\ -\operatorname{Im}\lambda & \operatorname{Re}\lambda\end{array}\right]$. As for (3.6)–(3.8), if r is even set

$$U:=\hat{U}\operatorname{diag}(I_r,\hat{P}_{\frac{r}{2}}\otimes\Sigma_{1,1})\Upsilon_r,$$

and if r is odd then set

$$U := \hat{U}\,\mathrm{diag}(I_{r+1}, \hat{P}_{\frac{r-1}{2}} \otimes \Sigma_{1,1}) \begin{bmatrix} \frac{\sqrt{2}}{2}I_{r-1} & 0 & -\frac{\sqrt{2}}{2}I_{r-1} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ \frac{\sqrt{2}}{2}I_{r-1} & 0 & \frac{\sqrt{2}}{2}I_{r-1} & 0 \end{bmatrix}.$$

By Proposition 2.1 we have that

$$(\hat{P}_s\otimes\Sigma_{1,1})^{-1} = \hat{P}_s\otimes\Sigma_{1,1} = (\hat{P}_s\otimes\Sigma_{1,1})^T, \quad (\hat{P}_s\otimes\Sigma_{1,1})^T(N_s(\Lambda))(\hat{P}_s\otimes\Sigma_{1,1}) = (N_s(\Lambda))^T,$$

and some simple calculations yield

$$U^T\Sigma_{p,q}U = \begin{bmatrix} I_r & 0 \\ 0 & -I_r \end{bmatrix}, \quad \mathcal{C}U = U\begin{bmatrix} A & T \\ -T^T & B \end{bmatrix},$$

where, if $r = 2s$, then

$$A = N_s^+((\mathrm{Re}\,\lambda)I_2) + E_r, \quad B = N_s^+((\mathrm{Re}\,\lambda)I_2) - E_r, \quad T = -N_s^-((\mathrm{Im}\,\lambda)J_1) + E_r,$$

with $E_r = \frac{1}{2}\begin{bmatrix} 0 & 0 \\ 0 & \Sigma_{1,1} \end{bmatrix}$; and if $r = 2s+1$, then with $J_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$,

$$A = \begin{bmatrix} N_s^+((\mathrm{Re}\,\lambda)I_2) & \frac{\sqrt{2}}{2}e_{r-2} \\ \frac{\sqrt{2}}{2}e_{r-2}^T & \mathrm{Re}\,\lambda \end{bmatrix}, \quad B = \begin{bmatrix} N_s^+((\mathrm{Re}\,\lambda)I_2) & \frac{\sqrt{2}}{2}e_{r-1} \\ \frac{\sqrt{2}}{2}e_{r-1}^T & \mathrm{Re}\,\lambda \end{bmatrix},$$
$$T = \begin{bmatrix} -N_s^-((\mathrm{Im}\,\lambda)J_1) & -\frac{\sqrt{2}}{2}e_{r-1} \\ \frac{\sqrt{2}}{2}e_{r-2}^T & -\mathrm{Im}\,\lambda \end{bmatrix}.$$

Now as for the case of real eigenvalues in the proof for Theorem 3.3, for nonreal eigenvalues $\lambda_1, \ldots, \lambda_\mu$ there exists a real matrix $\mathcal{U}_c := [V_c, W_c]$ such that

$$\mathcal{U}_c^T\Sigma_{p,q}\mathcal{U}_c = \Sigma_{m,m}, \qquad \mathcal{C}\mathcal{U}_c = \mathcal{U}_c\begin{bmatrix} R_c^+ & T_c \\ -T_c^T & R_c^- \end{bmatrix},$$

where R_c^+, T_c, R_c^- are in the asserted forms and $m = \sum_{k=1}^{\mu}(\sum_{j=1}^{s_k} 2p_{kj} + \sum_{j=1}^{x_k}(2l_{kj}+1))$.

Set $\mathcal{U} = [V_c, V_r;\ W_c, W_r]$ then analogously we get that $\mathcal{U}$ is real $\Sigma_{p,q}$-orthogonal and $\mathcal{U}^{-1}\mathcal{C}\mathcal{U}$ has the form (3.11). $\square$

In this section we have obtained real and complex structured Jordan canonical forms for $\Sigma_{p,q}$-Hermitian matrices. In the next section we present analogous results for $\Sigma_{p,q}$-skew Hermitian matrices.

4. $\Sigma_{p,q}$-skew Hermitian matrices. In this section we discuss structured Jordan canonical forms for $\Sigma_{p,q}$-skew Hermitian matrices. The construction is similar to that for $\Sigma_{p,q}$-Hermitian matrices discussed in Section 3 and therefore we omit much of the detail. The essential difference is that the role of the real eigenvalues is now taken by the purely imaginary eigenvalues.

Analogous to the $\Sigma_{p,q}$-Hermitian matrices by employing the results in Proposition 2.3 we have the following Jordan canonical forms both for complex and real $\Sigma_{p,q}$-skew Hermitian matrices.

THEOREM 4.1. *Let $\mathcal{C}$ be a $\Sigma_{p,q}$-skew Hermitian matrix with pairwise different purely imaginary eigenvalues $\sigma_1,\ldots,\sigma_\nu$ and pairwise different eigenvalues $\lambda_1,\ldots,\lambda_\mu$ with positive real parts. Then there exists a nonsingular matrix $\mathcal{U}$ such that*

$$\mathcal{U}^{-1}\mathcal{C}\mathcal{U} = \operatorname{diag}(R_c^+, R_c^-, R_g),$$

i) The diagonal blocks with index c, associated with eigenvalues not on the imaginary axis, are

$$R_c^+ = \operatorname{diag}(H_1(\lambda_1),\ldots,H_\mu(\lambda_\mu)),\ R_c^- = \operatorname{diag}(H_1(-\overline{\lambda_1}),\ldots,H_\mu(-\overline{\lambda_\mu})),$$

where for $k=1,\ldots,\mu$ we have

$$H_k(\lambda_k) = \lambda_k I + H_k,\ H_k(-\overline{\lambda_k}) = -\overline{\lambda_k} I + H_k,\ H_k = \operatorname{diag}(N_{p_{k,1}},\ldots,N_{p_{k,s_k}}).$$

ii) The block R_g, associated with purely imaginary eigenvalues, has the form

$$R_g = \operatorname{diag}(M_1(\sigma_1),\ldots,M_\nu(\sigma_\nu)),$$

where $M_k(\sigma_k) = \sigma_k I + M_k$ and for $k=1,\ldots,\nu$ we have $M_k = \operatorname{diag}(N_{q_{k,1}},\ldots,N_{q_{k,t_k}})$.

The matrix $\mathcal{U}$ has the form

$$\mathcal{U}^H \Sigma_{p,q} \mathcal{U} = \begin{bmatrix} 0 & W_c & 0 \\ W_c^H & 0 & 0 \\ 0 & 0 & W_g \end{bmatrix},$$

where

$$W_c = \operatorname{diag}(P_{H_1},\ldots,P_{H_\mu}), \qquad W_g = \operatorname{diag}(W_1^g,\ldots,W_\nu^g),$$

and for $k=1,\ldots,\mu$ we have $P_{H_k} = \operatorname{diag}(P_{p_{k,1}},\ldots,P_{p_{k,s_k}})$, and with $\operatorname{Ind}(\sigma_k) = \{\pi_{k,1},\ldots,\pi_{k,t_k}\}$ for $k=1,\ldots,\nu$ we have $W_k^g = \operatorname{diag}(\pi_{k,1}P_{q_{k,1}},\ldots,\pi_{k,t_k}P_{q_{k,t_k}})$.

THEOREM 4.2. *Let $\mathcal{C}$ be a real $\Sigma_{p,q}$-skew symmetric matrix with pairwise different nonzero purely imaginary eigenvalues $\sigma_1,\ldots,\sigma_\nu$ with positive imaginary parts, pairwise different eigenvalues $\lambda_1,\ldots,\lambda_\mu$ with positive real and imaginary parts and pairwise different real positive eigenvalues $\alpha_1,\ldots,\alpha_\eta$. (Note that when the spectrum contains σ_k, it also contains $-\sigma_k$, if it contains α_j then also $-\alpha_j$ and if it contains λ_j then also $-\lambda_j, \overline{\lambda_j}, -\overline{\lambda_j}$. Furthermore 0 may be an eigenvalue.) Then there exists a real nonsingular matrix $\mathcal{U}$ such that*

$$\mathcal{U}^{-1}\mathcal{C}\mathcal{U} = \operatorname{diag}(R_c^+, R_c^-, R_g).$$

i) The blocks with index c, associated with eigenvalues not on the imaginary axis, are $R_c^+ = \operatorname{diag}(\hat{R}_c^+, \tilde{R}_c^+)$, with $\hat{R}_c^+ = \operatorname{diag}(K_1(\alpha_1),\ldots,K_\eta(\alpha_\eta))$ where for $k=1,\ldots,\eta$ we have $K_k(\alpha_k) = \alpha_k I + K_k$ and $K_k = \operatorname{diag}(N_{f_{k,1}},\ldots,N_{f_{k,l_k}})$. Analogously $\tilde{R}_c^+ = \operatorname{diag}(H_1(\Lambda_1),\ldots,H_\mu(\Lambda_\mu))$, where for $k=1,\ldots,\mu$ we have $\Lambda_k = \begin{bmatrix} \operatorname{Re}\lambda_k & \operatorname{Im}\lambda_k \\ -\operatorname{Im}\lambda_k & \operatorname{Re}\lambda_k \end{bmatrix}$ and $H_k(\Lambda_k) = \operatorname{diag}(N_{p_{k,1}}(\Lambda_k),\ldots,N_{p_{k,s_k}}(\Lambda_k))$.

The block $R_c^- = \operatorname{diag}(\hat{R}_c^-, \tilde{R}_c^-)$, has the same substructure as R_c^+ just replacing α_j with $-\alpha_j$ and Λ_j by $-\Lambda_j$.

ii) The block R_g, associated with purely imaginary eigenvalues, has the structure

$$R_g = \operatorname{diag}(M_1((\operatorname{Im}\sigma_1)J_1),\ldots,M_\nu((\operatorname{Im}\sigma_\nu)J_1), M_0),$$

where for $k = 1, \ldots, \nu$ we have

$$M_k((\operatorname{Im}\sigma_k)J_1) = \operatorname{diag}(N_{q_{k,1}}((\operatorname{Im}\sigma_k)J_1), \ldots, N_{q_{k,t_k}}((\operatorname{Im}\sigma_k)J_1)),$$

and where

$$M_0 = \operatorname{diag}(N_{2g_1+1}, \ldots, N_{2g_a+1}, N_{2h_1}, N_{2h_1}, \ldots, N_{2h_b}, N_{2h_b})$$

is the structure associated with the eigenvalue 0.

The matrix $\mathcal{U}$ *has the form*

$$\mathcal{U}^T\Sigma_{p,q}\mathcal{U} = \begin{bmatrix} 0 & W_c & 0 \\ W_c^T & 0 & 0 \\ 0 & 0 & W_g \end{bmatrix},$$

where $W_c = \operatorname{diag}(\hat{W}_c, \tilde{W}_c)$ *with*

$$\hat{W}_c = \operatorname{diag}(P_{K_1}, \ldots, P_{K_\eta}), \quad \tilde{W}_c = \operatorname{diag}(P_{H_1} \otimes \Sigma_{1,1}, \ldots, P_{H_\mu} \otimes \Sigma_{1,1}),$$

and where

$$P_{K_k} = \operatorname{diag}(P_{f_{k,1}}, \ldots, P_{f_{k,l_k}}), \qquad P_{H_k} = \operatorname{diag}(P_{p_{k,1}}, \ldots, P_{p_{k,s_k}}).$$

The block W_g *has the form* $W_g = \operatorname{diag}(W_1^g, \ldots, W_\nu^g, W_0)$*, where for* $k = 1, \ldots, \nu$ *and* $\operatorname{Ind}(\sigma_k) = \{\pi_{k,1}, \ldots, \pi_{k,t_k}\}$ *we have* $W_k^g = \operatorname{diag}(P_{q_{k,1}} \otimes \Xi_{k,1}, \ldots, P_{q_{k,t_k}} \otimes \Xi_{k,t_k})$*, with* $\Xi_{k,j} = \pi_{k,j}I_2$ *if* $q_{k,j}$ *is odd and* $\Xi_{k,j} = (\operatorname{Im}\pi_{k,j})J_1$ *if* $q_{k,j}$ *is even.*

Finally for $\operatorname{Ind}(0) = \{\pi_1^0, \ldots, \pi_a^0, i, -i, \ldots, i, -i\}$ *we have*

$$W_0 = \operatorname{diag}(\pi_1^0 P_{2g_1+1}, \ldots, \pi_a^0 P_{2g_a+1}, \begin{bmatrix} 0 & P_{2h_1} \\ P_{2h_1}^T & 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 & P_{2h_b} \\ P_{2h_b}^T & 0 \end{bmatrix}).$$

After determining the Jordan structure under non $\Sigma_{p,q}$-unitary similarity transformations we now derive the corresponding structured canonical form under $\Sigma_{p,q}$-unitary transformations.

THEOREM 4.3. *Let* $\mathcal{C}$ *be a* $\Sigma_{p,q}$*-skew Hermitian matrix with pairwise distinct eigenvalues* $\lambda_1, \ldots, \lambda_\mu$ *with positive real parts and pairwise distinct* $\sigma_1, \ldots, \sigma_\nu$ *with real part zero. Then there exists a* $\Sigma_{p,q}$*-unitary matrix* $\mathcal{U}$*, such that*

$$\mathcal{U}^{-1}\mathcal{C}\mathcal{U} = \begin{bmatrix} R_c & & T_c & \\ & R_g^+ & & T_g \\ T_c^H & & R_c & \\ & T_g^H & & R_g^- \end{bmatrix}. \tag{4.1}$$

For the different blocks we have the following substructures.

i) The blocks with index c, associated with eigenvalues not on the imaginary axis, are $R_c = \operatorname{diag}(R_1^c, \ldots, R_\mu^c)$ *and* $T_c = \operatorname{diag}(T_1^c, \ldots, T_\mu^c)$ *where for* $k = 1, \ldots, \mu$

$$R_k^c = \operatorname{diag}(N_{p_{k,1}}^-(i\operatorname{Im}\lambda_k), \ldots, N_{p_{k,s_k}}^-(i\operatorname{Im}\lambda_k)),$$
$$T_k^c = -\operatorname{diag}(N_{p_{k,1}}^+(\operatorname{Re}\lambda_k), \ldots, N_{p_{k,s_k}}^+(\operatorname{Re}\lambda_k)).$$

ii) The blocks with index g, associated with purely imaginary eigenvalues, are

$$R_g^+ = \operatorname{diag}(C_1, \ldots, C_\nu),\ R_g^- = \operatorname{diag}(D_1, \ldots, D_\nu),\ T_g = \operatorname{diag}(F_1, \ldots, F_\nu),$$

where for $k = 1, \ldots, \nu$ the substructures are

$$C_k = \mathrm{diag}(C_k^e, C_k^+, C_k^-),\ D_k = \mathrm{diag}(D_k^e, D_k^+, D_k^-),\ F_k = \mathrm{diag}(F_k^e, F_k^+, F_k^-),$$

with further partitioning

$$\begin{aligned}
C_k^e &= \mathrm{diag}(N^-_{q_{k,1}}(\sigma_k) + \frac{1}{2} i\beta_{k,1} e_{q_{k,1}} e^H_{q_{k,1}}, \ldots, N^-_{q_{k,t_k}}(\sigma_k) + \frac{1}{2} i\beta_{k,t_k} e_{q_{k,t_k}} e^H_{q_{k,t_k}}),\\
D_k^e &= \mathrm{diag}(N^-_{q_{k,1}}(\sigma_k) - \frac{1}{2} i\beta_{k,1} e_{q_{k,1}} e^H_{q_{k,1}}, \ldots, N^-_{q_{k,t_k}}(\sigma_k) - \frac{1}{2} i\beta_{k,t_k} e_{q_{k,t_k}} e^H_{q_{k,t_k}}),\\
F_k^e &= \mathrm{diag}(-N^+_{q_{k,1}} + \frac{1}{2} i\beta_{k,1} e_{q_{k,1}} e^H_{q_{k,1}}, \ldots, -N^+_{q_{k,t_k}} + \frac{1}{2} i\beta_{k,t_k} e_{q_{k,t_k}} e^H_{q_{k,t_k}}),\\
C_k^+ &= \mathrm{diag}\left(\begin{bmatrix} N^-_{u_{k,1}}(\sigma_k) & \frac{\sqrt{2}}{2} e_{u_{k,1}} \\ -\frac{\sqrt{2}}{2} e^H_{u_{k,1}} & \sigma_k \end{bmatrix}, \ldots, \begin{bmatrix} N^-_{u_{k,w_k}}(\sigma_k) & \frac{\sqrt{2}}{2} e_{u_{k,w_k}} \\ -\frac{\sqrt{2}}{2} e^H_{u_{k,w_k}} & \sigma_k \end{bmatrix}\right),\\
D_k^+ &= \mathrm{diag}(N^-_{u_{k,1}}(\sigma_k), \ldots, N^-_{u_{k,w_k}}(\sigma_k)),\\
F_k^+ &= -\mathrm{diag}\left(\begin{bmatrix} N^+_{u_{k,1}} \\ \frac{\sqrt{2}}{2} e^H_{u_{k,1}} \end{bmatrix}, \ldots, \begin{bmatrix} N^+_{u_{k,w_k}} \\ \frac{\sqrt{2}}{2} e^H_{u_{k,w_k}} \end{bmatrix}\right),\\
C_k^- &= \mathrm{diag}(N^-_{v_{k,1}}(\sigma_k), \ldots, N^-_{v_{k,z_k}}(\sigma_k)),\\
D_k^- &= \mathrm{diag}\left(\begin{bmatrix} \sigma_k & \frac{\sqrt{2}}{2} e^H_{v_{k,1}} \\ -\frac{\sqrt{2}}{2} e_{v_{k,1}} & N^-_{v_{k,1}}(\sigma_k) \end{bmatrix}, \ldots, \begin{bmatrix} \sigma_k & \frac{\sqrt{2}}{2} e^H_{v_{k,z_k}} \\ -\frac{\sqrt{2}}{2} e_{v_{k,z_k}} & N^-_{v_{k,z_k}}(\sigma_k) \end{bmatrix}\right),\\
F_k^- &= \mathrm{diag}([\frac{\sqrt{2}}{2} e_{v_{k,1}}, -N^+_{v_{k,1}}], \ldots [\frac{\sqrt{2}}{2} e_{v_{k,z_k}}, -N^+_{v_{k,z_k}}]).
\end{aligned}$$

Each λ_k ($-\overline{\lambda_k}$) has s_k Jordan blocks of sizes $p_{k,1}, \ldots, p_{k,s_k}$. Each purely imaginary eigenvalue σ_k has
a) t_k even sized Jordan blocks of sizes $2q_{k,1}, \ldots, 2q_{k,t_k}$ with the corresponding structure inertia indices $i(-1)^{q_{k,1}+1}\beta_{k,1}, \ldots, i(-1)^{q_{k,t_k}+1}\beta_{k,t_k}$;
b) w_k odd sized Jordan blocks of sizes $2u_{k,1}+1, \ldots, 2u_{k,w_k}+1$ corresponding to the structure inertia indices $(-1)^{u_{k,1}+1}, \ldots, (-1)^{u_{k,w_k}+1}$;
c) z_k odd sized Jordan blocks of sizes $2v_{k,1}+1, \ldots, 2v_{k,z_k}+1$ corresponding to the structure indices $(-1)^{v_{k,1}}, \ldots, (-1)^{v_{k,z_k}}$.

Proof. For all eigenvalues $\lambda_1, \ldots, \lambda_\mu$, by Theorem 4.1 there is a matrix $\hat{\mathcal{U}}_c$ such that

$$\hat{\mathcal{U}}_c^H \Sigma_{p,q} \hat{\mathcal{U}} = \begin{bmatrix} 0 & W_c \\ W_c^H & 0 \end{bmatrix}, \qquad \mathcal{C}\hat{\mathcal{U}}_c = \hat{\mathcal{U}}_c \begin{bmatrix} R_c^+ & 0 \\ 0 & R_c^- \end{bmatrix},$$

where W_c, R_c^+, R_c^- are defined in Theorem 4.1. Let $\mathcal{U}_c := \hat{\mathcal{U}}_c \,\mathrm{diag}(I_m, W_c^{-1})\Upsilon_m$, where Υ_m is defined in (2.1) and $m = \sum_{k=1}^{\mu}\sum_{j=1}^{s_k} p_{kj}$. By using i), ii) of Proposition 2.2 and (2.2), (2.3) we have

$$\mathcal{U}_c^H \Sigma_{p,q} \mathcal{U}_c = \Sigma_{m,m}, \qquad \mathcal{C}\mathcal{U}_c = \mathcal{U}_c \begin{bmatrix} R_c & T_c \\ T_c^H & R_c \end{bmatrix}, \tag{4.2}$$

where R_c, T_c are as asserted.

Now we consider the purely imaginary eigenvalues. As for the real eigenvalues of $\Sigma_{p,q}$-Hermitian matrices we first focus on one Jordan block $N_r(\sigma)$ with σ purely imaginary. According to Proposition 2.3 for this block there is a matrix $\hat{U}$ such that

$$\hat{U}^H \Sigma_{p,q} \hat{U} = \pi P_r, \qquad \mathcal{C}\hat{U} = \hat{U} N_r(\sigma),$$

where $\pi = \pm 1$ if r is odd and $\pi = \pm i$ if r is even. Similarly if r is even let $U := \hat{U}\,\mathrm{diag}(I_{\frac{r}{2}}, (\pi P_{\frac{r}{2}})^{-1})\Upsilon_{\frac{r}{2}}$. Then

$$U^H \Sigma_{p,q} U = \begin{bmatrix} I_{\frac{r}{2}} & 0 \\ 0 & -I_{\frac{r}{2}} \end{bmatrix},\ \mathcal{C}U = U \begin{bmatrix} N^-_{\frac{r}{2}}(\sigma) + \frac{1}{2} i\beta e_{\frac{r}{2}} e^H_{\frac{r}{2}} & -N^+_{\frac{r}{2}} + \frac{1}{2} i\beta e_{\frac{r}{2}} e^H_{\frac{r}{2}} \\ -N^+_{\frac{r}{2}} - \frac{1}{2} i\beta e_{\frac{r}{2}} e^H_{\frac{r}{2}} & N^-_{\frac{r}{2}}(\sigma) - \frac{1}{2} i\beta e_{\frac{r}{2}} e^H_{\frac{r}{2}} \end{bmatrix},$$

where $\beta = (-1)^{\frac{r}{2}} i\pi$. If r is odd let $U := \hat{U}\,\mathrm{diag}(I_{\frac{r+1}{2}}, (\pi P_{\frac{r-1}{2}})^{-1})\hat{\Upsilon}_{\frac{r-1}{2}}$. Then

$$U^H \Sigma_{p,q} U = \begin{bmatrix} I_{\frac{r-1}{2}} & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & -I_{\frac{r-1}{2}} \end{bmatrix},\ \mathcal{C}U = U \begin{bmatrix} N^-_{\frac{r-1}{2}}(\sigma) & \frac{\sqrt{2}}{2} e_{\frac{r-1}{2}} & -N^+_{\frac{r-1}{2}} \\ -\frac{\sqrt{2}}{2}\beta e^H_{\frac{r-1}{2}} & \sigma & -\frac{\sqrt{2}}{2}\beta e^H_{\frac{r-1}{2}} \\ -N^+_{\frac{r-1}{2}} & -\frac{\sqrt{2}}{2} e_{\frac{r-1}{2}} & N^-_{\frac{r-1}{2}}(\sigma) \end{bmatrix},$$

where $\beta = (-1)^{\frac{r+1}{2}}\pi$. Note that $\beta = \pm 1$, so here $U^H \Sigma_{p,q} U$ is either $\Sigma_{\frac{r+1}{2},\frac{r-1}{2}}$ or $\Sigma_{\frac{r-1}{2},\frac{r+1}{2}}$ depending on the sign of β.

Applying these formulas to all purely imaginary eigenvalues $\sigma_1, \ldots, \sigma_\nu$, analogous to the real eigenvalue case in Theorem 3.3 for $\Sigma_{p,q}$-Hermitian matrices we can construct a matrix $\mathcal{U}_g$ such that

$$\mathcal{U}_g^H \Sigma_{p,q} \mathcal{U}_g = \Sigma_{n_1,n_2}, \qquad \mathcal{C}\mathcal{U}_g = \mathcal{U}_g \begin{bmatrix} R_g^+ & T_g \\ T_g^H & R_g^- \end{bmatrix},$$

where R_g^+, R_g^-, T_g are defined in the theorem and n_1, n_2 are the sizes of R_g^+, R_g^-, respectively.

The $\Sigma_{p,q}$-unitary matrix $\mathcal{U}$ can then be generated from $\mathcal{U}_c, \mathcal{U}_g$, and by combining above relation with (4.2) we have (4.1). $\square$

As the final result in this section we present the real version of Theorem 4.3.

THEOREM 4.4. *Let $\mathcal{C}$ be a real $\Sigma_{p,q}$-skew symmetric matrix with pairwise distinct real positive eigenvalues $\alpha_1, \ldots, \alpha_\eta$, pairwise distinct eigenvalues $\lambda_1, \ldots, \lambda_\mu$ with positive real and imaginary parts and pairwise distinct purely imaginary eigenvalues $\sigma_1, \ldots, \sigma_\nu$ with positive imaginary parts. (Note that we then also have eigenvalues $-\alpha_1, \ldots, -\alpha_\eta$, $\overline{\lambda_1}, \ldots, \overline{\lambda_\mu}$, $-\lambda_1, \ldots, -\lambda_\mu$, $-\overline{\lambda_1}, \ldots, -\overline{\lambda_\mu}$ and $-\sigma_1, \ldots, -\sigma_\eta$ and also 0 may be another eigenvalue.)*

Then there exists a real $\Sigma_{p,q}$-orthogonal matrix $\mathcal{U}$, such that

$$\mathcal{U}^{-1}\mathcal{C}\mathcal{U} = \begin{bmatrix} R_c & & T_c & \\ & R_g^+ & & T_g \\ T_c^T & & R_c & \\ & T_g^T & & R_g^- \end{bmatrix}, \tag{4.3}$$

where the different blocks have the following substructures:
i) The blocks with index c, associated with the eigenvalues with nonzero real part, are

$$\begin{aligned} R_c &= \mathrm{diag}(\hat{R}_c, \tilde{R}_c), \quad T_c = \mathrm{diag}(\hat{T}_c, \tilde{T}_c), \\ \hat{R}_c &= \mathrm{diag}(\hat{R}^c_1, \ldots, \hat{R}^c_\eta), \quad \tilde{R}_c = \mathrm{diag}(\tilde{R}^c_1, \ldots, \tilde{R}^c_\mu), \\ \hat{T}_c &= \mathrm{diag}(\hat{T}^c_1, \ldots, \hat{T}^c_\eta), \quad \tilde{T}_c = \mathrm{diag}(\tilde{T}^c_1, \ldots, \tilde{T}^c_\mu), \end{aligned}$$

where for $k = 1, \ldots, \eta$ the substructures are

$$\hat{R}^c_k = \mathrm{diag}(N^-_{f_{k,1}}, \ldots, N^-_{f_{k,l_k}}), \quad \hat{T}^c_k = -\,\mathrm{diag}(N^+_{f_{k,1}}(\alpha_k), \ldots, N^+_{f_{k,l_k}}(\alpha_k)),$$

and for $k=1,\dots,\mu$,

$$\tilde{R}_k^c = \operatorname{diag}(N^-_{p_{k,1}}((\operatorname{Im}\lambda_k)J_1),\dots,N^-_{p_{k,s_k}}((\operatorname{Im}\lambda_k)J_1)),$$
$$\tilde{T}_k^c = -\operatorname{diag}(N^+_{p_{k,1}}((\operatorname{Re}\lambda_k)I_2),\dots,N^+_{p_{k,s_k}}((\operatorname{Re}\lambda_k)I_2)).$$

ii) The blocks with index g*, associated with the purely imaginary eigenvalues, are*

$$R_g^+ = \operatorname{diag}(C_1,\dots,C_\nu,C_0),\ R_g^- = \operatorname{diag}(D_1,\dots,D_\nu,D_0),\ T_g = \operatorname{diag}(F_1,\dots,F_\nu,F_0),$$

with the partitioning

$$C_k = \operatorname{diag}(C_k^e,C_k^+,C_k^-),\quad D_k = \operatorname{diag}(D_k^e,D_k^+,D_k^-),\quad F_k = \operatorname{diag}(F_k^e,F_k^+,F_k^-),$$

and for $k=1,\dots,\nu$ *the blocks have the further substructure*

$$C_k^e = \operatorname{diag}(N^-_{q_{k,1}}((\operatorname{Im}\sigma_k)J_1)+E_{k,1}\dots,N^-_{q_{k,t_k}}((\operatorname{Im}\sigma_k)J_1)+E_{k,t_k}),$$
$$D_k^e = \operatorname{diag}(N^-_{q_{k,1}}((\operatorname{Im}\sigma_k)J_1)-E_{k,1},\dots,N^-_{q_{k,t_k}}((\operatorname{Im}\sigma_k)J_1)-E_{k,t_k}),$$
$$F_k^e = \operatorname{diag}(-N^+_{q_{k,1}}(0_2)+E_{k,1},\dots,-N^+_{q_{k,t_k}}(0_2)+E_{k,t_k}),$$

$$C_k^+ = \operatorname{diag}\left(\left[\begin{array}{c|c} N^-_{u_{k,1}}((\operatorname{Im}\sigma_k)J_1) & \begin{array}{c}0\\ \frac{\sqrt2}{2}I_2\end{array} \\ \hline 0\quad -\frac{\sqrt2}{2}I_2 & (\operatorname{Im}\sigma_k)J_1\end{array}\right],\right.$$
$$\left.\dots,\left[\begin{array}{c|c} N^-_{u_{k,w_k}}((\operatorname{Im}\sigma_k)J_1) & \begin{array}{c}0\\ \frac{\sqrt2}{2}I_2\end{array} \\ \hline 0\quad -\frac{\sqrt2}{2}I_2 & (\operatorname{Im}\sigma_k)J_1\end{array}\right]\right),$$

$$D_k^+ = \operatorname{diag}(N^-_{u_{k,1}}((\operatorname{Im}\sigma_k)J_1),\dots,N^-_{u_{k,w_k}}((\operatorname{Im}\sigma_k)J_1)),$$

$$F_k^+ = -\operatorname{diag}\left(\left[\begin{array}{c} N^+_{u_{k,1}}(0_2) \\ \hline 0\quad \frac{\sqrt2}{2}I_2\end{array}\right],\dots,\left[\begin{array}{c} N^+_{u_{k,w_k}}(0_2) \\ \hline 0\quad \frac{\sqrt2}{2}I_2\end{array}\right]\right),$$

$$C_k^- = \operatorname{diag}(N^-_{v_{k,1}}((\operatorname{Im}\sigma_k)J_1),\dots,N^-_{v_{k,z_k}}((\operatorname{Im}\sigma_k)J_1)),$$

$$D_k^- = \operatorname{diag}\left(\left[\begin{array}{c|c} (\operatorname{Im}\sigma_k)J_1 & 0\quad \frac{\sqrt2}{2}I_2 \\ \hline \begin{array}{c}0\\ -\frac{\sqrt2}{2}I_2\end{array} & N^-_{v_{k,1}}((\operatorname{Im}\sigma_k)J_1)\end{array}\right],\right.$$
$$\left.\dots,\left[\begin{array}{c|c} (\operatorname{Im}\sigma_k)J_1 & 0\quad \frac{\sqrt2}{2}I_2 \\ \hline \begin{array}{c}0\\ -\frac{\sqrt2}{2}I_2\end{array} & N^-_{v_{k,z_k}}((\operatorname{Im}\sigma_k)J_1)\end{array}\right]\right),$$

$$F_k^- = \operatorname{diag}\left(\left[\begin{array}{c|c} \begin{array}{c}0\\ \frac{\sqrt2}{2}I_2\end{array} & -N^+_{v_{k,1}}(0_2)\end{array}\right],\dots,\left[\begin{array}{c|c} \begin{array}{c}0\\ \frac{\sqrt2}{2}I_2\end{array} & -N^+_{v_{k,z_k}}(0_2)\end{array}\right]\right).$$

Here for $j=1,\dots,t_k$, $E_{k,j} = \frac12\beta_{k,j}\begin{bmatrix}0&0\\0&J_1\end{bmatrix}$.

Finally, the blocks with index 0*, associated to the eigenvalue* 0*, are*

$$C_0 = \operatorname{diag}(C_0^e,C_0^+,C_0^-),\quad D_0 = \operatorname{diag}(D_0^e,D_0^+,D_0^-),\quad F_0 = \operatorname{diag}(F_0^e,F_0^+,F_0^-),$$

with substructures

$$C_0^e = D_0^e = \operatorname{diag}(N^-_{2x_1},\dots,N^-_{2x_c}),\quad F_0^e = -\operatorname{diag}(N^+_{2x_1},\dots,N^+_{2x_c}),$$

$$C_0^+ = \operatorname{diag}\left(\begin{bmatrix} N_{g_1}^- & \frac{\sqrt{2}}{2}e_{g_1} \\ -\frac{\sqrt{2}}{2}e_{g_1}^T & 0 \end{bmatrix}, \ldots, \begin{bmatrix} N_{g_a}^- & \frac{\sqrt{2}}{2}e_{g_a} \\ -\frac{\sqrt{2}}{2}e_{g_a}^T & 0 \end{bmatrix}\right),$$

$$D_0^+ = \operatorname{diag}(N_{g_1}^-, \ldots, N_{g_a}^-), \quad F_0^+ = -\operatorname{diag}\left(\begin{bmatrix} N_{g_1}^+ \\ \frac{\sqrt{2}}{2}e_{g_1}^T \end{bmatrix}, \ldots, \begin{bmatrix} N_{g_a}^+ \\ \frac{\sqrt{2}}{2}e_{g_a}^T \end{bmatrix}\right),$$

$$C_0^- = \operatorname{diag}(N_{h_1}^-, \ldots, N_{h_b}^-), \ F_0^- = \operatorname{diag}\left([\frac{\sqrt{2}}{2}e_{h_1}, -N_{h_1}^+], \ldots, [\frac{\sqrt{2}}{2}e_{h_b}, -N_{h_b}^+]\right),$$

$$D_0^- = \operatorname{diag}\left(\begin{bmatrix} 0 & \frac{\sqrt{2}}{2}e_{h_1}^T \\ -\frac{\sqrt{2}}{2}e_{h_1} & N_{h_1}^- \end{bmatrix}, \ldots, \begin{bmatrix} 0 & \frac{\sqrt{2}}{2}e_{h_b}^T \\ -\frac{\sqrt{2}}{2}e_{h_b} & N_{h_b}^- \end{bmatrix}\right).$$

Each nonzero real eigenvalue α_k *(*$-\alpha_k$*) has* l_k *Jordan blocks of sizes* $f_{k,1}, \ldots, f_{k,l_k}$ *and each nonreal eigenvalue* λ_k *(*$-\lambda_k, \overline{\lambda_k}, -\overline{\lambda_k}$*) that is not on the imaginary axis has* s_k *Jordan blocks with sizes* $p_{k,1}, \ldots, p_{k,s_k}$*.*

For each nonzero purely imaginary eigenvalue σ_k *(*$-\sigma_k$*) we have*
a) t_k *even sized Jordan blocks of sizes* $2q_{k,1}, \ldots, 2q_{k,t_k}$ *with the corresponding structure inertia indices* $i(-1)^{q_{k,1}+1}\beta_{k,1}, \ldots, i(-1)^{q_{k,t_k}+1}\beta_{k,t_k}$ *for* σ_k *and* $i(-1)^{q_{k,1}}\beta_{k,1}, \ldots,$ $i(-1)^{q_{k,t_k}}\beta_{k,t_k}$ *for* $-\sigma_k$*;*
b) w_k *odd sized Jordan blocks of sizes* $2u_{k,1}+1, \ldots, 2u_{k,w_k}+1$ *corresponding to the structure inertia indices* $(-1)^{u_{k,1}+1}, \ldots, (-1)^{u_{k,w_k}+1}$*;*
c) z_k *odd sized Jordan blocks of sizes* $2v_{k,1}+1, \ldots, 2v_{k,z_k}+1$ *corresponding to the structure indices* $(-1)^{v_{k,1}}, \ldots, (-1)^{v_{k,z_k}}$*.*

The zero eigenvalue has $2c$ *even sized Jordan blocks with sizes of* $2x_1, 2x_1, \ldots,$ $2x_c, 2x_c$ *with corresponding structure inertia indices* $i, -i, \ldots, i, -i$ *, and* $a+b$ *odd sized Jordan blocks, where* a *of them have sizes* $2g_1+1, \ldots, 2g_a+1$ *with the corresponding structure inertia indices* $(-1)^{g_1+1}, \ldots, (-1)^{g_a+1}$ *and* b *of them have sizes* $2h_1+1, \ldots, 2h_b+1$ *with the corresponding structure inertia indices* $(-1)^{h_1}, \ldots, (-1)^{h_b}$*.*

Proof. As in the previous proofs we need to study the canonical forms of the non purely imaginary and purely imaginary eigenvalues separately. Here for the latter case we have to deal with two subcases, the nonzero and zero eigenvalues. For non purely imaginary eigenvalues by Theorem 4.2 there is a real matrix $\hat{\mathcal{U}}_c$ such that

$$\hat{\mathcal{U}}_c^T \Sigma_{p,q} \hat{\mathcal{U}}_c = \begin{bmatrix} 0 & W_c \\ W_c^T & 0 \end{bmatrix}, \quad \mathcal{C}\hat{\mathcal{U}}_c = \hat{\mathcal{U}}_c \begin{bmatrix} R_c^+ & 0 \\ 0 & R_c^- \end{bmatrix}.$$

Let $\mathcal{U}_c := \hat{\mathcal{U}}_c \operatorname{diag}(I_m, W_c^{-1})\Upsilon_m$, where $m := \sum_{k=1}^{\eta}\sum_{j=1}^{t_k} f_{k,j} + \sum_{k=1}^{\mu}\sum_{j=1}^{s_k} 2p_{k,j}$. By using Proposition 2.2 we can verify that

$$\mathcal{U}_c^T \Sigma_{p,q} \mathcal{U}_c = \Sigma_{m,m}, \qquad \mathcal{C}\mathcal{U}_c = \mathcal{U}_c \begin{bmatrix} R_c & T_c \\ T_c^T & R_c \end{bmatrix},$$

where R_c,T_c are as asserted.

For nonzero purely imaginary eigenvalues $\sigma_1, \ldots \sigma_\nu$ we first consider a single Jordan block $N_r(\operatorname{Im}\sigma J_1)$. By Proposition 2.3 there exists a real matrix $\hat{U}$ such that

$$\hat{U}^T \Sigma_{p,q} \hat{U} = P_r \otimes \Xi, \quad \mathcal{C}\hat{U} = \hat{U} N_r(\operatorname{Im}\sigma J_1),$$

where $\Xi = \pi I_2$ if r is odd and $\Xi = (\operatorname{Im}\pi)J_1$ is r even, and π is the structure inertia index corresponding to $N_r(\sigma)$. If r is even, then we set $U := \hat{U}\operatorname{diag}(I_r, ((\operatorname{Im}\pi)P_{\frac{r}{2}} \otimes J_1)^{-1})\Upsilon_{\frac{r}{2}}$ and obtain

$$U^T \Sigma_{p,q} U = \begin{bmatrix} I_r & 0 \\ 0 & -I_r \end{bmatrix}, \ \mathcal{C}U = U \begin{bmatrix} N_s^-((\operatorname{Im}\sigma)J_1) + E_r & -N_s^+(0_2) + E_r \\ -N_s^+(0_2) - E_r & N_s^-((\operatorname{Im}\sigma)J_1) - E_r \end{bmatrix},$$

where $E_r = \frac{1}{2}\beta \begin{bmatrix} 0 & 0 \\ 0 & J_1 \end{bmatrix}$ and $\beta = (-1)^{\frac{r}{2}} i\pi$. If r is odd, then we set

$$U := \hat{U}\,\mathrm{diag}(I_{r+1}, (\pi P_{\frac{r-1}{2}} \otimes I_2)^{-1})(\hat{\Upsilon}_{\frac{r-1}{2}} \otimes I_2)$$

we have

$$U^T \Sigma_{p,q} U = \begin{bmatrix} I_{r-1} & & \\ & \beta I_2 & \\ & & -I_{r-1} \end{bmatrix},$$

$$\mathcal{C}U = U \left[\begin{array}{c|c|c} N^-_{\frac{r-1}{2}}((\mathrm{Im}\,\sigma)J_1) & \begin{matrix} 0 \\ \frac{\sqrt{2}}{2} I_2 \end{matrix} & -N^+_{\frac{r-1}{2}}(0_2) \\ \hline \begin{matrix} 0 & -\frac{\sqrt{2}}{2}\beta I_2 \end{matrix} & (\mathrm{Im}\,\sigma)J_1 & \begin{matrix} 0 & -\frac{\sqrt{2}}{2}\beta I_2 \end{matrix} \\ \hline -N^+_{\frac{r-1}{2}}(0_2) & \begin{matrix} 0 \\ -\frac{\sqrt{2}}{2} I_2 \end{matrix} & N^-_{\frac{r-1}{2}}((\mathrm{Im}\,\sigma)J_1) \end{array}\right],$$

where $\beta = (-1)^{\frac{s+1}{2}}\pi$. Based on these properties we can construct a real matrix $\mathcal{U}_g$ such that

$$\mathcal{U}_g^T \Sigma_{p,q} \mathcal{U}_g = \Sigma_{n_1,n_2}, \qquad \mathcal{C}\mathcal{U}_g = \mathcal{U}_g \begin{bmatrix} \hat{R}_g^+ & \hat{T}_g \\ \hat{T}_g^T & \hat{R}_g^- \end{bmatrix},$$

where

$$\hat{R}_g^+ = \mathrm{diag}(C_1, \ldots, C_\nu), \quad \hat{R}_g^- = \mathrm{diag}(D_1, \ldots, D_\nu), \quad \hat{T}_g = \mathrm{diag}(F_1, \ldots, F_\nu),$$

and C_k, D_k, F_k $(k = 1, \ldots, \nu)$ are in the asserted forms and

$$n_1 = 2\sum_{k=1}^{\nu}(w_k + \sum_{j=1}^{t_k} q_{k,j} + \sum_{j=1}^{w_k} u_{k,j} + \sum_{j=1}^{z_k} v_{k,j}),$$

$$n_2 = 2\sum_{k=1}^{\nu}(z_k + \sum_{j=1}^{t_k} q_{k,j} + \sum_{j=1}^{w_k} u_{k,j} + \sum_{j=1}^{z_k} v_{k,j}).$$

For the eigenvalue zero we have distinguished between even and odd sized Jordan blocks. For odd sized Jordan blocks as N_r there exists a real matrix $\hat{U}$ such that

$$\hat{U}^T \Sigma_{p,q} \hat{U} = \pi P_r, \quad \mathcal{C}\hat{U} = \hat{U} N_r.$$

As in the purely imaginary case in the proof of Theorem 4.3 we then can generate a real matrix U from $\hat{U}$ such that

$$U^T \Sigma_{p,q} U = \begin{bmatrix} I_{\frac{r-1}{2}} & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & -I_{\frac{r-1}{2}} \end{bmatrix}, \mathcal{C}U = U \begin{bmatrix} N^-_{\frac{r-1}{2}} & \frac{\sqrt{2}}{2} e_{\frac{r-1}{2}} & -N^+_{\frac{r-1}{2}} \\ -\frac{\sqrt{2}}{2}\beta e^T_{\frac{r-1}{2}} & 0 & -\frac{\sqrt{2}}{2}\beta e^T_{\frac{r-1}{2}} \\ -N^+_{\frac{r-1}{2}} & -\frac{\sqrt{2}}{2} e_{\frac{r-1}{2}} & N^-_{\frac{r-1}{2}} \end{bmatrix},$$

with $\beta = (-1)^{\frac{r+1}{2}}\pi$. By Proposition 2.3, even sized Jordan blocks N_r must appear in pairs. More precisely for each pair of N_r, N_r there is real matrix $\hat{U}$ such that

$$\hat{U}^T \Sigma_{p,q} \hat{U} = \begin{bmatrix} 0 & P_r \\ P_r^T & 0 \end{bmatrix}, \qquad \mathcal{C}\hat{U} = \hat{U} \begin{bmatrix} N_r & 0 \\ 0 & N_r \end{bmatrix}.$$

Hence with $U := \hat{U}\,\mathrm{diag}(I_r, P_r^{-1})\Upsilon_r$ we have

$$U^T\Sigma_{p,q}U = \begin{bmatrix} I_r & 0 \\ 0 & -I_r \end{bmatrix}, \quad \mathcal{C}U = U\begin{bmatrix} N_r^- & -N_r^+ \\ -N_r^+ & N_r^- \end{bmatrix}.$$

Based on these facts for the eigenvalue zero there exists a real matrix $\mathcal{U}_z$ such that

$$\mathcal{U}_z^T\Sigma_{p,q}\mathcal{U}_z = \Sigma_{n_1^0,n_2^0}, \qquad \mathcal{C}\mathcal{U}_z = \mathcal{U}_z\begin{bmatrix} C_0 & F_0 \\ F_0^T & D_0 \end{bmatrix},$$

where C_0, F_0, D_0 are in the asserted forms, and $n_1^0 = a + \sum_{k=1}^{c} 2x_k + \sum_{k=1}^{a} g_k + \sum_{k=1}^{b} h_k$, $n_2^0 = b + \sum_{k=1}^{c} 2x_k + \sum_{k=1}^{a} g_k + \sum_{k=1}^{b} h_k$.

Finally by combining all these cases we can generate a real $\Sigma_{p,q}$-orthogonal matrix $\mathcal{U}$ from $\mathcal{U}_c, \mathcal{U}_g, \mathcal{U}_z$ which satisfies (4.3). □

We have seen that the results for $\Sigma_{p,q}$-Hermitian and $\Sigma_{p,q}$-skew Hermitian matrices are quite similar, which was to be expected, since both classes have an algebra structure. In the next section we now study the canonical forms for matrices in the associated Lie group of $\Sigma_{p,q}$-unitary matrices.

5. $\Sigma_{p,q}$-unitary matrices. In the previous two sections we have studied structured Jordan canonical forms for $\Sigma_{p,q}$-Hermitian and $\Sigma_{p,q}$-skew Hermitian matrices. Each class has an algebra structure, the $\Sigma_{p,q}$-Hermitian matrices form a Jordan algebra and the $\Sigma_{p,q}$-skew Hermitian matrices a Lie algebra. The Lie group associated with these two algebras is the class of $\Sigma_{p,q}$-unitary matrices. In order to derive structured canonical forms for this group analogous to the results for the algebras, we can make use of the Cayley transformation.

LEMMA 5.1. *If $\mathcal{A}$ is $\Sigma_{p,q}$-unitary and $1 \notin \Lambda(\mathcal{A})$ then the Cayley transformation of $\mathcal{B}$*

$$\mathcal{B} = \rho(\mathcal{A}) = (\mathcal{A}+I)(\mathcal{A}-I)^{-1} \tag{5.1}$$

is $\Sigma_{p,q}$-skew Hermitian. Conversely, if $\mathcal{A}$ is $\Sigma_{p,q}$-skew Hermitian then $\mathcal{B}$ as in (5.1) is $\Sigma_{p,q}$-unitary.

Proof. We only prove the result for the case that $\mathcal{A}$ is $\Sigma_{p,q}$-unitary. The other direction follows form the fact that $\rho(\rho(\mathcal{A})) = \mathcal{A}$.

Since $\mathcal{A}$ is $\Sigma_{p,q}$-unitary, $\Sigma_{p,q}\mathcal{A} = \mathcal{A}^{-H}\Sigma_{p,q}$. By this relation

$$\begin{aligned}\Sigma_{p,q}\mathcal{B} &= \Sigma_{p,q}(\mathcal{A}+I)(\mathcal{A}-I)^{-1} = (\mathcal{A}^{-H}+I)\Sigma_{p,q}(\mathcal{A}-I)^{-1} \\ &= (\mathcal{A}^{-H}+I)(\mathcal{A}^{-H}-I)^{-1}\Sigma_{p,q} = (I+\mathcal{A}^H)(\mathcal{A}^{-H})(\mathcal{A}^H)(I-\mathcal{A}^H)^{-1}\Sigma_{p,q} \\ &= (\mathcal{A}+I)^H(I-\mathcal{A})^{-H}\Sigma_{p,q} = -\mathcal{B}^H\Sigma_{p,q} = -(\Sigma_{p,q}\mathcal{B})^H.\end{aligned}$$

Therefore $\mathcal{B}$ is $\Sigma_{p,q}$-skew Hermitian. □

Using the Cayley transformation ρ the canonical forms of $\Sigma_{p,q}$-unitary matrices (if 1 is not an eigenvalue) can be easily obtained from the canonical form of the corresponding $\Sigma_{p,q}$-skew Hermitian matrix discussed in Section 4. However, if we Cayley transform the canonical form it is usually not a canonical form again and we need further reductions to obtain again the canonical form. But, obviously it suffices to further reduce each Jordan block separately. Before discussing these reductions, we first split the Jordan structure of a $\Sigma_{p,q}$-unitary matrix $\mathcal{G}$ into two parts, the part related to the eigenvalue 1 and the rest.

LEMMA 5.2. *Let $\mathcal{G}$ be a $\Sigma_{p,q}$-unitary matrix that has 1 as an eigenvalue. Then, there exists a nonsingular matrix $\mathcal{Y}$, such that*

$$\mathcal{Y}^H\mathcal{Y} = \operatorname{diag}(\Sigma_{p_1,q_1}, \Sigma_{p_2,q_2}), \quad \mathcal{Y}^{-1}\mathcal{G}\mathcal{Y} = \operatorname{diag}(\mathcal{G}_1, \mathcal{G}_2),$$

where $p_1+p_2 = p, q_1+q_2 = q$, $\mathcal{G}_1$ is Σ_{p_1,q_1}-unitary with $1 \notin \Lambda(\mathcal{G}_1)$ and $\mathcal{G}_2$ is Σ_{p_2,q_2}-unitary and has 1 as only eigenvalue.
Furthermore, if $\mathcal{G}$ is real, then $\mathcal{Y}$ can be chosen real, so that $\mathcal{G}_1, \mathcal{G}_2$ are also real.

Proof. Let $\hat{\mathcal{Y}}$ be a nonsingular matrix such that

$$\mathcal{G}\hat{\mathcal{Y}} = \hat{\mathcal{Y}}\operatorname{diag}(\hat{\mathcal{G}}_1, \hat{\mathcal{G}}_2) =: \hat{\mathcal{Y}}\hat{\mathcal{G}},$$

with $1 \notin \Lambda(\hat{\mathcal{G}}_1)$ and $\Lambda(\hat{\mathcal{G}}_2) = \{1\}$. Then we have $\hat{\mathcal{Y}}^H\mathcal{G}^H = \hat{\mathcal{G}}^H\hat{\mathcal{Y}}^H$ and, using the $\Sigma_{p,q}$-unitarity of $\mathcal{G}$ we have the discrete Lyapunov (or Stein) equation

$$\hat{\mathcal{G}}^H(\hat{\mathcal{Y}}^H\Sigma_{p,q}\hat{\mathcal{Y}})\hat{\mathcal{G}} = \hat{\mathcal{Y}}^H\Sigma_{p,q}\hat{\mathcal{Y}}. \tag{5.2}$$

By the diagonal block form of $\hat{\mathcal{G}}$ and the eigenvalue splitting, the solution of (5.2) has also block diagonal form, i.e., $\hat{\mathcal{Y}}^H\Sigma_{p,q}\hat{\mathcal{Y}} = \operatorname{diag}(T_1, T_2)$. Note that $\hat{\mathcal{Y}}^H\Sigma_{p,q}\hat{\mathcal{Y}}$ as well as T_1, T_2 are nonsingular Hermitian. Therefore, there exist nonsingular matrices Z_1, Z_2 such that

$$Z_1^HT_1Z_1 = \Sigma_{p_1,q_1}, \qquad Z_2^HT_2Z_2 = \Sigma_{p_2,q_2}.$$

To finish the proof, we set $\mathcal{Y} = \hat{\mathcal{Y}}\operatorname{diag}(Z_1, Z_2)$, $\mathcal{G}_1 = Z_1^{-1}\hat{\mathcal{G}}_1Z_1$ and $\mathcal{G}_2 = Z_2^{-1}\hat{\mathcal{G}}_2Z_2$.

The real case is clear, since 1 is a real eigenvalue. □

It is well known, that Cayley transformation directly leads to a rational relationship between the eigenvalues, i.e., if $\gamma \neq 1$ is an eigenvalue of a $\Sigma_{p,q}$-unitary matrix $\mathcal{G}$, then $\lambda = \rho(\gamma) = \frac{\gamma+1}{\gamma-1}$ is an eigenvalue of the Cayley transformation $\rho(\mathcal{G})$ and we have the following well-known facts.

PROPOSITION 5.3. *Let $\mathcal{G}$ be $\Sigma_{p,q}$-unitary with $1 \notin \Lambda(\mathcal{G})$. Set $\mathcal{C} = \rho(\mathcal{G})$ and let $\gamma \in \Lambda(\mathcal{G})$ and $\lambda = \rho(\gamma) \in \Lambda(\mathcal{C})$. Then*
i) $\lambda \neq 1, -1$.
ii) γ and λ have the same algebraic and geometric multiplicities.
iii) $|\gamma| = 1$ if and only if λ is purely imaginary.
iv) If $\lambda \in \Lambda(\mathcal{C})$ is not purely imaginary, then $-\overline{\lambda} = \rho(\overline{\gamma}^{-1})$ and, furthermore, $\lambda, -\overline{\lambda} \in \Lambda(\mathcal{C})$ if and only if $\gamma, \overline{\gamma}^{-1} \in \Lambda(\mathcal{G})$. In order to further reduce Cayley transformed Jordan blocks we need the following result.

LEMMA 5.4. *Let $N_r(\lambda)$ be a Jordan block with $\lambda \neq 1$ and let $\gamma = \rho(\lambda)$. Then there exists a nonsingular upper triangular matrix X_r such that*

$$X_r^{-1}\rho(N_r(\lambda))X_r = N_r(\gamma),$$

and $e_r^TX_re_r \neq 0$.

Proof. See, e.g., [8]. □

We are now prepared to present block by block the transformations of the results in Section 4.

LEMMA 5.5. *Let $\mathcal{G}$ be a $\Sigma_{p,q}$-unitary matrix and let $N(\gamma) = \gamma I + N$ with $N = \operatorname{diag}(N_{r_1}, \ldots, N_{r_s})$ be the Jordan structure of $\mathcal{G}$ corresponding to $\gamma \in \Lambda(\mathcal{G})$ with $|\gamma| \neq 1$. Then there exists a full rank matrix U such that*

$$U^H\Sigma_{p,q}U = \begin{bmatrix} 0 & \hat{P}_N \\ \hat{P}_N^H & 0 \end{bmatrix}, \quad \mathcal{G}U = U\begin{bmatrix} N(\gamma) & 0 \\ 0 & N(\overline{\gamma})^{-1} \end{bmatrix}$$

and $\overline{\gamma}^{-1} \in \Lambda(\mathcal{G})$ *has the same algebraic and geometric multiplicities as* γ.

If $\mathcal{G}$ *is real then we have two cases:*
i) If γ *is real then there exists a real full rank matrix* U *such that*

$$U^T \Sigma_{p,q} U = \begin{bmatrix} 0 & \hat{P}_N \\ \hat{P}_N^T & 0 \end{bmatrix}, \quad \mathcal{G}U = U \begin{bmatrix} N(\gamma) & 0 \\ 0 & (N(\gamma))^{-1} \end{bmatrix}.$$

ii) If γ *is nonreal then there exists a real full rank matrix* U *such that*

$$U^T \Sigma_{p,q} U = \begin{bmatrix} 0 & \hat{P}_N \otimes \Sigma_{1,1} \\ \hat{P}_N^T \otimes \Sigma_{1,1} & 0 \end{bmatrix}, \quad \mathcal{G}U = U \begin{bmatrix} N(\Gamma) & 0 \\ 0 & (N(\Gamma))^{-1} \end{bmatrix},$$

with $\Gamma = \begin{bmatrix} \operatorname{Re}\gamma & \operatorname{Im}\gamma \\ -\operatorname{Im}\gamma & \operatorname{Re}\gamma \end{bmatrix}$.

Proof. We may assume without loss of generality that $1 \notin \Lambda(\mathcal{G})$. Otherwise by Lemma 5.2 we can consider the smaller size matrix $\mathcal{G}_1$. If ρ is the Cayley transformation, then by Lemma 5.1, $\mathcal{C} = \rho(\mathcal{G})$ is $\Sigma_{p,q}$-skew Hermitian. Furthermore, $\lambda = \rho(\gamma) \in \Lambda(\mathcal{C})$ and by Proposition 5.3 ii), iv), λ is not purely imaginary and the associated Jordan structure associated with λ is $\lambda I + N$. Applying Proposition 2.3 there exists a matrix $\hat{U}$ such that

$$\hat{U}^H \Sigma_{p,q} \hat{U} = \begin{bmatrix} 0 & P_N \\ P_N^H & 0 \end{bmatrix}, \quad \mathcal{C}\hat{U} = \hat{U} \begin{bmatrix} N(\lambda) & 0 \\ 0 & N(-\overline{\lambda}) \end{bmatrix}.$$

With $\tilde{U} = \hat{U}\operatorname{diag}(I, P_N^{-1})$ and, since $P_N N P_N^H = -N^H$, we have

$$\tilde{U}^H \Sigma_{p,q} \tilde{U} = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}, \mathcal{C}\tilde{U} = \tilde{U} \begin{bmatrix} N(\lambda) & 0 \\ 0 & -(N(\lambda))^H \end{bmatrix}.$$

Using the Cayley transformation then we have

$$\mathcal{G}\tilde{U} = \tilde{U} = \begin{bmatrix} \rho(N(\lambda)) & 0 \\ 0 & \rho(-(N(\lambda))^H) \end{bmatrix}.$$

Note that

$$\begin{aligned} \rho(-N(\lambda))^H) &= (-N(\lambda)^H + I)(-N(\lambda)^H - I)^{-1} \\ &= \{(N(\lambda) - I)(N(\lambda) + I)^{-1})\}^H \\ &= \{\rho(N(\lambda))\}^{-H}. \end{aligned}$$

Applying Lemma 5.4, there exists a nonsingular matrix $X = \operatorname{diag}(X_{r_1}, \ldots, X_{r_s})$ such that $X^{-1}\rho(N(\lambda))X = N(\gamma)$. Obviously $X^H\{\rho(N(\lambda))\}^{-H}X^{-H} = N(\gamma)^{-H}$. Setting $V = \tilde{U}\operatorname{diag}(X, X^{-H})$ we obtain

$$V^H \Sigma_{p,q} V = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}, \quad \mathcal{G}V = V \begin{bmatrix} N(\gamma) & 0 \\ 0 & N(\gamma)^{-H} \end{bmatrix},$$

and taking $U = V\operatorname{diag}(I, \hat{P}_N)$ finishes the proof in the complex case..

Since the Cayley transformation of a real matrix is also real, we can apply Proposition 2.3 to get the result for the real case. ∎

This result shows that for the eigenvalues of a $\Sigma_{p,q}$-unitary matrix that are not of modulus 1, the structured canonical form cannot be of the form of a usual Jordan matrix, only half of these eigenvalues have the classical Jordan structure, while for the other half of the eigenvalues we have to involve the inverses of Jordan blocks.

For eigenvalues with $|\gamma| = 1$ the canonical structure is even more complicated. If we restrict the chains of root vectors to have the proper structures coming form a $\Sigma_{p,q}$-skew Hermitian matrices as in Proposition 2.3 then no Jordan block will appear in the canonical form. We can do further reductions for which we will need the following simple result.

LEMMA 5.6. *Given a vector $t = [t_1, \dots, t_r]^T$ and $t_r \neq 0$ then there exists a nonsingular upper triangular Toeplitz matrix T such that $T^{-1}t = e_r$.*

Proof. See [8]. □

We now study the reduction of Cayley transformed blocks arising form unimodular eigenvalues.

LEMMA 5.7. *Let $\mathcal{G}$ be a $\Sigma_{p,q}$-unitary matrix and let $\gamma \in \Lambda(\mathcal{G})$ with $|\gamma| = 1$ and $\gamma \neq 1$. Let $N_r(\gamma)$ be a single Jordan block, then there exists a full rank matrix U such that*

$$U^H \Sigma_{p,q} U = \hat{P}_r, \quad \mathcal{G}U = U \begin{bmatrix} N_s(\gamma) & i\beta e_s e_1^H N_s(\overline{\gamma})^{-1} \\ 0 & N_s(\overline{\gamma})^{-1} \end{bmatrix}, \tag{5.3}$$

if $r = 2s$ and

$$\begin{aligned} & U^H \Sigma_{p,q} U = \beta \hat{P}_r, \\ & \mathcal{G}U = U \begin{bmatrix} N_s(\gamma) & \gamma e_s & \frac{\gamma}{1-\gamma} e_s e_1^H N_s(\overline{\gamma})^{-1} \\ 0 & \gamma & -e_1^H N_s(\overline{\gamma})^{-1} \\ 0 & 0 & N_s(\overline{\gamma})^{-1} \end{bmatrix}, \end{aligned} \tag{5.4}$$

if $r = 2s + 1$.

Here $\beta = (-1)^s i\pi$ with $\pi \in \{\pm i\}$ if $r = 2s$ and $\beta = (-1)^{s+1}\pi$, $\pi \in \{\pm 1\}$ if $r = 2s + 1$ where π is the structure inertia index of the corresponding eigenvalue $\lambda = \rho(\gamma)$.

If $\mathcal{G}$ is real then we have two cases:

i) If $\gamma \neq -1$, then with $\Gamma = \begin{bmatrix} \operatorname{Re}\gamma & \operatorname{Im}\gamma \\ -\operatorname{Im}\gamma & \operatorname{Re}\gamma \end{bmatrix}$, $\hat{P}_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and

$$S(\gamma) = -\frac{1}{2} \begin{bmatrix} 1 & \frac{\operatorname{Im}\gamma}{1-\operatorname{Re}\gamma} \\ \frac{\operatorname{Im}\gamma}{1-\operatorname{Re}\gamma} & -1 \end{bmatrix},$$

there exists a real full rank matrix U such that if $r = 2s$, then

$$U^T \Sigma_{p,q} U = \hat{P}_r \otimes \Sigma_{1,1}, \quad \mathcal{G}U = U \begin{bmatrix} N_s(\Gamma) & -\beta \begin{bmatrix} 0 & 0 \\ \hat{P}_2 & 0 \end{bmatrix} N_s(\Gamma)^{-1} \\ 0 & N_s(\Gamma)^{-1} \end{bmatrix}, \tag{5.5}$$

and if $r = 2s + 1$, then

$$U^T \Sigma_{p,q} U = \beta \begin{bmatrix} 0 & 0 & \hat{P}_s \otimes \Sigma_{1,1} \\ 0 & I_2 & 0 \\ \hat{P}_s \otimes \Sigma_{1,1} & 0 & 0 \end{bmatrix},$$

(5.6)

$$\mathcal{G}U = U\left[\begin{array}{c|c|c} N_s(\Gamma) & \begin{array}{c} 0 \\ \Gamma \end{array} & \left[\begin{array}{cc} 0 & 0 \\ S(\gamma) & 0 \end{array}\right] N_s(\Gamma)^{-1} \\ \hline 0 & \Gamma & [-\Sigma_{1,1}, 0] N_s(\Gamma)^{-1} \\ \hline 0 & 0 & N_s(\Gamma)^{-1} \end{array}\right].$$

ii) If $\gamma = -1$, then there exists a real full rank matrix U such that

$$U^T \Sigma_{p,q} U = \begin{bmatrix} 0 & \hat{P}_r \\ \hat{P}_r^T & 0 \end{bmatrix}, \quad \mathcal{G}U = U \begin{bmatrix} N_r(-1) & 0 \\ 0 & N_r(-1)^{-1} \end{bmatrix}, \tag{5.7}$$

if r is even and

$$U^T \Sigma_{p,q} U = \beta \hat{P}_r, \quad \mathcal{G}U = U \begin{bmatrix} N_s(-1) & -e_s & -\frac{1}{2} e_s e_1^T N_s(-1)^{-1} \\ 0 & -1 & -e_1^T N_s(-1)^{-1} \\ 0 & 0 & N_s(-1)^{-1} \end{bmatrix}, \tag{5.8}$$

if $r = 2s + 1$. Here $\beta = (-1)^{s+1}\pi$ and π is the structure inertia index of 0 *corresponding to $\rho(\mathcal{G})$.*

Proof. We may again assume without loss of generality that $1 \notin \Lambda(\mathcal{G})$ and set $\mathcal{C} = \rho(\mathcal{G})$. By Proposition 5.3 the corresponding $\lambda = \rho(\gamma)$ now is purely imaginary, and $\mathcal{C}$ has the Jordan block $\lambda I + N_r$. Applying Proposition 2.3 there exists a matrix $\hat{U}$ such that

$$\hat{U}^H \Sigma_{p,q} \hat{U} = \pi P_r, \quad \mathcal{C}\hat{U} = \hat{U} N_r(\lambda). \tag{5.9}$$

If $r = 2s$ then $\pi \in \{\pm i\}$ and we partition

$$\hat{U}^H \Sigma_{p,q} \hat{U} = \begin{bmatrix} 0 & \pi P_s \\ (\pi P_s)^H & 0 \end{bmatrix}, \quad N_r(\lambda) = \begin{bmatrix} N_s(\lambda) & e_s e_1^H \\ 0 & N_s(\lambda) \end{bmatrix}.$$

Applying the Cayley transformation we obtain

$$\mathcal{G}\hat{U} = \hat{U}\rho(N_r(\lambda)).$$

Using the notation $\hat{N}_s(\gamma) = \rho(N_s(\lambda))$ and the property that $(N_s(\lambda) - I)^{-1} = \frac{1}{2}(\hat{N}_s(\gamma) - I)$ we obtain

$$\rho(N_r(\lambda)) = \begin{bmatrix} \hat{N}_s(\gamma) & \frac{1}{2}(I - \hat{N}_s(\gamma)) e_s e_1^H (\hat{N}_s(\gamma) - I) \\ 0 & \hat{N}_s(\gamma) \end{bmatrix}.$$

Setting $\tilde{U} = \hat{U} \operatorname{diag}(I_s, (\pi P_s)^{-1})$, then

$$\tilde{U}^H \tilde{U} = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}, \quad \mathcal{G}\tilde{U} = \tilde{U} \begin{bmatrix} \hat{N}_s(\gamma) & \frac{i\beta}{2}(I - \hat{N}_s(\gamma)) e_s e_s^H \hat{N}_s(\gamma)^{-H} (I - \hat{N}_s(\gamma))^H \\ 0 & \hat{N}_s(\gamma)^{-H} \end{bmatrix}.$$

By Lemma 5.2, there exists a nonsingular upper triangular matrix X such that $X^{-1}\hat{N}_s(\gamma)X = N_s(\gamma)$. Since the last component of $t := X^{-1}(\frac{\sqrt{2}}{2} e_s)$ is nonzero, by Lemma 5.5 there exists a nonsingular upper triangular Toeplitz matrix T such that $T^{-1}t = e_s$. Setting $Y = X(I - N_s(\gamma))T$ and $U = \tilde{U} \operatorname{diag}(Y, Y^{-H}\hat{P}_s)$, we obtain (5.3), since $(I - N_s(\gamma))T$ commutes with $N_s(\gamma)$ and $\hat{P}_s^{-1} N_s(\gamma)^{-H} \hat{P}_s = N_s(\overline{\gamma})^{-1}$.

If r is odd, following (5.9) we partition

$$\hat{U}^H \Sigma_{p,q} \hat{U} = \begin{bmatrix} 0 & 0 & \pi P_s \\ 0 & \beta & 0 \\ (\pi P_s)^H & 0 & 0 \end{bmatrix}, \quad N_r(\lambda) = \begin{bmatrix} N_s(\lambda) & e_s & 0 \\ 0 & \lambda & e_1^H \\ 0 & 0 & N_s(\lambda) \end{bmatrix}$$

and as before we obtain

$$\mathcal{G}\hat{U} = \hat{U} \begin{bmatrix} \hat{N}_s(\gamma) & \frac{\gamma-1}{2}(I - \hat{N}_s(\gamma))e_s & \frac{1-\gamma}{4}(I - \hat{N}_s(\gamma))e_s e_1^H(\hat{N}_s(\gamma) - I) \\ 0 & \gamma & \frac{1-\gamma}{2}e_1^H(\hat{N}_s(\gamma) - I) \\ 0 & 0 & \hat{N}_s(\gamma) \end{bmatrix}.$$

With $\tilde{U} = \hat{U}\,\mathrm{diag}(I_{s+1}, (\pi P_s)^{-1})$ we then have

$$\tilde{U}^H \Sigma_{p,q} \tilde{U} = \begin{bmatrix} 0 & 0 & I_s \\ 0 & \beta & 0 \\ I_s & 0 & 0 \end{bmatrix},$$

$$\mathcal{C}\tilde{U} = \tilde{U} \begin{bmatrix} N_s(\lambda) & \frac{\gamma-1}{2}(I - \hat{N}_s(\gamma))e_s & \frac{\gamma-1}{4}\beta(I - \hat{N}_s(\gamma))e_s e_s^H \hat{N}_s(\gamma)^{-H}(I - \hat{N}_s(\gamma)^H) \\ 0 & \gamma & \beta\frac{\gamma-1}{2}e_s^H \hat{N}_s(\gamma)^{-H}(I - \hat{N}_s(\gamma)^H) \\ 0 & 0 & \hat{N}_s(\gamma)^{-H} \end{bmatrix}.$$

Setting $Y = \frac{1-\overline{\gamma}}{2}X(I - N_s(\gamma))T$ and $U = \tilde{U}\,\mathrm{diag}(Y, 1, Y^{-H}\beta\hat{P}_s)$, we have (5.4).

If $\mathcal{G}$ is real, then the real forms (5.5), (5.6) and (5.7), (5.8) can be derived in the similar way. Note that if $\gamma = -1$ then the corresponding eigenvalue of $\mathcal{C} = \rho(\mathcal{G})$ is 0. □

So far we have restricted ourselves to the Jordan structure associated with eigenvalues not equal to 1. For the eigenvalue 1 we give a separate analysis.

LEMMA 5.8. *Let $\mathcal{G}$ be a $\Sigma_{p,q}$-unitary matrix and let $N_r(1)$ be a Jordan block of $\mathcal{G}$. Then there exists a full rank matrix U such that*

$$U^H \Sigma_{p,q} U = \hat{P}_r, \quad \mathcal{G}U = U \begin{bmatrix} N_s(1) & -i\beta e_s e_1^H N_s(1)^{-1} \\ 0 & N_s(1)^{-1} \end{bmatrix},$$

if $r = 2s$ and if $r = 2s+1$, then

$$U^H \Sigma_{p,q} U = \beta\hat{P}_r, \quad \mathcal{G}U = U \begin{bmatrix} N_s(1) & e_s & -\frac{1}{2}e_s e_1^H N_s(1)^{-1} \\ 0 & 1 & -e_1^H N_s(1)^{-1} \\ 0 & 0 & N_s(1)^{-1} \end{bmatrix}. \tag{5.10}$$

Here $\beta = (-1)^s i\pi$ with $\pi \in \{\pm i\}$ if $r = 2s$ and $\beta = (-1)^{s+1}\pi$ with $\pi \in \{\pm 1\}$ if $r = 2s+1$.

If $\mathcal{G}$ is real, then there exists a real matrix U such that

$$U^T \Sigma_{p,q} U = \hat{P}_{2r}, \quad \mathcal{G}U = U \begin{bmatrix} N_r(1) & 0 \\ 0 & N_r(1)^{-1} \end{bmatrix},$$

if r is even and if $r = 2s+1$ we have again (5.10).

Proof. By Lemma 5.2 we may assume without loss of generality that $\Lambda(\mathcal{G}) = \{1\}$. Otherwise we work on the small size matrix $\mathcal{G}_2$. We cannot use the Cayley transformation ρ but a different rational transformation $\hat{\rho}(z) = (1-z)(1+z)^{-1}$. If $\mathcal{A}$ is $\Sigma_{p,q}$-unitary then $\mathcal{B} = \hat{\rho}(\mathcal{A})$ is $\Sigma_{p,q}$-skew Hermitian and conversely. With this new transformation we obtain the proof analogous to the proof of Lemma 5.7. □

Using these results, analogous to the case for the Jordan and Lie algebras we can show the following structured canonical forms for both complex and real $\Sigma_{p,q}$-unitary matrices, respectively. The proofs are similar to that in the previous sections so we omit them here.

THEOREM 5.9. *Let $\mathcal{G}$ be a $\Sigma_{p,q}$-unitary matrix $\mathcal{G}$, let $\lambda_1,\ldots,\lambda_\mu$ be the pairwise different eigenvalues of modulus less than one and let $\sigma_1,\ldots,\sigma_\nu$ be the pairwise different eigenvalues of modulus one. Then there exists a nonsingular matrix $\mathcal{U}$ such that*

$$\mathcal{U}^{-1}\mathcal{G}\mathcal{U} = \operatorname{diag}(R_c, R_c^i, R_u).$$

i) The diagonal blocks R_c, R_c^i, associated with eigenvalues not on the unit circle, are

$$R_c = \operatorname{diag}(H_1(\lambda_1),\ldots,H_\mu(\lambda_\mu)), \quad R_c^i = \operatorname{diag}(H_1(\overline{\lambda_1})^{-1},\ldots,H_\mu(\overline{\lambda_\mu})^{-1}),$$

where for $k=1,\ldots,\mu$ we have $H_k(\lambda_k)=\lambda_k I + H_k$, $H_k(\overline{\lambda_k})=\overline{\lambda_k}I+H_k$ and $H_k = \operatorname{diag}(N_{p_{k,1}},\ldots,N_{p_{k,s_k}})$.
ii) The diagonal block R_u associated with the unimodular eigenvalues are $R_u = \operatorname{diag}(M_1,\ldots,M_\nu)$, where for $k=1,\ldots,\nu$, we have

$$M_k = \operatorname{diag}(A_{k,1},\ldots,A_{k,t_k}; B_{k,1},\ldots,B_{k,w_k}).$$

Here for $j=1,\ldots,t_k$ we have

$$A_{k,j} = \begin{bmatrix} N_{q_{k,1}}(\sigma_k) & i\delta_k\beta^e_{k,j}e_{q_{k,j}}e_1^H N_{q_{k,j}}(\overline{\sigma_k})^{-1} \\ 0 & N_{q_{k,j}}(\overline{\sigma_k})^{-1} \end{bmatrix},$$

with $\delta_k=1$ if $\sigma_k\neq 1$ and $\delta_k=-1$ if $\sigma_k=1$ and furthermore $\beta^e_{k,j}=(-1)^{p_{k,j}}i\pi^e_{k,j}$ with $\pi^e_{k,j}\in\{\pm i\}$.

Moreover, for $j=1,\ldots,w_k$, we have

$$B_{k,j} = \begin{bmatrix} N_{r_{k,j}}(\sigma_k) & \sigma_k e_{r_{k,j}} & s(\sigma_k)e_{r_{k,j}}e_1^H N_{r_{k,j}}(\overline{\sigma_k})^{-1} \\ 0 & \sigma_k & -e_1^H N_{r_{k,j}}(\overline{\sigma_k})^{-1} \\ 0 & 0 & N_{r_{k,j}}(\overline{\sigma_k})^{-1} \end{bmatrix},$$

with $s(\sigma_k)=\frac{\sigma_k}{1-\sigma_k}$ if $\sigma_k\neq 1$ and $s(1)=-\frac{1}{2}$.

The matrix $\mathcal{U}$ has the form

$$\mathcal{U}^H\Sigma_{p,q}\mathcal{U} = \begin{bmatrix} 0 & W_c & 0 \\ W_c^H & 0 & 0 \\ 0 & 0 & W_u \end{bmatrix},$$

with $W_c=\operatorname{diag}(\hat{P}_{H_1},\ldots,\hat{P}_{H_\mu})$ and $W_u=\operatorname{diag}(W_1^u,\ldots,W_\nu^u)$, where for $k=1,\ldots,\mu$ we have $\hat{P}_{H_k}=\operatorname{diag}(\hat{P}_{p_{k,1}},\ldots,\hat{P}_{p_{k,s_k}})$ and for $k=1,\ldots,\nu$ we have

$$W_k^u = \operatorname{diag}(\hat{P}_{2q_{k,1}},\ldots,\hat{P}_{2q_{k,t_k}}; \beta^o_{k,1}\hat{P}_{2r_{k,1}+1},\ldots,\beta^o_{k,w_k}\hat{P}_{2r_{k,w_k}+1}).$$

Here for $j=1,\ldots,w_k$ we have $\beta^o_{k,j}=(-1)^{r_{k,j}+1}\pi^o_{k,j}$ with $\pi^o_{kj}\in\{\pm 1\}$.

Each eigenvalue λ_k ($\overline{\lambda_k}^{-1}$) has s_k Jordan blocks of sizes $p_{k,1},\ldots,p_{k,s_k}$ and each unimodular eigenvalue σ_k has
a) t_k even sized Jordan blocks of sizes $2q_{k,1},\ldots,2q_{k,t_k}$ corresponding to the structure inertia indices $(-1)^{q_{k,1}+1}i\beta^e_{k,1},\ldots,(-1)^{q_{k,t_k}+1}i\beta^e_{k,t_k}$ and
b) w_k odd sized Jordan blocks of sizes $2r_{k,1}+1,\ldots,2r_{k,w_k}+1$ corresponding to the structure inertia indices $(-1)^{r_{k,1}+1}\beta^o_{k,1},\ldots,(-1)^{r_{k,w_k}+1}\beta^o_{k,w_k}$.

THEOREM 5.10. *Let $\mathcal{G}$ be a real $\Sigma_{p,q}$-orthogonal matrix, let $\alpha_1,\ldots,\alpha_\eta$ be pairwise different real eigenvalues of modulus less than one, let $\lambda_1,\ldots,\lambda_\mu$ be pairwise different* *nonreal eigenvalues with positive imaginary parts of modulus less than one, and let* $\gamma_1, \ldots, \gamma_\nu$ *be pairwise different nonreal eigenvalues of modulus* 1*, also with positive imaginary parts. (Note that then also* $\alpha_1^{-1}, \ldots, \alpha_\eta^{-1}$, $\overline{\lambda_1}, \ldots, \overline{\lambda_\mu}; \lambda_1^{-1}, \ldots, \lambda_\mu^{-1}; \overline{\lambda_1}^{-1}$, $\ldots$, $\overline{\lambda_\mu}^{-1}$, $\overline{\gamma_1}, \ldots, \overline{\gamma_\nu}$ *and possibly also* $-1, 1$ *are eigenvalues.) Then there exists a real nonsingular matrix* $\mathcal{U}$ *such that*

$$\mathcal{U}^{-1}\mathcal{G}\mathcal{U} = \operatorname{diag}(R_c, R_c^i, R_u).$$

i) The blocks with index c*, associated with eigenvalues not on the unit circle, are* $R_c = \operatorname{diag}(\hat{R}_c, \tilde{R}_c)$ *and* $R_c^i = \operatorname{diag}(\hat{R}_c^i, \tilde{R}_c^i)$*, with*

$$\begin{aligned}
\hat{R}_c &= \operatorname{diag}(K_1(\alpha_1), \ldots, K_\eta(\alpha_\eta)), \quad \hat{R}_c^i = \operatorname{diag}(K_1(\alpha_1)^{-1}, \ldots, K_\eta(\alpha_\eta)^{-1}),\\
\tilde{R}_c &= \operatorname{diag}(H_1(\Lambda_1), \ldots, H_\mu(\Lambda_\mu)), \quad \tilde{R}_c^i = \operatorname{diag}(H_1(\Lambda_1)^{-1}, \ldots, H_\mu(\Lambda_\mu)^{-1}),
\end{aligned}$$

where for $k = 1, \ldots, \eta$ *we have* $K_k(\alpha_k) = \alpha_k I + K_k$ *and* $K_k = \operatorname{diag}(N_{f_{k,1}}, \ldots, N_{f_{k,l_k}})$ *and for* $k = 1, \ldots, \mu$ *we have* $H_k(\Lambda_k) = \operatorname{diag}(N_{p_{k,1}}(\Lambda_k), \ldots, N_{p_{k,s_k}}(\Lambda_k))$*, with* $\Lambda_k = \begin{bmatrix} \operatorname{Re}\lambda_k & \operatorname{Im}\lambda_k \\ -\operatorname{Im}\lambda_k & \operatorname{Re}\lambda_k \end{bmatrix}$.
ii) The block R_u*, associated with the unimodular eigenvalues, is*
$R_u = \operatorname{diag}(M_1, \ldots, M_\nu, M_-, M_+)$ *with*

$$\begin{aligned}
M_k &= \operatorname{diag}(A_{k,1}, \ldots, A_{k,t_k}; B_{k,1}, \ldots, B_{k,w_k}),\\
M_- &= \operatorname{diag}(A_1^-, \ldots, A_{t_-}^-, B_1^-, \ldots, B_{w_-}^-),\\
M_+ &= \operatorname{diag}(A_1^+, \ldots, A_{t_+}^+, B_1^+, \ldots, B_{w_+}^+).
\end{aligned}$$

Here we have the following substructures:
a) For $j = 1, \ldots, t_k$

$$A_{k,j} = \left[\begin{array}{c|c} N_{q_{k,1}}(\Gamma_k) & -\beta_{k,j}^e \begin{bmatrix} 0 & 0 \\ \hat{P}_2 & 0 \end{bmatrix} N_{q_{k,j}}(\Gamma_k)^{-1} \\ \hline 0 & N_{q_{k,j}}(\Gamma_k)^{-1} \end{array}\right],$$

with $\beta_{k,j}^e = (-1)^{p_{k,j}} i\pi_{k,j}^e$ *and* $\pi_{k,j}^e \in \{\pm i\}$.
b) For $j = 1, \ldots, w_k$

$$B_{k,j} = \left[\begin{array}{c|c|c} N_{r_{k,j}}(\Gamma_k) & \begin{array}{c} 0 \\ \Gamma_k \end{array} & \begin{bmatrix} 0 & 0 \\ S(\Gamma_k) & 0 \end{bmatrix} N_{r_{k,j}}(\Gamma_k)^{-1} \\ \hline 0 & \Gamma_k & [-\Sigma_{1,1}, 0] N_{r_{k,j}}(\Gamma_k)^{-1} \\ \hline 0 & 0 & N_{r_{k,j}}(\Gamma_k)^{-1} \end{array}\right],$$

with

$$\Gamma_k = \begin{bmatrix} \operatorname{Re}\gamma_k & \operatorname{Im}\gamma_k \\ -\operatorname{Im}\gamma_k & \operatorname{Re}\gamma_k \end{bmatrix}, \quad S(\Gamma_k) = -\frac{1}{2}\begin{bmatrix} 1 & \frac{\operatorname{Im}\gamma_k}{1-\operatorname{Re}\gamma_k} \\ \frac{\operatorname{Im}\gamma_k}{1-\operatorname{Re}\gamma_k} & -1 \end{bmatrix}.$$

c) For $k = 1, \ldots, t_-$

$$A_k^- = \begin{bmatrix} N_{q_k^-}(-1) & 0 \\ 0 & N_{q_k^-}(-1)^{-1} \end{bmatrix},$$

d) For $k=1,\ldots,w_-$

$$B_k^- = \begin{bmatrix} N_{r_k^-}(-1) & -e_{r_k^-} & -\frac{1}{2}e_{r_k^-}e_1^T N_{r_k^-}(-1)^{-1} \\ 0 & -1 & -e_1^T N_{r_k^-}(-1)^{-1} \\ 0 & 0 & N_{r_k^-}(-1)^{-1} \end{bmatrix}.$$

e) For $k=1,\ldots,t_+$

$$A_k^+ = \begin{bmatrix} N_{q_k^+}(1) & 0 \\ 0 & N_{q_k^+}(1)^{-1} \end{bmatrix}.$$

f) For $k=1,\ldots,w_+$

$$B_k^+ = \begin{bmatrix} N_{r_k^+}(1) & e_{r_k^+} & -\frac{1}{2}e_{r_k^+}e_1^T N_{r_k^+}(1)^{-1} \\ 0 & 1 & -e_1^T N_{r_k^+}(1)^{-1} \\ 0 & 0 & N_{r_k^+}(1)^{-1} \end{bmatrix}.$$

The matrix $\mathcal{U}$ *has the form*

$$\mathcal{U}^T\Sigma_{p,q}\mathcal{U} = \begin{bmatrix} 0 & W_c & 0 \\ W_c^T & 0 & 0 \\ 0 & 0 & W_u \end{bmatrix},$$

where

$$W_c = \operatorname{diag}(\hat{W}_c, \tilde{W}_c), \quad W_u = \operatorname{diag}(W_1^u,\ldots,W_\nu^u, W_-^u, W_+^u),$$
$$\hat{W}_c = \operatorname{diag}(\hat{P}_{K_1},\ldots,\hat{P}_{k_\eta}), \quad \tilde{W}_c = \operatorname{diag}(\hat{P}_{H_1}\otimes\Sigma_{1,1},\ldots,\hat{P}_{H_\mu}\otimes\Sigma_{1,1})$$

and as substructures we have for $k=1,\ldots,\eta$ *that* $\hat{P}_{K_k} = \operatorname{diag}(\hat{P}_{f_{k,1}},\ldots,\hat{P}_{f_{k,l_k}})$ *and for* $k=1,\ldots,\mu$ *that* $\hat{P}_{H_k} = \operatorname{diag}(\hat{P}_{p_{k,1}},\ldots,\hat{P}_{p_{k,s_k}})$.

The substructure for the blocks with index u is as follows:

1) For $k=1,\ldots,\nu$ *we have*

$$W_k^u = \operatorname{diag}\left(\hat{P}_{2q_{k,1}}\otimes\Sigma_{1,1},\ldots,\hat{P}_{2q_{k,t_k}}\otimes\Sigma_{1,1};\beta_{k,1}^o\begin{bmatrix} 0 & 0 & \hat{P}_{r_{k,1}}\otimes\Sigma_{1,1} \\ 0 & I_2 & 0 \\ \hat{P}_{r_{k,1}}^T\otimes\Sigma_{1,1} & 0 & 0 \end{bmatrix},\right.$$
$$\left.\ldots,\beta_{k,w_k}^o\begin{bmatrix} 0 & 0 & \hat{P}_{r_{k,w_k}}\otimes\Sigma_{1,1} \\ 0 & I_2 & 0 \\ \hat{P}_{r_{k,w_k}}^T\otimes\Sigma_{1,1} & 0 & 0 \end{bmatrix}\right),$$

with $\beta_{k,j}^o = (-1)^{r_{k,j}+1}\pi_{k,j}^o$ *and* $\pi_{k,j}^o \in \{\pm 1\}$, *for* $j=1,\ldots,w_k$.

2)

$$W_-^u = \operatorname{diag}(\hat{P}_{2q_1^-},\ldots,\hat{P}_{2q_{t_-}^-};\beta_1^-\hat{P}_{2r_1^-+1},\ldots,\beta_{w_-}^-\hat{P}_{2r_{w_-}^-+1}),$$

with $\beta_k^- = (-1)^{r_k^-+1}\pi_k^-$ *and* $\pi_k^- \in \{\pm 1\}$ *for* $k=1,\ldots,w_-$.

3)

$$W_+^u = \operatorname{diag}(\hat{P}_{2q_1^+},\ldots,\hat{P}_{2q_{t_+}^+};\beta_1^+\hat{P}_{2r_1^++1},\ldots,\beta_{w_+}^+\hat{P}_{2r_{w_+}^++1}),$$

with $\beta_k^+ = (-1)^{r_k^+ + 1}\pi_k^+$, *and* $\pi_k^+ \in \{\pm 1\}$, *for* $k = 1, \ldots, w_+$.

Each real eigenvalue α_k (α_k^{-1}) *has* l_k *Jordan blocks of sizes* $f_{k,1}, \ldots, f_{k,l_k}$ *and each eigenvalue* λ_k $(\overline{\lambda_k}, \lambda_k^{-1}, \overline{\lambda_k}^{-1})$ *has* s_k *Jordan blocks of sizes* $p_{k,1}, \ldots, p_{k,s_k}$.

Each nonreal unimodular eigenvalue γ_k $(\overline{\gamma_k})$ *has* t_k *even sized Jordan blocks of sizes* $2q_{k,1}, \ldots, 2q_{k,t_k}$ *corresponding to the structure inertia indices* $(-1)^{q_{k,1}+1} i\beta_{k,1}^e$, $\ldots$, $(-1)^{q_{k,t_k}+1} i\beta_{k,t_k}^e$ *and* w_k *odd sized Jordan blocks of sizes* $2r_{k,1}+1, \ldots, 2r_{k,w_k}+1$ *corresponding to the structure inertia indices* $(-1)^{r_{k,1}+1}\beta_{k,1}^o$, $\ldots$, $(-1)^{r_{k,w_k}+1}\beta_{k,w_k}^o$.

The eigenvalue -1 *has* $2t_-$ *even sized Jordan blocks of sizes* $q_1^-, q_1^-, \ldots, q_{t_-}^-, q_{t_-}^-$ *corresponding to the structure inertia indices* $i, -i, \ldots, i, -i$, *and* w_- *odd sized Jordan blocks of sizes* $2r_1^- + 1, \ldots, 2r_{w_-}^- + 1$ *corresponding to the indices* $(-1)^{r_1^- + 1}\beta_1^-$, $\ldots$, $(-1)^{r_{w_-}^- + 1}\beta_{w_-}^-$.

The eigenvalue 1 *has* $2t_+$ *even sized Jordan blocks of sizes* $q_1^+, q_1^+ \ldots, q_{t_+}^+, q_{t_+}^+$ *corresponding to the structure inertia indices* $i, -i, \ldots, i, -i$ *and* w_+ *odd size Jordan blocks of sizes* $2r_1^+ + 1, \ldots, 2r_{w_+}^+ + 1$ *corresponding to the indices* $(-1)^{r_1^+ + 1}\beta_1^+$, $\ldots$, $(-1)^{r_{w_+}^+ + 1}\beta_{w_+}^+$.

Note that the structure inertia indices actually arise through the Cayley transformation in the associated $\Sigma_{p,q}$-skew Hermitian matrices, but they inherently describe also the associated structure for the unimodular eigenvalues of $\mathcal{G}$.

Finally we will give the canonical forms under $\Sigma_{p,q}$-unitary similarity transformations. To simplify the notation which is even more technical, we now introduce for a nonzero scalar γ the blocks

$$N_r^+(\gamma) := \frac{1}{2}(N_r(\gamma) + N_r(\gamma)^{-H}), \quad N_r^-(\gamma) := \frac{1}{2}(N_r(\gamma) - N_r(\gamma)^{-H}),$$

and similarly for a 2×2 real nonsingular matrix Γ we set

$$N_r^+(\Gamma) = \frac{1}{2}(N_r(\Gamma) + N_r(\Gamma)^{-T}), \quad N_r^-(\Gamma) = \frac{1}{2}(N_r(\Gamma) - N_r(\Gamma)^{-T}).$$

THEOREM 5.11. *Let* $\mathcal{G}$ *be a* $\Sigma_{p,q}$*-unitary matrix with pairwise distinct eigenvalues* $\lambda_1, \ldots, \lambda_\mu$ *of modulus less than one and pairwise distinct eigenvalues* $\gamma_1, \ldots, \gamma_\nu$ *of modulus one. Note that then also* $\overline{\lambda_1}^{-1}, \ldots, \overline{\lambda_\mu}^{-1}$ *are eigenvalues. Then there exists a* $\Sigma_{p,q}$*-unitary matrix* $\mathcal{U}$, *such that*

$$\mathcal{U}^{-1}\mathcal{G}\mathcal{U} = \begin{bmatrix} R_c & & T_c & \\ & R_u^+ & & T_u \\ T_c & & R_c & \\ & Y_u & & R_u^- \end{bmatrix}.$$

i) The blocks with index c, *associated with the eigenvalues that do not have modulus one, have the form* $R_c = \operatorname{diag}(R_1^c, \ldots, R_\mu^c)$ *and* $T_c = \operatorname{diag}(T_1^c, \ldots, T_\mu^c)$, *where for* $k = 1, \ldots, \mu$

$$R_k^c = \operatorname{diag}(N_{p_{k,1}}^+(\lambda_k), \ldots, N_{p_{k,s_k}}^+(\lambda_k)), \quad T_k^c = -\operatorname{diag}(N_{p_{k,1}}^-(\lambda_k), \ldots, N_{p_{k,s_k}}^-(\lambda_k)).$$

ii) The blocks with index u, *associated with the unimodular eigenvalues, are*

$$R_u^+ = \operatorname{diag}(C_1, \ldots, C_\nu), \quad R_u^- = \operatorname{diag}(D_1, \ldots, D_\nu),$$
$$T_u = \operatorname{diag}(F_1, \ldots, F_\nu), \quad Y_u = \operatorname{diag}(G_1, \ldots, G_\nu).$$

Here for $k = 1, \ldots, \nu$ the blocks are

$$C_k = \operatorname{diag}(C_k^e, C_k^+, C_k^-), \quad D_k = \operatorname{diag}(D_k^e, D_k^+, D_k^-),$$
$$F_k = \operatorname{diag}(F_k^e, F_k^+, F_k^-), \quad G_k = \operatorname{diag}(G_k^e, G_k^+, G_k^-),$$

and with $\delta_k = 1$ for $\gamma_k \neq 1$ and $\delta_k = -1$ if $\gamma_k = 1$ the substructures are

$$\begin{aligned}
C_k^e &= \operatorname{diag}(N_{q_{k,1}}^+(\gamma_k) + \frac{1}{2} i\delta_k \beta_{k,1} e_{q_{k,1}} e_{q_{k,1}}^H N_{q_{k,1}}(\gamma_k)^{-H}, \\
&\qquad \ldots, N_{q_{k,t_k}}^+(\gamma_k) + \frac{1}{2} i\delta_k \beta_{k,t_k} e_{q_{k,t_k}} e_{q_{k,t_k}}^H N_{q_{k,t_k}}(\gamma_k)^{-H}), \\
D_k^e &= \operatorname{diag}(N_{q_{k,1}}^+(\gamma_k) - \frac{1}{2} i\delta_k \beta_{k,1} e_{q_{k,1}} e_{q_{k,1}}^H N_{q_{k,1}}(\gamma_k)^{-H}, \\
&\qquad \ldots, N_{q_{k,t_k}}^+(\gamma_k) - \frac{1}{2} i\delta_k \beta_{k,t_k} e_{q_{k,t_k}} e_{q_{k,t_k}}^H N_{q_{k,t_k}}(\gamma_k)^{-H}), \\
F_k^e &= \operatorname{diag}(-N_{q_{k,1}}^-(\gamma_k) + \frac{1}{2} i\delta_k \beta_{k,1} e_{q_{k,1}} e_{q_{k,1}}^H N_{q_{k,1}}(\gamma_k)^{-H}, \\
&\qquad \ldots, -N_{q_{k,t_k}}^-(\gamma_k) + \frac{1}{2} i\delta_k \beta_{k,t_k} e_{q_{k,t_k}} e_{q_{k,t_k}}^H N_{q_{k,t_k}}(\gamma_k)^{-H}), \\
G_k^e &= -\operatorname{diag}(N_{q_{k,1}}^-(\gamma_k) + \frac{1}{2} i\delta_k \beta_{k,1} e_{q_{k,1}} e_{q_{k,1}}^H N_{q_{k,1}}(\gamma_k)^{-H}, \\
&\qquad \ldots, N_{q_{k,t_k}}^-(\gamma_k) + \frac{1}{2} i\delta_k \beta_{k,t_k} e_{q_{k,t_k}} e_{q_{k,t_k}}^H N_{q_{k,t_k}}(\gamma_k)^{-H}), \\
C_k^+ &= \operatorname{diag}\left(\begin{bmatrix} N_{u_{k,1}}^+(\gamma_k) + \frac{s(\gamma_k)}{2} e_{u_{k,1}} e_{u_{k,1}}^H N_{u_{k,1}}(\gamma_k)^{-H} & \frac{\sqrt{2}}{2}\gamma_k e_{u_{k,1}} \\ -\frac{\sqrt{2}}{2} e_{u_{k,1}}^H N_{u_{k,1}}(\gamma_k)^{-H} & \gamma_k \end{bmatrix}, \right. \\
&\qquad \left. \ldots, \begin{bmatrix} N_{u_{k,w_k}}^+(\gamma_k) + \frac{s(\gamma_k)}{2} e_{u_{k,w_k}} e_{u_{k,w_k}}^H N_{u_{k,w_k}}(\gamma_k)^{-H} & \frac{\sqrt{2}}{2}\gamma_k e_{u_{k,w_k}} \\ -\frac{\sqrt{2}}{2} e_{u_{k,w_k}}^H N_{u_{k,w_k}}(\gamma_k)^{-H} & \gamma_k \end{bmatrix} \right), \\
D_k^+ &= \operatorname{diag}(N_{u_{k,1}}^+(\gamma_k) - \frac{s(\gamma_k)}{2} e_{u_{k,1}} e_{u_{k,1}}^H N_{u_{k,1}}(\gamma_k)^{-H}, \\
&\qquad \ldots, N_{u_{k,w_k}}^+(\gamma_k) - \frac{s(\gamma_k)}{2} e_{u_{k,w_k}} e_{u_{k,w_k}}^H N_{u_{k,w_k}}(\gamma_k)^{-H}), \\
F_k^+ &= -\operatorname{diag}\left(\begin{bmatrix} N_{u_{k,1}}^-(\gamma_k) - \frac{s(\gamma_k)}{2} e_{u_{k,1}} e_{u_{k,1}}^H N_{u_{k,1}}(\gamma_k)^{-H} \\ \frac{\sqrt{2}}{2} e_{u_{k,1}}^H N_{u_{k,1}}(\gamma_k)^{-H} \end{bmatrix}, \right. \\
&\qquad \left. \ldots, \begin{bmatrix} N_{u_{k,w_k}}^-(\gamma_k) - \frac{s(\gamma_k)}{2} e_{u_{k,w_k}} e_{u_{k,w_k}}^H N_{u_{k,w_k}}(\gamma_k)^{-H} \\ \frac{\sqrt{2}}{2} e_{u_{k,w_k}}^H N_{u_{k,w_k}}(\gamma_k)^{-H} \end{bmatrix} \right), \\
G_k^+ &= -\operatorname{diag}([N_{u_{k,1}}^-(\gamma_k) + \frac{s(\gamma_k)}{2} e_{u_{k,1}} e_{u_{k,1}}^H N_{u_{k,1}}(\gamma_k)^{-H}, \frac{\sqrt{2}}{2}\gamma_k e_{u_{k,1}}^H], \\
&\qquad \ldots, [N_{u_{k,1}}^-(\gamma_k) + \frac{s(\gamma_k)}{2} e_{u_{k,w_k}} e_{u_{k,w_k}}^H N_{u_{k,w_k}}(\gamma_k)^{-H}, \frac{\sqrt{2}}{2}\gamma_k e_{u_{k,w_k}}^H]), \\
C_k^- &= \operatorname{diag}(N_{v_{k,1}}^+(\gamma_k) - \frac{s(\gamma_k)}{2} e_{v_{k,1}} e_{v_{k,1}}^H N_{v_{k,1}}(\gamma_k)^{-H}, \\
&\qquad \ldots, N_{v_{k,z_k}}^+(\gamma_k) - \frac{s(\gamma_k)}{2} e_{v_{k,z_k}} e_{v_{k,z_k}}^H N_{v_{k,z_k}}(\gamma_k)^{-H}), \\
D_k^- &= \operatorname{diag}\left(\begin{bmatrix} \gamma_k & \frac{\sqrt{2}}{2} e_{v_{k,1}}^H N_{v_{k,1}}(\gamma_k)^{-H} \\ -\frac{\sqrt{2}}{2}\gamma_k e_{v_{k,1}} & N_{v_{k,1}}^+(\gamma_k) + \frac{s(\gamma_k)}{2} e_{v_{k,1}} e_{v_{k,1}}^H N_{v_{k,1}}(\gamma_k)^{-H} \end{bmatrix}, \right.
\end{aligned}$$

$$\ldots, \begin{bmatrix} \gamma_k & \frac{\sqrt{2}}{2} e^H_{v_{k,z_k}} N_{v_{k,z_k}}(\gamma_k)^{-H} \\ -\frac{\sqrt{2}}{2}\gamma_k e_{v_{k,z_k}} & N^+_{v_{k,z_k}}(\gamma_k) + \frac{s(\gamma_k)}{2} e_{v_{k,z_k}} e^H_{v_{k,z_k}} N_{v_{k,z_k}}(\gamma_k)^{-H} \end{bmatrix}\Bigg),$$

$$F_k^- = \mathrm{diag}([\frac{\sqrt{2}}{2}\gamma_k e_{v_{k,1}}, -N^-_{v_{k,1}}(\gamma_k) - \frac{s(\gamma_k)}{2} e_{v_{k,1}} e^H_{v_{k,1}} N_{v_{k,1}}(\gamma_k)^{-H}],$$
$$\ldots, [\frac{\sqrt{2}}{2}\gamma_k e_{v_{k,z_k}}, -N^-_{v_{k,z_k}}(\gamma_k) - \frac{s(\gamma_k)}{2} e_{v_{k,z_k}} e^H_{v_{k,z_k}} N_{v_{k,z_k}}(\gamma_k)^{-H}]),$$

$$G_k^- = \mathrm{diag}\left(\begin{bmatrix} \frac{\sqrt{2}}{2} e^H_{v_{k,1}} N_{v_{k,1}}(\gamma_k)^{-H} \\ -N^-_{v_{k,1}}(\gamma_k) + \frac{s(\gamma_k)}{2} e_{v_{k,1}} e^H_{v_{k,1}} N_{v_{k,1}}(\gamma_k)^{-H} \end{bmatrix},\right.$$
$$\left.\ldots, \begin{bmatrix} \frac{\sqrt{2}}{2} e^H_{v_{k,z_k}} N_{v_{k,z_k}}(\gamma_k)^{-H} \\ -N^-_{v_{k,z_k}}(\gamma_k) + \frac{s(\gamma_k)}{2} e_{v_{k,z_k}} e^H_{v_{k,z_k}} N_{v_{k,z_k}}(\gamma_k)^{-H} \end{bmatrix}\right).$$

In these formulas we have used $s(\gamma_k) = \frac{\gamma_k}{1-\gamma_k}$ *if* $\gamma_k \neq 1$ *and* $s(1) = -\frac{1}{2}$.

Each λ_k $(\overline{\lambda_k}^{-1})$ *has* s_k *Jordan blocks of sizes* $p_{k,1}, \ldots, p_{k,s_k}$. *For each unimodular eigenvalue* γ_k *we have*
a) t_k *even sized Jordan blocks of sizes* $2q_{k,1}, \ldots, 2q_{k,t_k}$ *with the corresponding structure inertia indices* $i(-1)^{q_{k,1}+1}\beta_{k,1}, \ldots, i(-1)^{q_{k,t_k}+1}\beta_{k,t_k}$;
b) w_k *odd sized Jordan blocks of sizes* $2u_{k,1}+1, \ldots, 2u_{k,w_k}+1$ *corresponding to the indices* $(-1)^{u_{k,1}+1}, \ldots, (-1)^{u_{k,w_k}+1}$;
c) z_k *odd sized Jordan blocks of sizes* $2v_{k,1}+1, \ldots, 2v_{k,z_k}+1$ *corresponding to the indices* $(-1)^{v_{k,j}}, \ldots, (-1)^{v_{k,z_k}}$.

THEOREM 5.12. *Let* $\mathcal{G}$ *be a real* $\Sigma_{p,q}$*-orthogonal matrix with pairwise distinct real eigenvalues* $\alpha_1, \ldots, \alpha_\eta$ *of modulus less than one, pairwise distinct nonreal eigenvalues* $\lambda_1, \ldots, \lambda_\mu$ *of modulus less than one with positive imaginary parts, and pairwise different nonreal eigenvalues* $\gamma_1, \ldots, \gamma_\nu$ *of modulus one also with positive imaginary parts. (Note that we then also have the eigenvalues* $\alpha_1^{-1}, \ldots, \alpha_\eta^{-1}$, $\overline{\lambda_1}, \ldots, \overline{\lambda_\mu}$, $\lambda_1^{-1}, \ldots, \lambda_\mu^{-1}$, $\overline{\lambda_1}^{-1}, \ldots, \overline{\lambda_\mu}^{-1}$ *and* $\overline{\gamma_1}, \ldots, \overline{\gamma_\nu}$ *as well as possibly* $-1, 1$.*)*

Then there exists a real $\Sigma_{p,q}$*-orthogonal matrix* $\mathcal{U}$ *such that*

$$\mathcal{U}^{-1}\mathcal{G}\mathcal{U} = \begin{bmatrix} R_c & & T_c & \\ & R_u^+ & & T_u \\ T_c & & R_c & \\ & Y_u & & R_u^- \end{bmatrix}.$$

i) The blocks with index c, associated with eigenvalues not on the unit circle, are split further as $R_c = \mathrm{diag}(\hat{R}_c, \tilde{R}_c)$ *and* $T_c = \mathrm{diag}(\hat{T}_c, \tilde{T}_c)$ *with*

$$\hat{R}_c = \mathrm{diag}(\hat{R}_1^c, \ldots, \hat{R}_\eta^c), \quad \tilde{R}_c = \mathrm{diag}(\tilde{R}_1^c, \ldots, \tilde{R}_\mu^c),$$
$$\hat{T}_c = \mathrm{diag}(\hat{T}_1^c, \ldots, \hat{T}_\eta^c), \quad \tilde{T}_c = \mathrm{diag}(\tilde{T}_1^c, \ldots, \tilde{T}_\mu^c)$$

and for $k = 1, \ldots, \eta$ *we have*

$$\hat{R}_k^c = \mathrm{diag}(N^+_{f_{k,1}}(\alpha_k), \ldots, N^+_{f_{k,l_k}}(\alpha_k)), \quad \hat{T}_k^c = -\,\mathrm{diag}(N^-_{f_{k,1}}(\alpha_k), \ldots, N^-_{f_{k,l_k}}(\alpha_k)),$$

while for $k = 1, \ldots, \mu$

$$\tilde{R}_k^c = \mathrm{diag}(N^+_{p_{k,1}}(\Lambda_k), \ldots, N^+_{p_{k,s_k}}(\Lambda_k)), \quad \tilde{T}_k^c = -\,\mathrm{diag}(N^-_{p_{k,1}}(\Lambda_k), \ldots, N^-_{p_{k,s_k}}(\Lambda_k)).$$

ii) The blocks with index u, associated with the unimodular eigenvalues, are split further in real and nonreal eigenvalues, as

$$R_u^+ = \operatorname{diag}(C_1,\dots,C_\nu,C_-,C_+),\quad R_u^- = \operatorname{diag}(D_1,\dots,D_\nu,D_-,D_+),$$
$$T_u = \operatorname{diag}(F_1,\dots,F_\nu,F_-,F_+),\quad Y_u = \operatorname{diag}(G_1,\dots,G_\nu,G_-,G_+)$$

and have for $k=1,\dots,\nu$ the partitioning

$$C_k = \operatorname{diag}(C_k^e,C_k^+,C_k^-),\quad D_k = \operatorname{diag}(D_k^e,D_k^+,D_k^-),$$
$$F_k = \operatorname{diag}(F_k^e,F_k^+,F_k^-),\quad G_k = \operatorname{diag}(G_k^e,G_k^+,G_k^-).$$

In these blocks we have with

$$E_{k,j} = \frac{1}{2}\beta_{k,j}\begin{bmatrix} 0 & 0\\ 0 & J_1\end{bmatrix} N_{q_{k,j}}(\Gamma_k)^{-T},\quad \tilde{E}_{k,j} = \frac{1}{2}\begin{bmatrix} 0 & 0\\ 0 & E_1\end{bmatrix} N_{v_{k,j}}(\Gamma_k)^{-T},$$
$$\hat{E}_{k,j} = \frac{1}{2}\begin{bmatrix} 0 & 0\\ 0 & E_1\end{bmatrix} N_{u_{k,j}}(\Gamma_k)^{-T},\quad E_1 = -\frac{1}{2}\begin{bmatrix} 1 & -\frac{\operatorname{Im}\gamma_k}{1-\operatorname{Re}\gamma_k}\\ \frac{\operatorname{Im}\gamma_k}{1-\operatorname{Re}\gamma_k} & 1\end{bmatrix},$$

the following substructures.

$$\begin{aligned}
C_k^e &= \operatorname{diag}(N_{q_{k,1}}^+(\Gamma_k)+E_{k,1}\dots,N_{q_{k,t_k}}^+(\Gamma_k)+E_{k,t_k}),\\
D_k^e &= \operatorname{diag}(N_{q_{k,1}}^+(\Gamma_k)-E_{k,1},\dots,N_{q_{k,t_k}}^+(\Gamma_k)-E_{k,t_k}),\\
F_k^e &= \operatorname{diag}(-N_{q_{k,1}}^-(\Gamma_k)+E_{k,1},\dots,-N_{q_{k,t_k}}^-(\Gamma_k)+E_{k,t_k}),\\
G_k^e &= -\operatorname{diag}(N_{q_{k,1}}^-(\Gamma_k)+E_{k,1},\dots,N_{q_{k,t_k}}^-(\Gamma_k)+E_{k,t_k}),\\
C_k^+ &= \operatorname{diag}\left(\left[\begin{array}{c|c} N_{u_{k,1}}^+(\Gamma_k)+\hat{E}_{k,1} & \begin{matrix}0\\ \frac{\sqrt{2}}{2}\Gamma_k\end{matrix}\\ \hline [0,-\frac{\sqrt{2}}{2}I_2]N_{u_{k,1}}(\Gamma_k)^{-T} & \Gamma_k\end{array}\right],\right.\\
&\qquad\left.\dots,\operatorname{diag}(\left[\begin{array}{c|c} N_{u_{k,w_k}}^+(\Gamma_k)+\hat{E}_{k,w_k} & \begin{matrix}0\\ \frac{\sqrt{2}}{2}\Gamma_k\end{matrix}\\ \hline [0,-\frac{\sqrt{2}}{2}I_2]N_{u_{k,w_k}}(\Gamma_k)^{-T} & \Gamma_k\end{array}\right]\right),\\
D_k^+ &= \operatorname{diag}(N_{u_{k,1}}^+(\Gamma_k)-\hat{E}_{k,1},\dots,N_{u_{k,w_k}}^+(\Gamma_k)-\hat{E}_{k,w_k}),\\
F_k^+ &= -\operatorname{diag}\left(\left[\begin{array}{c} N_{u_{k,1}}^-(\Gamma_k)-\hat{E}_{k,1}\\ \hline [0,\frac{\sqrt{2}}{2}I_2]N_{u_{k,1}}(\Gamma_k)^{-T}\end{array}\right],\dots,\left[\begin{array}{c} N_{u_{k,w_k}}^-(\Gamma_k)-\hat{E}_{k,w_k}\\ \hline [0,\frac{\sqrt{2}}{2}I_2]N_{u_{k,w_k}}(\Gamma_k)^{-T}\end{array}\right]\right),\\
G_k^+ &= -\operatorname{diag}([N_{u_{k,1}}^-(\Gamma_k)+\hat{E}_{k,1}|\ \begin{matrix}0\\ \frac{\sqrt{2}}{2}\Gamma_k\end{matrix}\ ],\dots,[N_{u_{k,w_k}}^-(\Gamma_k)+\hat{E}_{k,w_k}|\ \begin{matrix}0\\ \frac{\sqrt{2}}{2}\Gamma_k\end{matrix}\ ]),\\
C_k^- &= \operatorname{diag}(N_{v_{k,1}}^+(\Gamma_k)-\tilde{E}_{k,1},\dots,N_{v_{k,z_k}}^+(\Gamma_k)-\tilde{E}_{k,z_k}),\\
D_k^- &= \operatorname{diag}\left(\left[\begin{array}{c|c} \Gamma_k & [0,\frac{\sqrt{2}}{2}I_2]N_{v_{k,1}}(\Gamma_k)^{-T}\\ \hline \begin{matrix}0\\ -\frac{\sqrt{2}}{2}\Gamma_k\end{matrix} & N_{v_{k,1}}^+(\Gamma_k)+\tilde{E}_{k,z_k}\end{array}\right],\right.\\
&\qquad\left.\dots,\left[\begin{array}{c|c} \Gamma_k & [0,\frac{\sqrt{2}}{2}I_2]N_{v_{k,z_k}}(\Gamma_k)^{-T}\\ \hline \begin{matrix}0\\ -\frac{\sqrt{2}}{2}\Gamma_k\end{matrix} & N_{v_{k,z_k}}^+(\Gamma_k)+\tilde{E}_{k,z_k}\end{array}\right]\right),
\end{aligned}$$

$$F_k^- = \operatorname{diag}\left([\begin{array}{c} 0 \\ \frac{\sqrt{2}}{2} I_2 \end{array} | -N^-_{v_{k,1}}(\gamma_k) - \tilde{E}_{k,1}], \ldots, [\begin{array}{c} 0 \\ \frac{\sqrt{2}}{2} I_2 \end{array} | -N^-_{v_{k,z_k}}(\Gamma_k) - \tilde{E}_{k,z_k}] \right),$$

$$G_k^- = \operatorname{diag}\left(\left[\begin{array}{c} [0, \frac{\sqrt{2}}{2} I_2] N_{v_{k,1}}(\Gamma_k)^{-T} \\ -N^-_{v_{k,1}} + \tilde{E}_{k,1} \end{array} \right], \ldots, \left[\begin{array}{c} [0, \frac{\sqrt{2}}{2} I_2] N_{v_{k,z_k}}(\Gamma_k)^{-T} \\ -N^-_{v_{k,z_k}} + \tilde{E}_{k,z_k} \end{array} \right] \right).$$

The blocks associated with eigenvalues $1, -1$ are partitioned further as

$$C_\pm = \operatorname{diag}(C^e_\pm, C^+_\pm, C^-_\pm), \quad D_\pm = \operatorname{diag}(D^e_\pm, D^+_\pm, D^-_\pm),$$
$$F_\pm = \operatorname{diag}(F^e_\pm, F^+_\pm, F^-_\pm), \quad G_\pm = \operatorname{diag}(G^e_\pm, G^+_\pm, G^-_\pm),$$

and have with $E_k^\pm = \frac{1}{4} e_{g_k^\pm} e^T_{g_k^\pm} N_{g_k^\pm}(\pm 1)^{-T}$ and $\hat{E}_k^\pm = \frac{1}{4} e_{h_k^\pm} e^T_{h_k^\pm} N_{h_k^\pm}(\pm 1)^{-T}$ the substructures

$$C^e_\pm = D^e_\pm = \operatorname{diag}(N^+_{2x_1^\pm}(\pm 1), \ldots, N^+_{2x^\pm_{c_\pm}}(\pm 1)),$$

$$F^e_\pm = G^e_\pm = -\operatorname{diag}(N^-_{2x_1^\pm}(\pm 1), \ldots, N^-_{2x^\pm_{c_\pm}}(\pm 1)),$$

$$C^+_\pm = \operatorname{diag}\left(\left[\begin{array}{cc} N^+_{g_1^\pm}(\pm 1) - E_1^\pm & \pm \frac{\sqrt{2}}{2} e_{g_1^\pm} \\ -\frac{\sqrt{2}}{2} e^T_{g_1^\pm} N_{g_1^\pm}(\pm 1)^{-T} & \pm 1 \end{array} \right], \right.$$
$$\left. \ldots, \left[\begin{array}{cc} N^+_{g^\pm_{a_\pm}}(\pm 1) - E^\pm_{a_\pm} & \pm \frac{\sqrt{2}}{2} e_{g^\pm_{a_\pm}} \\ -\frac{\sqrt{2}}{2} e^T_{g^\pm_{a_\pm}} N_{g^\pm_{a_\pm}}(\pm 1)^{-T} & \pm 1 \end{array} \right] \right),$$

$$D^+_\pm = \operatorname{diag}(N^+_{g_1^\pm}(\pm 1) + E_1^\pm, \ldots, N^+_{g^\pm_{a_\pm}}(\pm 1) + E^\pm_{a_\pm}),$$

$$F^+_\pm = -\operatorname{diag}\left(\left[\begin{array}{c} N^-_{g_1^\pm}(\pm 1) + E_1^\pm \\ \frac{\sqrt{2}}{2} e^T_{g_1^\pm} N_{g_1^\pm}(\pm 1)^{-T} \end{array} \right], \ldots, \left[\begin{array}{c} N^-_{g^\pm_{a_\pm}}(\pm 1) + E^\pm_{a_\pm} \\ \frac{\sqrt{2}}{2} e^T_{g^\pm_{a_\pm}} N_{g^\pm_{a_\pm}}(\pm 1)^{-T} \end{array} \right] \right),$$

$$G^+_\pm = -\operatorname{diag}([N^-_{g_1^\pm}(\pm 1) - E_1^\pm | \pm \frac{\sqrt{2}}{2} e_{g_1^\pm}], \ldots, [N^-_{g^\pm_{a_\pm}}(\pm 1) - E^\pm_{a_\pm} | \pm \frac{\sqrt{2}}{2} e_{g^\pm_{a_\pm}}]),$$

$$C^-_\pm = \operatorname{diag}(N^+_{h_1^\pm}(\pm 1) + \hat{E}_1^\pm, \ldots, N^+_{h^\pm_{b_\pm}}(\pm 1) + \hat{E}^\pm_{b_\pm}),$$

$$D^-_\pm = \operatorname{diag}\left(\left[\begin{array}{cc} \pm 1 & \frac{\sqrt{2}}{2} e^T_{h_1^\pm} N_{h_1^\pm}(\pm 1)^{-T} \\ \mp \frac{\sqrt{2}}{2} e_{h_1^\pm} & N^+_{h_1^\pm}(\pm 1) - \hat{E}_1^\pm \end{array} \right], \right.$$
$$\left. \ldots, \left[\begin{array}{cc} \pm 1 & \frac{\sqrt{2}}{2} e^T_{h^\pm_{b_\pm}} N_{h^\pm_{b_\pm}}(\pm 1)^{-T} \\ \mp \frac{\sqrt{2}}{2} e_{h^\pm_{b_\pm}} & N^+_{h^\pm_{b_\pm}}(\pm 1) - \hat{E}^\pm_{b_\pm} \end{array} \right] \right),$$

$$F^-_\pm = \operatorname{diag}([\pm \frac{\sqrt{2}}{2} e_{h_1^\pm}, -N^-_{h_1^\pm}(\pm 1) + \hat{E}_1^\pm], \ldots, [\pm \frac{\sqrt{2}}{2} e_{h^\pm_{b_\pm}}, -N^-_{h^\pm_{b_\pm}}(\pm 1) + \hat{E}^\pm_{b_\pm}]),$$

$$G^-_\pm = \operatorname{diag}\left(\left[\begin{array}{c} \frac{\sqrt{2}}{2} e^T_{h_1^\pm} N_{h_1^\pm}(\pm 1)^{-T} \\ -N^-_{h_1^\pm}(\pm 1) - \hat{E}_1^\pm \end{array} \right], \ldots, \left[\begin{array}{c} \frac{\sqrt{2}}{2} e^T_{h^\pm_{b_\pm}} N_{h^\pm_{b_\pm}}(\pm 1)^{-T} \\ -N^-_{h^\pm_{b_\pm}}(\pm 1) - \hat{E}^\pm_{b_\pm} \end{array} \right] \right).$$

Each real eigenvalue α_k (α_k^{-1}) has l_k Jordan blocks of sizes $f_{k,1}, \ldots, f_{k,l_k}$ and each λ_k (λ_k^{-1}, $\overline{\lambda_k}$, $\overline{\lambda_k}^{-1}$) has s_k Jordan blocks of sizes $p_{k,1}, \ldots, p_{k,s_k}$.

Each nonreal unimodular eigenvalue γ_k $(\overline{\gamma_k})$ *has*
a) t_k *even sized Jordan blocks of sizes* $2q_{k,1}, \ldots, 2q_{k,t_k}$ *with the corresponding structure inertia indices* $i(-1)^{q_{k,1}+1}\beta_{k,1}, \ldots, i(-1)^{q_{k,t_k}+1}\beta_{k,t_k}$ *associated with* γ_k *and* $i(-1)^{q_{k,1}}\beta_{k,1}, \ldots, i(-1)^{q_{k,t_k}}\beta_{k,t_k}$ *associated with* $\overline{\gamma_k}$*;*
b) w_k *odd sized Jordan blocks of sizes* $2u_{k,1}+1, \ldots, 2u_{k,w_k}+1$ *corresponding to the structure inertia indices* $(-1)^{u_{k,1}+1}, \ldots, (-1)^{u_{k,w_k}+1}$*;*
c) z_k *odd sized Jordan blocks of sizes* $2v_{k,1}+1, \ldots, 2v_{k,z_k}+1$ *corresponding to the structure inertia indices* $(-1)^{v_{k,1}}, \ldots, (-1)^{v_{k,z_k}}$*.*

The eigenvalue 1 *has* $2c_+$ *even sized Jordan blocks of sizes* $2x_1^+, 2x_1^+, \ldots, 2x_{c_+}^+, 2x_{c_+}^+$ *corresponding to the structure inertia indices* $i, -i, \ldots, i, -i$*, and* $a_+ + b_+$ *odd sized Jordan blocks,* a_+ *of them of sizes* $2g_1^+ + 1, \ldots, 2g_{a_+}^+ + 1$ *with the corresponding structure inertia indices* $(-1)^{g_1^+ + 1}, \ldots, (-1)^{g_{a_+}^+ + 1}$ *and* b_+ *of them of sizes* $2h_1^+ + 1, \ldots, 2h_{b_+}^+ + 1$ *with the corresponding structure inertia indices* $(-1)^{h_1^+}, \ldots, (-1)^{h_{b_+}^+}$*.*

Similarly, the eigenvalue -1 *has* $2c_-$ *even sized Jordan blocks of sizes* $2x_1^-, 2x_1^-, \ldots, 2x_{c_-}^-, 2x_{c_-}^-$ *corresponding to the structure inertia indices* $i, -i, \ldots, i, -i$*, and* $a_- + b_-$ *odd sized Jordan blocks,* a_- *of them of sizes* $2g_1^- + 1, \ldots, 2g_{a_-}^- + 1$ *with the corresponding structure inertia indices* $(-1)^{g_1^- + 1}, \ldots, (-1)^{g_{a_-}^- + 1}$ *and* b_- *of them of sizes* $2h_1^- + 1, \ldots, 2h_{b_-}^- + 1$ *with the corresponding structure inertia indices* $(-1)^{h_1^-}, \ldots, (-1)^{h_{b_-}^-}$*.*

6. Conclusion. We have presented real and complex structured Jordan canonical forms under real $\Sigma_{p,q}$-orthogonal and $\Sigma_{p,q}$-unitary matrices, respectively. Combining these results with the structured canonical forms for Hamiltonian, skew Hamiltonian and symplectic matrices in [8] a complete list of the possible structured canonical forms is available.

The structured Jordan canonical forms for groups of structured matrices such as complex $\Sigma_{p,q}$-symmetric, skew symmetric and orthogonal matrices, complex J-symmetric, J-skew symmetric and J-orthogonal matrices, with the similarity matrices in the corresponding Lie groups, can be derived in a similar way, were $J = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$. We can also generalize these results to the cases that $\Sigma_{p,q}$ and J are replaced by general nonsingular Hermitian and skew Hermitian matrices, respectively. Due to the large amount of material that we have already presented we have refrained from presenting these results.

It is also possible to generalize all these results to the matrix pencil case with structures as it has been done for Hamiltonian pencils, and symplectic pencils in [8] and for skew Hamiltonian/Hamiltonian pencils in [9, 10]. This generalization can be done as follows: Suppose that for a matrix pencil $\mathcal{A} - \lambda\mathcal{B}$ with say $\mathcal{A} = \mathcal{A}^H$, $\mathcal{B} = \mathcal{B}^H$ the matrix $\mathcal{B}$ is invertible, then the matrix $\hat{A} = \mathcal{B}^{-1}\mathcal{A}$ satisfies $\mathcal{B}\hat{A} = \hat{A}^H\mathcal{B}$. So we can determine a nonsingular matrix $\mathcal{U}$ such that

$$\mathcal{U}^H\mathcal{B}\mathcal{U} = \mathcal{D}_b, \quad \mathcal{U}^{-1}\hat{\mathcal{A}}\mathcal{U} = \mathcal{D}_a.$$

Taking the product form of $\hat{\mathcal{A}}$ we have

$$\mathcal{U}^H\mathcal{B}\mathcal{U} = \mathcal{D}_b, \quad \mathcal{U}^H\mathcal{A}\mathcal{U} = \mathcal{D}_b\mathcal{D}_a,$$

which is just the result of Thompson [12] or Uhlig for the real case [14]. We can also easily obtain the canonical forms for all the pencils with $\mathcal{A} = \pm\mathcal{A}^H$, $\mathcal{B} = \pm\mathcal{B}^H$.

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