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POLAR DECOMPOSITION UNDER PERTURBATIONS OF THE SCALAR PRODUCT*

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Dedicated to our friend and teacher Angel Rafael Larotonda on his sixtieth birthday

Abstract. Let $\mathcal{A}$ be a unital C*-algebra with involution $*$ represented in a Hilbert space $\mathcal{H}$, G the group of invertible elements of $\mathcal{A}$, $\mathcal{U}$ the unitary group of $\mathcal{A}$, G^s the set of invertible selfadjoint elements of $\mathcal{A}$, $Q = \{\varepsilon \in G : \varepsilon^2 = 1\}$ the space of reflections and $P = Q \cap \mathcal{U}$. For any positive $a \in G$ consider the a-unitary group $\mathcal{U}_a = \{g \in G : a^{-1}g^*a = g^{-1}\}$, i.e., the elements which are unitary with respect to the scalar product $\langle \xi, \eta \rangle_a = \langle a\xi, \eta \rangle$ for $\xi,\ \eta \in \mathcal{H}$. If π denotes the map that assigns to each invertible element its unitary part in the polar decomposition, it is shown that the restriction $\pi|_{\mathcal{U}_a} : \mathcal{U}_a \to \mathcal{U}$ is a diffeomorphism, that $\pi(\mathcal{U}_a \cap Q) = P$, and that $\pi(\mathcal{U}_a \cap G^s) = \mathcal{U}_a \cap G^s = \{u \in G : u = u^* = u^{-1} \text{ and } au = ua\}$.

Key words. Polar decomposition, C*-algebras, positive operators, projections.

AMS subject classifications. Primary 47A30, 47B15

1. Introduction. If $\mathcal{A}$ is the algebra of bounded linear operators in a Hilbert space $\mathcal{H}$, denote by G the group of invertible elements of $\mathcal{A}$. Every $T \in G$ admits two *polar decompositions*

$$T = U_1P_1 = P_2U_2$$

where U_1, U_2 are unitary operators (i.e. $U_i^* = U_i^{-1}$) and P_1, P_2 are positive operators (i.e $\langle P_i\xi, \xi\rangle \geq 0$ for every $\xi \in \mathcal{H}$). It turns out that $U_1 = U_2$, $P_1 = (T^*T)^{1/2}$ and $P_2 = (TT^*)^{1/2}$. We shall call $U = U_1 = U_2$ *the unitary part* of T. Consider the map

$$\pi : G \to \mathcal{U} \qquad \pi(T) = U$$

where $\mathcal{U}$ is the unitary group of $\mathcal{A}$. If G^+ denotes the set of all positive invertible elements of $\mathcal{A}$, then every $A \in G^+$ defines an inner product $\langle\ ,\ \rangle_A$ on $\mathcal{H}$ which is equivalent to the original $\langle\ ,\ \rangle$; namely

$$\langle \xi, \eta \rangle_A = \langle A\xi, \eta \rangle \quad (\xi, \eta \in \mathcal{H}).$$

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Every $X \in \mathcal{A}$ admits an A-adjoint operator X^{*_A}, which is the unique $Y \in \mathcal{A}$ such that

$$\langle X\xi, \eta\rangle_A = \langle \xi, Y\eta\rangle_A \quad (\xi, \eta \in \mathcal{H}).$$

It is easy to see that $X^{*_A} = A^{-1}X^*A$. Together with the definition of $*_A$ one gets the sets of *A-Hermitian operators*

$$\mathcal{A}_A^h = \{X \in \mathcal{A} : X^{*_A} = X\} = \{X \in \mathcal{A} : AX = X^*A\},$$

A-unitary operators

$$\mathcal{U}_A = \{X \in G : X^{*_A} = X^{-1}\} = \{X \in \mathcal{A} : AX^{-1} = X^*A\}$$

and *A-positive operators*

$$G_A^+ = \{X \in \mathcal{A}_A^h \cap G : \langle X\xi, \xi\rangle_A \geq 0 \quad \forall\, \xi \in \mathcal{H}\}$$

As in the "classical" case, i.e. $A = I$, we get two polar decompositions for each $T \in G$

$$T = V_1R_1 = R_2V_2$$

with $V_i \in \mathcal{U}_A$, $R_i \in G_A^+$, $i = 1, 2$ and as before $V_1 = V_2 = V$. Thus, we get a map

$$\pi_A : G \to \mathcal{U}_A \quad , \quad \pi_A(T) = V \ , \quad T \in G.$$

This paper is devoted to a simultaneous study of the maps π_A $(A \in G^+)$, the way that

$$\mathcal{U}_A \ , \quad G_A^h \ , \quad G_A^+$$

intersect

$$\mathcal{U}_B \ , \quad G_B^h \ , \quad G_B^+$$

for different $A, B \in G^+$ and the intersections of these sets with

$$Q = \{S \in \mathcal{A} : S^2 = I\} \quad \text{and} \quad P_A = \{S \in Q : S^{*_A} = S\}$$

(reflections and A-Hermitian reflections of $\mathcal{A}$). The main result is the fact that, for every $A, B \in G^+$,

$$\pi_A|_{\mathcal{U}_B} : \mathcal{U}_B \to \mathcal{U}_A$$

is a bijection. The proof of this theorem is based on the form of the positive solutions of the operator equation $XAX = B$ for $A, B \in G^+$. This identity was first studied by G. K. Pedersen and M. Takesaki [10] in their study of the Radon-Nykodym theorems in von Neumann algebras. As a corollary we get a short proof of the equality

$$\pi_A(Q \cap \mathcal{U}_B) = P_A$$

for every $A, B \in G^+$, which was proven in [1] as a C^*-algebraic version of results of Pasternak-Winiarski [8] on the analyticity of the map $A \mapsto P_A^{\mathcal{M}}$, where $P_A^{\mathcal{M}}$ is the A-orthogonal projection on the closed subspace $\mathcal{M}$ of $\mathcal{H}$. We include a parametrization of all solutions of Pedersen-Takesaki equation. The results are presented in the context of unital C*-algebras.

2. Preliminaries. Let $\mathcal{A}$ be a unital C*-algebra, $G = G(\mathcal{A})$ the group of invertible elements of $\mathcal{A}$, $\mathcal{U} = \mathcal{U}(\mathcal{A})$ the unitary group of $\mathcal{A}$, $G^+ = G^+(\mathcal{A})$ the set of positive invertible elements of $\mathcal{A}$ and $G^s = G^s(\mathcal{A})$ the set of positive selfadjoint elements of $\mathcal{A}$. Let $Q = Q(\mathcal{A}) = \{\varepsilon \in G : \varepsilon^2 = 1\}$ be the space of reflections and

$$P = P(\mathcal{A}) = Q \cap G^s = Q \cap \mathcal{U} = \{\rho \in G : \rho = \rho^* = \rho^{-1}\}$$

the space of orthogonal reflections, also called the Grassmann manifold of $\mathcal{A}$.

Each $g \in G$ admits two *polar decompositions*

$$g = \lambda u = u'\lambda' \ , \quad \lambda, \lambda' \in G^+ \ , \quad u, u' \in \mathcal{U}.$$

In fact, $\lambda = (gg^*)^{1/2}$, $u = (gg^*)^{-1/2}g$, $\lambda' = (g^*g)^{1/2}$ and $u' = g(g^*g)^{-1/2}$. A simple exercise of functional calculus shows that $u = u'$. We shall say that u is *the unitary part* of g. Observe that in the decomposition $g = \lambda u$ (resp. $g = u\lambda'$) the components λ, u (resp. u, λ') are uniquely determined, for instance, if $\lambda u = \lambda_o u_0$, then $\lambda_0^{-1}\lambda = u_0 u^{-1}$ is a unitary element with positive spectrum: $\sigma(\lambda_0^{-1}\lambda) = \sigma(\lambda_0^{-1/2}\lambda\lambda_0^{1/2}) = \sigma(\lambda) \subseteq \mathbb{R}^+$. Then $\lambda_0^{-1}\lambda = u_0u^{-1} = 1$. The map

$$\pi : G \to \mathcal{U} \quad \text{given by} \quad \pi(g) = u \quad (g \in G)$$

is a fibration with very rich geometric properties (see [11], [2] and the references therein). We are interested in the way that the 0 $\pi^{-1}(u) = G^+u = uG^+$ intersect the base space of a similar fibration induced by a different involution. More precisely, each $a \in G^+$ induces a C^* involution on $\mathcal{A}$, namely

$$x^{\#_a} = a^{-1}x^*a. \tag{1}$$

If $\mathcal{A}$ is represented in the Hilbert space $\mathcal{H}$, then $a \in G^+$ induces the inner product $\langle , \rangle_a$ given by

$$\langle \xi, \eta \rangle_a = \langle a\xi, \eta \rangle \ , \quad \xi, \ \eta \in \mathcal{H}.$$

It is clear that $\langle x\xi, \eta \rangle_a = \langle \xi, x_a^{\#}\eta \rangle$ for all $x \in \mathcal{A}$ and ξ, η in $\mathcal{H}$. $\mathcal{A}$ is a C*-algebra with this involution and with the norm $\| \cdot \|_a$ associated to $\langle , \rangle_a$, $\|x\|_a = \|a^{1/2}xa^{-1/2}\|$, $x \in \mathcal{A}$. For each $a \in G^+$, consider the unitary group $\mathcal{U}_a$ corresponding to the involution $\#_a$:

$$\mathcal{U}_a = \{g \in G : g^{\#_a} = g^{-1}\} = \{g \in G : a^{-1}g^*a = g^{-1}\}.$$

We shall study the restriction $\pi\Big|_{\mathcal{U}_a}$ and the way that different $\mathcal{U}_a$, $\mathcal{U}_b$ are set in G. Moreover, we shall also consider the *a-hermitian* part of G,

$$G_a^s = \{g \in G : g^{\#_a} = g\} = \{g \in G : a^{-1}g^*a = g\},$$

the *a-positive* part of G_a^s

$$G_a^+ = \{g \in G_a^s : \sigma(g) \subseteq \mathbb{R}^+\}$$

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and the intersections of these sets when a varies in G^+. The reader is referred to [7] and [5] for a discussion of operators which are hermitian for some inner product.

Observe that each $a \in G^+$ induces a fibration $\pi_a : G \to \mathcal{U}_a$ with fibers homeomorphic to G_a^+. This paper can be seen in some sense as a simultaneous study of the fibrations π_a, $a \in G^+$.

Let us mention that, from an intrinsic viewpoint, $\mathcal{U}_a$ can be identified with $\mathcal{U}$. Indeed, consider the map $\varphi_a : \mathcal{A} \to \mathcal{A}$ given by

$$\varphi_a(b) = a^{-1/2} b a^{1/2} \quad (b \in \mathcal{A}). \tag{2}$$

Then $\varphi_a(\mathcal{U}) = \mathcal{U}_a$, $\varphi_a(G^s) = G_a^s$ and $\varphi_a(G^+) = G_a^+$, since $\varphi_a : (\mathcal{A}, *) \to (\mathcal{A}, \#_a)$ is an isomorphism of C*-algebras. We are concerned with the way in which the base space and fibers of different fibrations behave with respect to each other.

3. The polar decomposition. In [10], Pedersen and Takesaki proved a technical result which was relevant for their generalization of the Sakai's Radon-Nikodym theorem for von Neumann algebras [11]. More precisely, they determined the uniqueness and existence of positive solutions of the equation $THT = K$ for H, K positive bounded operators in a Hilbert space. We need a weak version of their result, namely when H, K are positive invertible operators. In this case it is possible to give an explicit solution.

LEMMA 3.1 ([10]). *If H, K are positive invertible bounded operators in a Hilbert space, the equation*

$$THT = K \tag{3}$$

has a unique solution, namely $T = H^{-1/2}(H^{1/2}KH^{1/2})^{1/2}H^{-1/2}$.

Proof. Multiply (3) at left and right by $H^{1/2}$ and factorize

$$H^{1/2}THTH^{1/2} = (H^{1/2}TH^{1/2})^2.$$

Then we get the equation $(H^{1/2}TH^{1/2})^2 = H^{1/2}KH^{1/2}$. Taking (positive) square roots and using the invertibility of $H^{1/2}$ we get the result. □

Returning to the map $\pi : G \to \mathcal{U}$, consider the fiber $\pi^{-1}(u) = \{\lambda u : \lambda \in G^+\}$. In order to compare the fibration π with π_a, the following is the key result

THEOREM 3.2. *Let $a \in G^+$. Then, for every $u \in \mathcal{U}$ the fiber $\pi^{-1}(u)$ intersects $\mathcal{U}_a$ at a single point, namely $a^{-1/2}(a^{1/2}uau^{-1}a^{1/2})^{1/2}a^{-1/2} \cdot u$. In other words, the restriction $\pi|_{\mathcal{U}_a} : \mathcal{U}_a \to \mathcal{U}$ is a homeomorphism.*

Proof. If $g = \lambda u \in \mathcal{U}_a$ then $a^{-1}g^*a = g^{-1}$ is equivalent to $a^{-1}u^{-1}\lambda a = u^{-1}\lambda^{-1}$, so, after a few manipulations,

$$\lambda a \lambda = uau^{-1}. \tag{4}$$

By Pedersen and Takesaki's result, there is a unique $\lambda \in G^+$ which satisfies equation (4) for fixed $a \in G^+$, $u \in \mathcal{U}$, namely $\lambda = a^{-1/2}(a^{1/2}uau^{-1}a^{1/2})^{1/2}a^{-1/2}$. Thus, $(\pi|_{\mathcal{U}_a})^{-1} : \mathcal{U} \to \mathcal{U}_a$ is given by

$$(\pi|_{\mathcal{U}_a})^{-1}(u) = a^{-1/2}(a^{1/2}uau^{-1}a^{1/2})^{1/2}a^{-1/2} \cdot u \tag{5}$$

which obviously is a continuous map. □

Let $a \in G^+$ and consider the involution $\#_a$ defined in equation (1). It is natural to look at those reflections $\varepsilon \in Q$ which are $\#_a$-orthogonal, i.e the so called $\#_a$-Grassmann manifold of $\mathcal{A}$. Let us denote this space by

$$P_a = \{\varepsilon \in Q : \varepsilon = \varepsilon^{\#_a} = \varepsilon^{-1}\} = Q \cap \mathcal{U}_a = Q \cap G_a^s.$$

In [8], Pasternak-Winiarski studied the behavior of the orthogonal projection onto a closed subspace of a Hilbert space when the inner product varies continuously. Note that we can identify naturally the space of idempotents q with the reflections of Q via the affine map $q \mapsto \varepsilon = 2q - 1$, which also maps the space of orthogonal projections onto P. Based on [8], a geometric study of the space Q is made in [1], where the characterization $\pi(P_a) = P$ is given (proposition 5.1 of [1]). In the following proposition we shall give a new proof of this fact by showing that the homeomorphism $\pi|_{\mathcal{U}_a} : \mathcal{U}_a \to \mathcal{U}$ maps $P_a \subseteq \mathcal{U}_a$ onto $P \subseteq \mathcal{U}$. Therefore the formula given in equation (5) for the inverse of $\pi|_{\mathcal{U}_a}$ extends the formula given in proposition 5.1 of [1] for $(\pi|_{P_a})^{-1}$, since they must coincide on P.

PROPOSITION 3.3. *Let $a \in G^+$. Then $\pi(P_a) = \pi(Q \cap \mathcal{U}_a) = P$. Therefore $\pi|_{P_a} : P_a \to P$ is a homeomorphism.*

Proof. By the previous remarks, we just need to show that $\pi(P_a) = P$. Observe that if $\varepsilon \in Q$ then $\rho = \pi(\varepsilon) \in P$: in fact, if $\varepsilon = \lambda\rho$ then $\varepsilon = \varepsilon^{-1} = \rho^{-1}\lambda^{-1}$; but, since the unitary part of ε corresponding to both right and left polar decompositions coincide, we get $\rho^{-1} = \rho$. Then $\rho^* = \rho^{-1} = \rho$ and $\rho \in P$. Thus, $\pi(\mathcal{U}_a \cap Q) \subseteq P$.

Let $\alpha = (\pi|_{\mathcal{U}_a})^{-1} : \mathcal{U} \to \mathcal{U}_a$. Then by (5)

$$\alpha(u) = a^{-1/2}(a^{1/2}uau^{-1}a^{1/2})^{1/2}a^{-1/2} \cdot u.$$

In order to prove the result we need to show that if $\rho \in P$ then $\alpha(\rho) \in P_a = Q \cap \mathcal{U}_a$, i.e. $\alpha(\rho) \in Q$. Indeed,

$$\begin{aligned} \alpha(\rho)^2 &= a^{-1/2}(a^{1/2}\rho a\rho a^{1/2})^{1/2}a^{-1/2}\rho a^{-1/2}(a^{1/2}\rho a\rho a^{1/2})^{1/2}a^{-1/2}\rho \\ &= a^{-1/2}((a^{1/2}\rho a^{1/2})^2)^{1/2}(a^{1/2}\rho a^{1/2})^{-1}((a^{1/2}\rho a^{1/2})^2)^{1/2}a^{-1/2}\rho. \end{aligned}$$

Thus, applying the continuous functional calculus (see e.g. [9]) to the selfadjoint element $a^{1/2}\rho a^{1/2}$, if $f(t) = |t| = (t^2)^{1/2}$ and $g(t) = \frac{1}{t}$, $t \in \mathbb{R} \setminus \{0\}$,

$$\begin{aligned} (\lambda\rho)^2 &= a^{-1/2}f(a^{1/2}\rho a^{1/2})g(a^{1/2}\rho a^{1/2})f(a^{1/2}\rho a^{1/2})a^{-1/2}\rho \\ &= a^{-1/2}[f(a^{1/2}\rho a^{1/2})]^2 g(a^{1/2}\rho a^{1/2})a^{-1/2}\rho \\ &= a^{-1/2}(a^{1/2}\rho a^{1/2})a^{-1/2}\rho \;=\; 1. \quad \square \end{aligned}$$

3.1. Positive parts. In order to complete the results on the relationship between polar decomposition and inner products, consider the complementary map of the decomposition $g = \lambda u$, namely

$$\pi^+ : G \to G^+ \ , \quad \pi^+(g) = (gg^*)^{1/2} \ , \quad (g \in G).$$

Of course, there is another "complementary map", namely $g \mapsto (g^*g)^{1/2}$ corresponding to the decomposition $g = u\lambda'$. We shall see that for every $a \in G^+$, the restriction

$$\pi^+|_{G_a^+} : G_a^+ \to G^+$$

is a homeomorphism. Indeed, given $\mu \in G^+$, consider the polar decomposition $a\mu = \lambda u$, with $\lambda \in G^+$ and $u \in \mathcal{U}$. Then $a^{-1}\lambda = \mu u^*$, so $\pi^+(a^{-1}\lambda) = \mu$ and $a^{-1}\lambda \in G_a^+$, since $a^{-1}(a^{-1}\lambda)^* a = a^{-1}\lambda$ and the spectrum $\sigma(a^{-1}\lambda) \subseteq \mathbb{R}^+$. Note that

$$(\pi^+|_{G_a^+})^{-1}(\mu) = a^{-1}\pi^+(a\mu),$$

which is clearly a continuous map. An interesting rewriting of the above statement is the following:

PROPOSITION 3.4. *If $\mathcal{A}$ is a unital C^*-algebra and $a, \lambda \in G^+$, then there exists a unique $u \in \mathcal{U}$ such that $a\lambda u \in G^+$.*

Proof. Indeed, if $g = (\pi^+|_{G_a^+})^{-1}(\lambda) \in G_a^+$ and $u = \pi(g)$, then $\lambda u = g \in G_a^+$ means exactly that $a\lambda u \in G^+$. $\square$

It is worth mentioning that $x \in G$ is the unique positive solution of Pedersen-Takesaki equation $xax = b$ if and only if $a^{1/2}xb^{-1/2} \in \mathcal{U}$. Changing a, b by a^2, b^{-2} respectively, we can write Lemma 3.1 as follows:

PROPOSITION 3.5. *If $\mathcal{A}$ is a unital C^*-algebra and $a, b \in G^+$, then there exists a unique $x \in G^+$ such that $axb \in \mathcal{U}$.*

3.2. Products of positive operators. The map $\Theta : G^+ \times G^+ \to \mathcal{U}$ given by

$$\Theta(a, b) = axb = a(a^{-1}(ab^2a)^{1/2}a^{-1})b = (ab^2a)^{1/2}a^{-1}b, \quad a, b \in G^+,$$

is not surjective: in fact, the image of Θ consists of those unitary elements which can be factorized as a product of three positive elements. On one side $\Theta(a, b) = axb \in \mathcal{U}$ is the product of three elements of G^+. On the other side, if $axb \in \mathcal{U}$ then by Pedersen-Takesaki's result x is the unique positive solution of $xa^2x = b^{-2}$.

It is easy to show that $-1 \in \mathcal{U}$ can not be decomposed as a product of four positive elements. See [12] and [13] for a complete bibliography on these factorization problems. See [3] for more results on factorization of elements of G and characterizations of $P_n = \{a_1 \dots a_n : a_i \in G^+\}$, at least in the finite dimensional case.

3.3. Parametrization of the solutions of Pedersen-Takesaki equations. Given $a, b \in G^+$, denote by $m = |b^{1/2}a^{1/2}| = (a^{1/2}ba^{1/2})^{1/2}$. Then the set of all solutions of the equation $xax = b$ is

$$\{a^{-1/2}\ m\ \varepsilon\ a^{-1/2}\ :\ \varepsilon \in Q \quad \text{and} \quad \varepsilon m = m\varepsilon\}.$$

In fact, $xax = b$ if and only if $(a^{1/2}xa^{1/2})^2 = m^2$ and the set of all solutions of $x^2 = c^2$ for $c \in G^+$ is $\{c\varepsilon\ :\ \varepsilon^2 = 1 \quad \text{and} \quad \varepsilon c = c\varepsilon\}$. The singular case, which is much more interesting, deserves a particular study that we intend to do in a forthcoming paper.

4. Intersections and unions. For any selfadjoint $c \in \mathcal{A}$ we shall consider the relative commutant subC*-algebra

$$\mathcal{A}_c = \mathcal{A} \cap \{c\}' = \{d \in \mathcal{A} : dc = cd\}$$

and denote by $\mathcal{U}(\mathcal{A}_c) = \mathcal{A}_c \cap \mathcal{U}$, the unitary group of $\mathcal{A}_c$ and, analogously $G^s(\mathcal{A}_c)$, $G^+(\mathcal{A}_c)$, $Q(\mathcal{A}_c)$ and $P(\mathcal{A}_c)$.

The space G^s has a deep relationship with Q (in [4] there is a partial description of it). Here we only need to notice that the unitary part of any $c \in G^s$ also belongs to P. Indeed, if $\lambda\rho$ is the polar decomposition of c, then $\lambda\rho = c = c^* = \rho^*\lambda$. By the uniqueness of the unitary part, $\rho = \rho^* = \rho^{-1} \in P$. Observe also that $\rho\lambda = \lambda\rho$. Moreover, since $\lambda = |c| = (c^2)^{-1}$, then $\rho = f(c)$ where $f(t) = t\ |t|$. So $\rho c = c\rho$.

THEOREM 4.1. *Let $\mathcal{A}$ a unital C*-algebra and $a \in G^+$. Then*

$$\mathcal{U}_a \cap G^s = P(\mathcal{A}_a) = \{u \in P : au = ua\}.$$

Proof. By the previous remarks, if $b \in G^s \cap \mathcal{U}_a$ and $b = \lambda\rho$ is its polar decomposition, then $\rho \in P$ and $\rho\lambda^{-1} = a^{-1}\rho\lambda a$. Using that $\rho\lambda = \lambda\rho$ we get easily

$$\lambda^{-1}a\lambda^{-1} = \rho a\rho = \lambda a\lambda.$$

By the uniqueness of the positive solution, $\lambda = \lambda^{-1}$ and, since $\lambda \in G^+$, this means that $\lambda = 1$. Thus $a = \rho a\rho$ and then $\rho \in P(\mathcal{A}_a)$. Conversely, if $\rho \in P(\mathcal{A}_a)$, then $\rho \in \mathcal{U}_a$, since $a^{-1}\rho^* a = a^{-1}\rho a = \rho = \rho^{-1}$. □

REMARK 4.2. Let $a \in G^+$. Then easy computations show that
1. $\mathcal{U}_a \cap \mathcal{U} = \mathcal{U} \cap \mathcal{A}_a = \mathcal{U}(\mathcal{A}_a)$.
2. $G^s_a \cap G^s = G^s \cap \mathcal{A}_a = G^s(\mathcal{A}_a)$.
3. $G^+_a \cap G^+ = G^+ \cap \mathcal{A}_a = G^+(\mathcal{A}_a)$.
4. $\mathcal{U}_a \cap G^+ = \{1\}$.

We shall give two proofs of item 4:
First proof: $\pi(G^+) = \{1\}$ but π restricted to $\mathcal{U}_a$ is one to one.
Second proof: if $x \in \mathcal{U}_a \cap G^+$, then its spectrum $\sigma(x) \subseteq S^1 \cap \mathbb{R}^+ = \{1\}$; on the other side, x is normal with respect to the involutions $\#_a$, so x is a normal element such that $\sigma(x) = \{1\}$ and it must be $x = 1$.

Let $b \in G^+$. Recall that the map φ_b defined in (2) changes the usual involution by $\#_b$ and also all the corresponding spaces (e.g $\varphi_b(G^s) = G^s_b$).

LEMMA 4.3. *Let $b \in G^+$. Then for any $c \in G^+$, if $d = b^{-1/2}cb^{-1/2}$,*

$$\varphi_b(\mathcal{U}_d) = \mathcal{U}_c, \quad \varphi_b(G^s_d) = G^s_c, \quad \textit{and} \quad \varphi_b(G^+_d) = G^+_c$$

Proof. Notice that $\mathcal{U}_d = \varphi_d(\mathcal{U})$, so $\varphi_b(\mathcal{U}_d) = \varphi_b \circ \varphi_d(\mathcal{U})$. But $\varphi_b \circ \varphi_d = \varphi_c \circ Ad(u^*)$ where $u = \pi(d^{1/2}b^{1/2})$, since $c = |d^{1/2}b^{1/2}|^2$ and $d^{1/2}b^{1/2} = uc^{1/2}$. As $Ad_{u^*}(\mathcal{U}) = \mathcal{U}$ (and the same happens for G^s and G^+), we get $\varphi_b(\mathcal{U}_d) = \mathcal{U}_c$ and the other two identities. □

Then we can generalize the results above for any pair $a, b \in G^+$ instead of a and 1:

COROLLARY 4.4. *Let $a, b \in G^+$ and $c = b^{-1/2}ab^{-1/2}$. Then*

1. $\mathcal{U}_a \cap G^s_b = \varphi_b(\mathcal{U}_c \cap G^s) = b^{-1/2}(P(\mathcal{A}_c))b^{1/2}$.
2. $\mathcal{U}_a \cap \mathcal{U}_b = \varphi_b(\mathcal{U}_c \cap \mathcal{U}) = b^{-1/2}\mathcal{U}(\mathcal{A}_c)b^{1/2}$.
3. $G^s_a \cap G^s_b = \varphi_b(G^s_c \cap G^s) = b^{-1/2}G^s(\mathcal{A}_c)b^{1/2}$.
4. $G^+_a \cap G^+_b = \varphi_b(G^+_c \cap G^+) = b^{-1/2}G^+(\mathcal{A}_a)b^{1/2}$.
5. $\mathcal{U}_a \cap G^+_b = \varphi_b(\mathcal{U}_c \cap G^+) = \{1\}$.

Proof. Use Proposition 4.1, Remark 4.2 and Lemma 4.3. □

In the following proposition we describe the set of elements of $\mathcal{A}$ which are unitary (resp. positive, Hermitian) for some involution $\#_a$ ($a \in G^+$). We state the result without proof.

PROPOSITION 4.5. *If $\mathcal{A}$ is a unital C^*-algebra, the following identities hold:*

$$\bigcup_{a\in G^+} \mathcal{U}_a = \bigcup_{g\in G} g\,\mathcal{U}g^{-1} = \bigcup_{a\in G^+} a\,\mathcal{U}a^{-1},$$

$$\bigcup_{a\in G^+} G^+_a = \bigcup_{g\in G} g\;G^+g^{-1} = \bigcup_{a\in G^+} a\;G^+a^{-1} = G^+G^+,$$

where $G^+G^+ = \{ab : a,b \in G^+\}$ and

$$\bigcup_{a\in G^+} G^s_a = \bigcup_{g\in G} g\;G^s g^{-1} = \bigcup_{a\in G^+} a\;G^s a^{-1} = G^sG^+ = G^+G^s.$$

The following example shows that there is no obvious spectral characterization of these subsets of G: if x is nilpotent, then $1 + x$ does not belong to any of them but $\sigma(1 + x) = \{1\} \subseteq \mathbb{R}, \mathbb{R}^+$ and the circle S^1.

4.1. Final geometric remarks. All subsets of $\mathcal{A}$ studied in this paper have a rich structure as differential manifolds. The reader is referred to [6] and [2] for the case of $\mathcal{U}$ and to [4] (and the references therein) for Q, P, G^s and G^+. The map φ_a defined in equation (2) is clearly a diffeomorphism which allows to get all the information on $\mathcal{U}_a, G^s_a, G^+_a$ from that available on $\mathcal{U}, G^s, G^+$, respectively. The main results of the paper say that the map π is a diffeomorphism between $\mathcal{U}_a$ and $\mathcal{U}$, P_a and P and so on.

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