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TIGHT BOUNDS ON THE ALGEBRAIC CONNECTIVITY OF A BALANCED BINARY TREE*

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Abstract. In this paper, quite tight lower and upper bounds are obtained on the algebraic connectivity, namely, the second-smallest eigenvalue of the Laplacian matrix, of an unweighted balanced binary tree with k levels and hence $n = 2^k - 1$ vertices. This is accomplished by considering the inverse of a matrix of order $k-1$ readily obtained from the Laplacian matrix. It is shown that the algebraic connectivity is $1/(2^k - 2k + 3) + O(1/2^{2k})$.

Key words. Binary trees, Laplacian matrix, Algebraic connectivity

AMS subject classifications. 5C50, 15A48

1. Introduction. In this paper we improve known lower and upper bounds on the *algebraic connectivity*, namely, the second-smallest eigenvalue of the Laplacian matrix[1], of the balanced binary tree $\mathcal{B}_k$ of k levels and hence $n = 2^k - 1$ vertices. Specifically, in [9, Lemma 6.1], Guattery and Miller quote an earlier result of theirs in [8] in which they have shown that the algebraic connectivity, $\nu(\mathcal{B}_k)$, of $\mathcal{B}_k$ satisfies

$$\frac{1}{n} \le \nu(\mathcal{B}_k) \le \frac{2}{n}.$$

Our new upper bound is

$$\nu(\mathcal{B}_k) \le \frac{1}{(2^k - 2k + 3) - \dfrac{2k-2}{2^{k-1}-1}}$$

and our new lower bound is

$$\frac{1}{(2^k - 2k + 2) - \dfrac{2k - \sqrt{2}\left(2k - 1 - 2^{k-1}\right)}{2^k - 1 - \sqrt{2}\left(2^{k-1} - 1\right)} + \dfrac{1}{3 - 2\sqrt{2}\cos\left(\dfrac{\pi}{2k-1}\right)}} \le \nu(\mathcal{B}_k).$$

We comment that for large k the difference between the denominators in the lower bound and the upper bound is approximately

$$-1 - \frac{\sqrt{2}}{2-\sqrt{2}} + \frac{1}{3 - 2\sqrt{2}} = 1 + \sqrt{2} = 2.414\ 213\ 6...,$$

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1 For precise definitions of all concepts mentioned in this section, see Section 2.

which implies that the algebraic connectivity of the balanced binary tree is $1/(2^k - 2k + 3) + O(1/2^{2k})$.

We prove our main results in Section 3, while in Section 2 we review necessary preliminaries and also describe some motivation for our work here.

We comment that two other lower estimates for $\nu(\mathcal{B}_k)$ can be obtained as special cases of two general lower estimates on the algebraic connectivity of a graph $\mathcal{G}$. The first lower estimate, equaling $1/[2(2^k - 3)^2]$, is due to Friedland [6, Theorem 2.6] and involves the so-called *Cheeger* lower bound. The second lower bound, equaling $3 - \sqrt{9 - 1/(2^{k-1} - 1)^2}$, is due to Berman and Zhang [1, Theorem 2.2]. It can be readily checked that our new lower bound is better than the two bounds just mentioned.

2. Preliminaries. Let $\mathcal{G}$ be a graph with vertices $1, 2, \ldots, n$. Denote the degree of vertex i by $\deg(i)$. The *Laplacian* (*matrix*) of $\mathcal{G}$ is the $n \times n$ matrix $L = [\ell_{ij}]$ with

$$\ell_{ij} = \begin{cases} \deg(i) & \text{if } i = j, \\ -1 & \text{if } i \neq j \text{ and } i \text{ is adjacent to } j, \\ 0 & \text{otherwise.} \end{cases}$$

The Laplacian L of a graph $\mathcal{G}$ is a useful algebraic tool for assessing certain properties of the graph. Perhaps the most well-known property of L is the *matrix-tree theorem* due to Cayley [2] (see also Chaiken [3]), which relates the Laplacian to the number of spanning trees of $\mathcal{G}$. Numerous other properties of $\mathcal{G}$, related to the connectivity and the isoperimetric number of $\mathcal{G}$, are reflected by the spectrum of L (see [14, 16, 17] and the references therein). Since L is a symmetric, positive semidefinite, and singular matrix, its eigenvalues are nonnegative real numbers, and so they can be arranged in nondecreasing order:

$$0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n.$$

Fiedler [5] observed that the second-smallest eigenvalue, $\nu(\mathcal{G}) := \lambda_2$, of L provides a measure of connectivity and he called $\nu(\mathcal{G})$ the *algebraic connectivity* of $\mathcal{G}$. In particular he has shown that $\nu(\mathcal{G}) > 0$ if and only if $\mathcal{G}$ is a connected graph.

The algebraic connectivity of a graph $\mathcal{G}$ and its applications have been extensively studied in the literature; we cite the following papers and the references quoted therein: Grone and Merris [7], Merris [14, 15], Powers [19], Pothen, Simon, and Liou [18], Guattery and Miller [9], and Kirkland, Neumann, and Shader [12, 11, 13].

As an example of the usage of algebraic connectivity we give the problem of finding separators for graphs. Separators are edges or vertices which, if removed from the graph, break it into separate components. This problem is an important component of many graph algorithms. Many popular separator algorithms involve spectral methods, in particular the Laplacian L of the underlying graph $\mathcal{G}$. We refer the reader to [9] for a more detailed discussion. Typically, the algebraic connectivity $\nu(\mathcal{G})$ and a corresponding eigenvector u are computed. The best threshold cut method can be described as follows. Let n be the number of vertices of $\mathcal{G}$. Given a nonempty, proper subset X of vertices of $\mathcal{G}$ we define the *cut quotient* of X to be

$$\frac{|\partial X|}{\min\{|X|, |\overline{X}|\}},$$

where $|\partial X|$ is the number of edges in $\mathcal{G}$ with exactly one vertex in X. The minimum of the cut quotients is the *isoperimetric number, $i(\mathcal{G})$, of $\mathcal{G}$*. It is well known that

$$i(\mathcal{G}) \geq \frac{\nu(\mathcal{G})}{2}.$$

The best threshold cut method strives to find a set X for which the cut quotient of X is close to the minimum $i(\mathcal{G})$. This is achieved as follows.

(i) Associate with each vertex, i, the value of the ith entry of u.

(ii) Sort the vertices according to their value. For each index $1 \leq i \leq n-1$, compute the cut quotient for the separator obtained by splitting the vertices into those with sorted index less than i and those with sorted index greater than i.

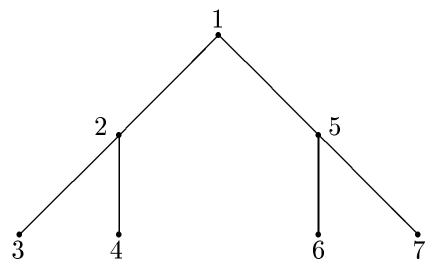
(iii) Choose the split from (ii) that provides the smallest cut quotient.

As noted in [9], until recently, there has not been a rigorous analysis of the quality of separators produced by such algorithms. In [9], the complete balanced binary trees are used as building blocks to construct graphs for which the best threshold cut method does poorly. An essential ingredient in their work is their bound

$$\frac{1}{n} \ \leq \ \nu(\mathcal{B}_k) \ \leq \ \frac{2}{n}$$

on the algebraic connectivity for a complete balanced binary tree.

3. Tight Bounds on $\nu\,(\mathcal{B}_k)$. As before, let $\mathcal{B}_k$ denote the balanced binary tree with $k \geq 2$ levels so that $\mathcal{B}_k$ has $n := 2^k - 1$ vertices. We now relabel the vertices of $\mathcal{B}_k$ so that 1 is the root vertex, the vertices on the left branch of 1 precede those on the right branch, and the vertices on the ith level precede those on level $i+1$ for $i = 2, 3, \ldots, k-1$. For example, for $k = 3$, we have the labeling illustrated below.



The following notation is used throughout this section. If N is an $m \times m$ matrix, each of whose eigenvalues is real, then the eigenvalues of N are denoted by $\lambda_i(N)$, $i = 1, 2, \ldots, m$, where

$$\lambda_1(N) \ \leq \ \lambda_2(N) \ \leq \ \cdots \ \leq \ \lambda_{m-1}(N) \ \leq \ \lambda_m(N).$$

The trace and adjoint of N are denoted by $\mathrm{tr}(N)$ and $\mathrm{adj}(N)$, respectively.

We begin by showing that the problem of determining $\nu(\mathcal{B}_k)$ can be transformed to that of determining the smallest eigenvalue of a principal submatrix of the Laplacian of order $2^{k-1}-1$.

PROPOSITION 3.1. *Let L_k be the Laplacian matrix of $\mathcal{B}_k$. Then*

$$\nu(\mathcal{B}_k) \;=\; \lambda_1(L_k(1,1)), \tag{1}$$

where $L_k(1,1)$ is the principal submatrix of L_k obtained by deleting its first row and column.

Proof. Note that L_k has the form

$$\left[\begin{array}{r|c|c} 2 & -1 \;\; 0 \;\; \cdots \;\; 0 & -1 \;\; 0 \;\; \cdots \;\; 0 \\ \hline -1 & & \\ 0 & & \\ \vdots & C & O \\ 0 & & \\ \hline -1 & & \\ 0 & & \\ \vdots & O & C \\ 0 & & \end{array}\right],$$

where C is a matrix of order $2^{k-1}-1$. It follows that each eigenvalue of $L_k(1,1)$ has multiplicity at least two and hence, by the Cauchy interlacing property (see [10]), $\lambda_1(L_k(1,1)) = \lambda_2(L_k)$. $\square$

Next, let $Q = (q_{i,j})$ be the $(n-1)\times(k-1)$ matrix with

$$q_{i,j} \;=\; \begin{cases} 1 & \text{if vertex } i+1 \text{ is on level } j+1 \text{ of } \mathcal{B}_k, \\ 0 & \text{otherwise,} \end{cases}$$

and define the $(k-1)\times(k-1)$ tridiagonal matrix

$$F_{k-1} \;:=\; \begin{bmatrix} 3 & -2 & 0 & \cdots & \cdots & \cdots & 0 \\ -1 & 3 & -2 & \ddots & & \ddots & \vdots \\ 0 & -1 & \ddots & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & & & 0 & -1 & 3 & -2 \\ 0 & \cdots & \cdots & \cdots & 0 & -1 & 1 \end{bmatrix}.$$

Concerning F_{k-1} we claim the following proposition.

PROPOSITION 3.2. *Let $L_k = [\ell_{i,j}]$ be the Laplacian matrix of $\mathcal{B}_k$. Then*
(a) $L_k(1,1)\,Q \;=\; QF_{k-1}$,

(b) each eigenvalue of F_{k-1} *is an eigenvalue of* $L_k(1,1)$,
(c) each eigenvalue of F_{k-1} *is real,*
(d) $\lambda_1(F_{k-1}) \;=\; \lambda_1(L_k(1,1))$, *and*
(e) $\mathrm{tr}(F_{k-1}^{-1}) \;=\; 2^k - (k+1)$.

Proof. Part (a) follows from the observation that for $1 \leq i \leq j \leq k$,
(i) If $|j-i| > 1$, then there are no edges joining a vertex in level i of $\mathcal{B}_k$ to a vertex in level j.
(ii) If $j - i = 1$, then each vertex in level i of $\mathcal{B}_k$ is joined to exactly two vertices in level j.
(iii) If $j - i = -1$, then each vertex in level i of $\mathcal{B}_k$ is joined to exactly one vertex in level j.
(iv) If v is in level i, then

$$\ell_{v,v} \;=\; \begin{cases} 2 & \text{if } i = 1, \\ 1 & \text{if } i = k, \\ 3 & \text{otherwise.} \end{cases}$$

Now let (λ, x) be an eigenpair of F_{k-1}. Then (a) implies that $L_k(1,1)(Qx) = \lambda(Qx)$. Since the columns of Q are linearly independent, it follows that $Qx \neq 0$. Hence (λ, Qx) is an eigenpair of $L_k(1,1)$ and so (b) holds. Part (c) follows from (b) and the fact that $L_k(1,1)$ is a real symmetric matrix. Adopting the notation in the proof of Proposition 3.1, we see that $L_k(1,1)$ is the direct sum of two copies of the matrix C. Since C is a principal submatrix of a Laplacian matrix and is irreducible, C is a nonsingular M-matrix. Hence C has a nonnegative eigenvector v corresponding to the eigenvalue $\lambda_1(C) = \lambda_1(L_k(1,1))$.

It is now easy to verify that

$$[v^T \quad 0]L_k(1,1) \;=\; \lambda_1(L_K(1,1))[v^T \quad 0].$$

The fact that v is a nonnegative eigenvector implies that $[v^T \quad 0]Q \neq 0$. This along with (a) implies that $[v^T \quad 0]Q$ is a left-eigenvector of F_{k-1} corresponding to the eigenvalue $\lambda_1(L_k(1,1))$. Part (d) now follows from (b).

Finally, we prove (e). As can be readily checked, the matrix F_{k-1} admits the following factorization as the product of an upper triangular matrix and a lower triangular matrix:

$$F_{k-1} \;=\; \begin{bmatrix} 1 & -2 & 0 & & .. & .. & & 0 \\ 0 & 1 & -2 & 0 & & & & 0 \\ & 0 & 1 & -2 & & & & \\ & & 0 & 1 & .. & & & \\ \vdots & & & \ddots & .. & .. & & \vdots \\ & & & & \ddots & .. & -2 & 0 \\ & & & & & 0 & 1 & -2 \\ 0 & 0 & & \cdots & & & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & & .. & .. & & 0 \\ -1 & 1 & & & & & \\ & -1 & 1 & & & & \\ & & .. & .. & & & \\ & & & .. & .. & & \vdots \\ & & & .. & .. & & \\ & & & & .. & 1 & 0 \\ & & & & & -1 & 1 \end{bmatrix},$$

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and thus we have the LU-factorization

$$F_{k-1}^{-1} = \begin{bmatrix} 1 & & & & & & \\ 1 & 1 & & & & & \\ & & 1 & & & & \\ \vdots & & & .. & & & \\ & & & & .. & & \\ 1 & & & & & 1 & \\ 1 & & & \dots & & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2^2 & 2^3 & .. & .. & 2^{k-2} \\ & 1 & 2 & 2^2 & .. & .. & 2^{k-3} \\ & & 1 & .. & .. & .. & .. \\ & & & .. & .. & .. & 2^3 \\ & & & & .. & .. & 2^2 \\ & & & & & 1 & 2 \\ & & & & & & 1 \end{bmatrix}.$$

But then $\left(F_{k-1}^{-1}\right)_{i,i} = 2^i - 1$, so that, easily, $\operatorname{tr}\left(F_{k-1}^{-1}\right) = \sum_{i=1}^{k-1}\left(2^i - 1\right) = 2^k - (k+1)$. $\square$

It follows from Proposition 3.2 that to either determine exactly or approximately $\lambda_1(L(1,1))$, it suffices to consider the problem of determining exactly or approximately the smallest eigenvalue of F_{k-1}. We begin by noting that for any $m \geq 1$, if D is the $m \times m$ diagonal matrix whose (i,i)th entry is $2^{i/2}$, $i = 1, 2, \dots, m$, then the matrix

$$(2) \quad G_m := DF_mD^{-1} = \begin{bmatrix} 3 & -\sqrt{2} & 0 & \cdots & \cdots & \cdots & 0 \\ -\sqrt{2} & 3 & -\sqrt{2} & \ddots & & \ddots & \vdots \\ 0 & -\sqrt{2} & \ddots & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & & & 0 & -\sqrt{2} & 3 & -\sqrt{2} \\ 0 & \cdots & \cdots & \cdots & 0 & -\sqrt{2} & 1 \end{bmatrix}$$

is (diagonally) similar to F_m and hence it has the same eigenvalues as F_m.

Now let $S_{m-1} = G_m(m,m)$ be the leading principal submatrix of G_m of order $m-1$. Note that S_{m-1} is a symmetric, tridiagonal matrix each of whose diagonal entries is 3 and each of whose super- and subdiagonal entries is $-\sqrt{2}$. Some additional properties of the matrices S_1, S_2, …, are the following.

LEMMA 3.3. *(a)* $\det(S_t) = 2^{t+1} - 1$.
(b) $\operatorname{tr}(S_t^{-1}) = t - 2 + \dfrac{2t+2}{2^{t+1}-1}$, *provided that* $t \geq 2$.

Proof. We prove (a) by induction on t. If $t = 1$, then $\det S_1 = 3 = 2^2 - 1$.

Assume that $t \geq 2$ and proceed by induction. Note that by Laplace expansion along the last row,

$$\begin{aligned} \det(S_t) &= 3\det(S_{t-1}) - (\sqrt{2})^2 \det(S_{t-2}) \\ &= 3(2^t - 1) - 2(2^{t-1} - 1) \\ &= 2^{t+1} - 1, \end{aligned}$$

as desired.

Next note that if we set $\det S_0 = 1$, then

$$\begin{aligned}
\operatorname{tr}(\operatorname{adj}(S_t)) &= \textstyle\sum_{j=0}^{t-1} \det(S_j)\det(S_{t-1-j}) \\
&= \textstyle\sum_{j=0}^{t-1} (2^{j+1}-1)(2^{t-j}-1) \\
&= \textstyle\sum_{j=0}^{t-1} 2^{t+1} + 1 - 2^{j+1} - 2^{t-j} \\
&= t2^{t+1} + t - 2(2 + 2^2 + \cdots + 2^t) \\
&= t2^{t+1} + t - 2(2^{t+1} - 2) \\
&= (t-2)2^{t+1} + t + 4.
\end{aligned}$$

Statement (b) now follows from (a). □

We shall now use Lemma 3.3 to obtain a lower bound on $\lambda_1(G_m)$. Since the eigenvalues of S_{m-1} interlace the eigenvalues of G_m, and as the eigenvalues of G_m are those of F_m, we can write that

$$\lambda_1(F_m) \le \lambda_1(S_{m-1}) \le \lambda_2(F_m) \le \cdots \le \lambda_{m-1}(S_{m-1}) \le \lambda_m(F_m).$$

Furthermore G_m is a nonsingular M-matrix; each of its eigenvalues is positive, as are those of S_{m-1}. Hence the eigenvalues of F_m^{-1} interlace those of S_{m-1}^{-1}. That is,

$$\lambda_1(F_m^{-1}) \le \lambda_1(S_{m-1}^{-1}) \le \lambda_2(F_m^{-1}) \le \cdots \le \lambda_{m-1}(S_{m-1}^{-1}) \le \lambda_m(F_m^{-1}).$$

Thus,

$$\begin{aligned}
\operatorname{tr}(F_m^{-1}) &= \sum_{i=1}^{m} \lambda_i(F_m^{-1}) \\
&\le \left(\sum_{i=1}^{m-1} \lambda_i(S_{m-1}^{-1})\right) + \lambda_m(F_m^{-1}) \\
&= \operatorname{tr}(S_{m-1}^{-1}) + \lambda_m(F_m^{-1}).
\end{aligned}$$

Hence we have that

$$\operatorname{tr}(F_m^{-1}) - \operatorname{tr}\left(S_{m-1}^{-1}\right) \le \lambda_m(F_m^{-1}). \tag{3}$$

This essentially proves the following lemma.

LEMMA 3.4. *Let G_m be the matrix given in* (2) *with $m \ge 3$. Then*

$$\lambda_1(G_m) \le \frac{1}{2^{m+1} - 2m + 1 - \dfrac{2m}{2^m - 1}}. \tag{4}$$

Proof. Recall that that the eigenvalues of G_m are the reciprocals of the eigenvalues of the matrix F_m^{-1} used above. Thus (4) follows from (3) and the trace formulas in Proposition 3.2 (with $m = k-1$) and Lemma 3.3b (with $t = m-1$). ◻

To obtain an upper bound on $\lambda_1(G_m)$, we consider the following $t \times t$ matrix which, for $t = m$, is readily seen to be a perturbation by a positive semidefinite matrix of G_m:

$$H_t := \begin{bmatrix} 3 & -\sqrt{2} & 0 & \cdots & \cdots & \cdots & 0 \\ -\sqrt{2} & 3 & -\sqrt{2} & \ddots & & \ddots & \vdots \\ 0 & -\sqrt{2} & \ddots & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & & & 0 & -\sqrt{2} & 3 & -\sqrt{2} \\ 0 & \cdots & \cdots & \cdots & 0 & -\sqrt{2} & 3-\sqrt{2} \end{bmatrix}. \tag{5}$$

Thus, from Weyl's theorem on the perturbation of eigenvalues of symmetric matrices (see p. 181 of Horn and Johnson [10]), we have that

$$\lambda_i(G_m) \le \lambda_i(H_m), \quad i = 1, \ldots, m. \tag{6}$$

We comment that from Elliot [4], it can be deduced that the eigenvalues of H_t are given by

$$\lambda_i(H_t) = 3 - 2\sqrt{2}\cos\left(\frac{(2i-1)\pi}{2t+1}\right), \quad i = 1, \ldots, t. \tag{7}$$

We proceed now to prove for H_t a result similar to Lemma 3.3.

LEMMA 3.5. *For H_t as given in* (5), *the following conditions hold:*

(a) $\det(H_t) = 2^{t+1} - 1 - \sqrt{2}(2^t - 1)$.

(b) $\operatorname{tr}\left(H_t^{-1}\right) = t - 2 + \dfrac{2t + 2 - \sqrt{2}\,(2t + 1 - 2^t)}{2^{t+1} - 1 - \sqrt{2}(2^t - 1)}$, *provided that* $t \ge 2$.

Proof. Statement (a) clearly holds if $t = 1$. If $t \ge 2$, then by the linearity of the determinant we see that $\det(H_t) = \det(S_t) - \sqrt{2}\det(S_{t-1})$. Thus part (a) follows from the formula for the determinant of S_{t-1} and S_t, which is given in Lemma 3.3a.

For $t \ge 2$, the (i,i)th entry of $\operatorname{adj}(H_t)$ is $\det(S_{t-1})\det(H_{t-i})$ (we set $\det(S_0) = 1$ and $\det(H_0) = 1$ here). Hence it follows that

$$\begin{aligned} \operatorname{tr}(\operatorname{adj}(H_t)) &= 2^t - 1 + \textstyle\sum_{i=1}^{t-1}(2^i - 1)(2^{t-i+1} - 1 - \sqrt{2}(2^{t-i} - 1)) \\ &= (t-2)2^{t+1} + t + 4 - \sqrt{2}((t-3)2^t + t + 3), \end{aligned}$$

with the last inequality coming from arithmetic simplification using formulas for geometric sums. Statement (b) now follows from the above trace formula, (a), and some additional arithmetic simplification. ◻

To obtain an upper bound on $\lambda_1(G_m)$, we once again resort to working with inverses. From (6) and the fact that the eigenvalues of F_m are the same as those of G_m, we have that

$$\lambda_i(F_m^{-1}) \;\geq\; \lambda_i(H_m^{-1}), \quad i = 1, 2, \ldots, m.$$

This implies that

$$\lambda_m(F_m^{-1}) + \operatorname{tr}(H_m^{-1}) - \lambda_m(H_m^{-1}) \;\leq\; \operatorname{tr}(F_m^{-1}).$$

Substituting in the appropriate trace formulas from Lemmas 3.3 and 3.5 yields

$$\lambda_m(F_m^{-1}) \;\leq\; 2^{m+1} - 2m - \frac{2m + 2 - \sqrt{2}\,(2m + 1 - 2^m)}{2^{m+1} - 1 - \sqrt{2}\,(2^m - 1)} + \lambda_m(H_m^{-1}).$$

If we now make use of the fact that $\lambda_m(H_m^{-1})$ is the reciprocal of the smallest eigenvalue of H_m, we arrive, using (7), at the following lower bound on the $\lambda_1(G_m)$.

LEMMA 3.6. *Let G_m be the matrix given in* (2) *with $m \geq 3$. Then*

$$\lambda_1(G_m) \;\geq\; \frac{1}{2^{m+1} - 2m - \dfrac{2m + 2 - \sqrt{2}\,(2m + 1 - 2^m)}{2^{m+1} - 1 - \sqrt{2}\,(2^m - 1)} + \dfrac{1}{3 - 2\sqrt{2}\cos\left(\dfrac{\pi}{2m+1}\right)}}.$$

The goal of our paper is to provide good lower and upper bounds on

$$\nu(\mathcal{B}_k) \;=\; \lambda_1(L_k(1,1)) \;=\; \lambda_1(F_{k-1}) \;=\; \lambda_1(G_{k-1}).$$

Thus, by using Lemmas 3.4 and 3.6 (with $m = k - 1$), we obtain the main result of this paper.

THEOREM 3.7. *Let $\mathcal{B}_k$ be the balanced binary tree on $k \geq 4$ levels. Then*

$$\nu(\mathcal{B}_k) \;\leq\; \frac{1}{(2^k - 2k + 3) - \dfrac{2k - 2}{2^{k-1} - 1}}$$

and

$$\nu(\mathcal{B}_k) \;\geq\; \frac{1}{(2^k - 2k + 2) - \dfrac{2k - \sqrt{2}\left(2k - 1 - 2^{k-1}\right)}{2^k - 1 - \sqrt{2}\,(2^{k-1} - 1)} + \dfrac{1}{3 - 2\sqrt{2}\cos\left(\dfrac{\pi}{2k-1}\right)}}.$$

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