The Electronic Journal of Linear Algebra.
A publication of the International Linear Algebra Society.
Volume 7, pp. 30-40, March 2000.
ISSN 1081-3810. http://math.technion.ac.il/iic/ela

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DIGRAPHS WITH LARGE EXPONENT*

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Abstract. Primitive digraphs on n vertices with exponents at least $\lfloor \omega_n/2 \rfloor + 2$, where $\omega_n = (n-1)^2+1$, are considered. For $n \geq 3$, all such digraphs containing a Hamilton cycle are characterized; and for $n \geq 6$, all such digraphs containing a cycle of length $n-1$ are characterized. Each eigenvalue of any stochastic matrix having a digraph in one of these two classes is proved to be geometrically simple.

Key words. primitive directed graph, exponent.

AMS subject classifications. 15A48, 05C20

1. Introduction. A directed graph (digraph) D is *primitive* if for some positive integer m there is a (directed) walk of length m between any two vertices u and v (including $u = v$). The minimum such m is the *exponent* of D, denoted by $exp(D)$. It is well known that D is primitive iff it is strongly connected and the *gcd* of its cycle lengths is 1. A nonnegative matrix A is primitive if A^m is entrywise positive for some positive integer m. If $D = D(A)$, the digraph of a primitive matrix A, then $exp(D) = exp(A)$, which is the minimum m such that A^m is entrywise positive.

Denoting $(n-1)^2+1$ by ω_n, the best upper bound for $exp(D)$ when a primitive digraph D has $n \geq 2$ vertices is given by $exp(D) \leq \omega_n$, with equality holding iff $D = D(W_n)$ where W_n is a Wielandt matrix; see, e.g., [2, Theorem 3.5.6]. When $n = 2$, then $D(W_2)$, consisting of a 1 cycle and a 2 cycle, has exponent equal to 2. Henceforth we assume that $n \geq 3$. The digraph $D(W_n)$ consists of a Hamilton cycle (i.e., a cycle of length n) and one more arc, between a pair of vertices that are distance two apart on the Hamilton cycle, giving a cycle of length $n-1$.

The following result of Lewin and Vitek [6, Theorem 3.1], see also [2, Theorem 3.5.8], is the basis for our discussion of digraphs with large exponent.

THEOREM 1.1. *If D has $n \geq 3$ vertices and is primitive with sufficiently large exponent, namely*

$$exp(D) \geq \lfloor \omega_n/2 \rfloor + 2, \textit{with } \omega_n = (n-1)^2+1, \tag{1}$$

then D has cycles of exactly two different lengths j, k with $n \geq k > j$.

We say that a primitive digraph D on n vertices satisfying (1) has a *large exponent.* Note that in Theorem 1.1, $gcd(j,k) = 1$ since D is primitive. If $gcd(j,k) = 1$, then

*Received by the editors on 26 January 2000. Accepted for publication on 21 February 2000. Handling editor: Richard A. Brualdi.

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every integer greater than or equal to $(j-1)(k-1)$ can be written as c_1j+c_2k, where c_i are nonnegative integers. The value $(j-1)(k-1)$ is the smallest such integer, and is called the Frobenius-Schur index for the two relatively prime integers j and k; see, e.g., [2, Lemma 3.5.5].

The Frobenius-Schur index is used to prove the following result that gives a necessary and sufficient condition for the existence of a primitive digraph with large exponent and cycles of two specified lengths.

THEOREM 1.2. *Let k and j be such that $gcd(j,k)=1$ and $n\geq k>j$. There exists a primitive digraph D on n vertices having only cycle lengths k and j and $exp(D)\geq\lfloor\omega_n/2\rfloor+2$ iff $j(k-2)\geq\lfloor\omega_n/2\rfloor+2-n$.*

Proof. Suppose that D is a digraph with large exponent and cycle lengths k and $j<k\leq n$. We claim that for any pair of vertices u and v, there is a walk from u to v of length at most $k+n-j-1\geq n$ that goes through a vertex on a k cycle and a vertex on a j cycle. To prove this claim, note that from the proof of Theorem 1 in [4], there are no pairs of vertex disjoint cycles in D; that is, any pair of cycles share at least one common vertex. If there is a walk from u to v of length less than or equal to n that passes through at least one vertex on a k cycle and at least one vertex on a j cycle, then the claim is proved.

So suppose that this is not the case. In particular, assume that u and v are only on k (resp. j) cycles, and any path from u to v passes only through vertices not on any j (resp. k) cycle. Consider the first case. Let l be the number of vertices *not* on a j cycle, and note that $2\leq l\leq n-j$. Since a shortest path from u to v goes only through vertices not on a j cycle, the length p of such a path satisfies $p\leq l-1$. Consider the walk from u to v formed by first traversing a k cycle at u (necessarily going through a vertex on a j cycle), then taking the path of length p from u to v. This generates a walk from u to v that goes through a vertex on a k cycle and one on a j cycle, and its length is $k+p\leq k+l-1\leq k+n-j-1$. The second case follows by interchanging k and j and noting that $j+n-k-1<k+n-j-1$. Thus the claim is proved. By the Frobenius-Schur index, there is a walk from u to v of length $k+n-j-1+(k-1)(j-1)=n+j(k-2)$ for any pair u,v. Thus $n+j(k-2)\geq exp(D)\geq\lfloor\omega_n/2\rfloor+2$, giving the condition on k and j.

For the converse, assume the condition on k and j, and consider the digraph D consisting of the k cycle $1\to k\to k-1\to\cdots\to k+j-n+1\to k+j-n\to k+j-n-1\to\cdots\to 2\to 1$, and arcs $1\to k+1\to k+2\to\cdots\to n-1\to n\to k+j-n$. Thus D has exactly one k cycle and one j cycle. Consider the length of a walk from k to $k+j-n+1$. Such a walk has length $n-j-1$ or $k+n-j-1+c_1k+c_2j$ for some nonnegative integers c_i, and (from the Frobenius-Schur index) there is no walk of length $k+n-j-1+(k-1)(j-1)-1$. Thus

$$exp(D)\geq k+n-j-1+(k-1)(j-1)=n+j(k-2)\geq\lfloor\omega_n/2\rfloor+2.\quad\square$$

Note that for D primitive with only cycles of lengths k and j with $j<k\leq n$, the bound on $exp(D)$ found in the above proof, namely $exp(D)\leq n+j(k-2)$, improves the bound in [4, Lemma 1] and includes the converse. Furthermore, Theorem 1.2 does not include additional assumptions as in [6, Theorem 4.1].

We assume that D has a large exponent and focus on the graph theoretic aspects of this condition. In Section 2, we characterize the case when D has a Hamilton cycle ($k = n \geq 3$); and in Section 3, we characterize the case $k = n - 1$. Our characterizations give some information on the case for general $k \leq n$ when $n \geq 4$, since a result of Beasley and Kirkland [1, Theorem 1] implies that any induced subdigraph on k vertices that is primitive also has large exponent (relative to $\lfloor \omega_k/2 \rfloor + 2$), so the structure of some such induced subdigraphs is known from our results. It is known from results in [6] exactly which numbers $\geq \lfloor \omega_n/2 \rfloor + 2$ are attainable as exponents of primitive digraphs. (Note that there are some gaps in this exponent set.) Our work in Sections 2 and 3 focuses on describing the corresponding digraphs when $k \geq n - 1$.

Some algebraic consequences of the large exponent condition (1) for a stochastic matrix A with $D(A) = D$ have been investigated in [4] and [5]. The characteristic polynomial of A has a simple form (see [4, Theorem 1]), and, if n is sufficiently large, then about half of the eigenvalues of A have modulus close to 1. Kirkland and Neumann [5] considered the magnitudes of the entries in the group generalized inverse of $I - A$ (which measures stability of the left Perron vector of A under perturbations). In Section 4 we use results of Sections 2 and 3 to investigate the multiplicities of eigenvalues of stochastic matrices with large exponents.

2. The Hamiltonian Case.

Assuming that D has large exponent and a Hamilton cycle, we begin by finding possible lengths for other cycles in D.

LEMMA 2.1. *Suppose that D is a primitive digraph on $n \geq 3$ vertices with $exp(D) \geq \lfloor \omega_n/2 \rfloor + 2$ and that D has a Hamilton cycle. Then D has precisely one Hamilton cycle, and all other cycles have length j, where $n > j \geq \lceil (n-1)/2 \rceil$.*

Proof. By Theorem 1.1, D contains cycles of exactly two lengths, $n = k > j$. W.l.o.g. take the given Hamilton cycle as $1 \to n \to n-1 \to \cdots \to 2 \to 1$, and assume that the arc $1 \to j$ lies on a second Hamilton cycle. Note that the only possible arcs from any vertex i are $i \to i-1 \pmod n$ and $i \to i+j-1 \pmod n$. Since the arc $j+1 \to j$ is not on the second Hamilton cycle, this cycle must include the arc $j+1 \to (j+1)+j-1 = 2j \pmod n$. Similarly, there is an arc on the second Hamilton cycle from $(m-1)j+1$ to $mj \pmod n$, for $m = 1, \ldots, n$. As $gcd(j, n) = 1$, D contains the digraph of a primitive circulant. By [3, Theorem 2.1], $exp(D) \leq (n-1)$ or $exp(D) \leq \lfloor n/2 \rfloor$, thus $exp(D) < \lfloor \omega_n/2 \rfloor + 2$. Hence, there is no second Hamilton cycle in D. For the lower bound on j, take $k = n$ in Theorem 1.2; see also [4, Theorem 1]. □

If D has large exponent and $k = n = 3$, then Lemma 2.1 implies that $j \in \{1, 2\}$. For $j = 1$, D consisting of a 3 cycle and a 1 cycle has exponent equal to $4 = \lfloor \omega_3/2 \rfloor + 2$. For $j = 2 = n-1$, either $D = D(W_3)$ with exponent equal to $5 = \omega_3$, or D consists of a 3 cycle with two 2 cycles and has exponent equal to 4. This last case is an example of the result that a digraph D on n vertices has $exp(D) = (n-1)^2$ iff D is isomorphic to an n cycle with two additional arcs from consecutive vertices forming two $n-1$ cycles; see, e.g., [2, pp. 82–83].

These observations motivate our next two theorems, which describe the Hamiltonian digraphs with large exponent. Most cases are covered in Theorem 2.2, but, if n

is odd, then the case $j = (n-1)/2$ is slightly different and is given in Theorem 2.3.

THEOREM 2.2. *Suppose that $j \geq n/2$. Then D is a primitive digraph on $n \geq 3$ vertices with $exp(D) \geq \lfloor \omega_n/2 \rfloor + 2$ and cycle lengths n and j iff D is isomorphic to a (primitive) subdigraph of the digraph formed by taking the cycle $1 \to n \to n-1 \to \cdots \to 2 \to 1$, and adding in the arcs $i \to i+j-1$ for $1 \leq i \leq n-j+1$.*

Proof. Assume that D is primitive with large exponent and has a Hamilton cycle. Then by Lemma 2.1, D has only one Hamilton cycle and other cycles of length j, which by assumption is at least $n/2$. W.l.o.g. assume that the Hamilton cycle is $1 \to n \to n-1 \to \cdots \to 2 \to 1$, and that D contains the arc $1 \to j$. Since D has cycles of just two different lengths, each vertex i of D has outdegree ≤ 2, and if the outdegree is 2, then the outarcs from vertex i are $i \to i-1$ and $i \to i+j-1$. Here and throughout the proof, all indices are mod n. As $1 \to j$, the outdegree of vertex i is 1 for each $i \in \{n-j+2, \ldots, j\}$, since otherwise $1 \to j \to j-1 \to \cdots \to i \to i+j-1-n \to i+j-2-n \to \cdots \to 2 \to 1$ is a cycle of length less than j. Consequently if the outdegree of vertex $i \in \{2, \ldots, j\}$ is 2, then in fact $i \in \{2, \ldots, n-j+1\}$. If there is no such i, then D has the desired structure, since D has at most $n-j+1$ consecutive vertices on the Hamilton cycle (namely 1 and $j+1, \ldots, n$) of outdegree 2. Henceforth suppose that there exists $i \in \{2, \ldots, n-j+1\}$ with outdegree 2, and let i_1 be the maximum such i; thus $i_1 \to i_1+j-1 \in \{j+1, \ldots, n\}$. As before, the outdegree is 1 for each vertex $\in \{n-j+i_1+1, \ldots, j+i_1-1\}$. In particular, if $n-j+i_1+1 \leq j+1$, then the only vertices that can have outdegree 2 are $1, \ldots, i_1$ and $j+i_1, \ldots, n$, that is $n-j+1$ consecutive vertices, as desired. So suppose henceforth that $n-j+i_1 > j$, that is $i_1 > 2j-n \geq 0$. Suppose also that there exists i_2 such that $n-j+i_1 \geq i_2 \geq j+1$ with i_2 having outdegree 2. Then $i_2 \to i_2+j-1$. Now $n+i_1-1 \geq i_2+j-1 \geq 2j$, so that $i_2+j-1 \pmod n = i_2+j-1-n \in \{2j-n, \ldots, i_1-1\}$. But then there is a cycle $i_1 \to i_1+j-1 \to i_1+j-2 \to \cdots \to i_2 \to i_2+j-1-n \to i_2+j-2-n \to \cdots \to 2 \to 1 \to j \to j-1 \to \cdots \to i_1+1 \to i_1$, which has length $3j-n$. As there is only one Hamilton cycle (Lemma 2.1), this implies that $3j-n = j$, giving a contradiction, since $gcd(n,j) = 1$. Thus again each of vertices $i_1+1, \ldots, j+i_1-1$ has outdegree 1, and so at most $n-j+1$ consecutive vertices have outdegree 2, as desired.

For the converse, consider the maximal such digraph D with the above Hamilton cycle and the $n-j+1$ additional arcs. Note that each of the vertices $n-j+2, \ldots, n$ has outdegree 1, and each of the vertices $1, 2, \ldots, j-1$ has indegree 1, so the only path from n to 1 is $n \to n-1 \to \cdots \to 1$ with length $n-1$. By Frobenius-Schur, it follows that there is no walk from n to 1 of length $n-1+(n-1)(j-1)-1$; hence $exp(D) \geq j(n-1)$. Since $gcd(n,j) = 1$, it follows that $j = n/2$ is inadmissible. Thus $j \geq n/2$ implies that $j \geq (n+1)/2$, and so $j(n-1) \geq (n^2-1)/2 \geq \lfloor \omega_n/2 \rfloor + 2$. Since D is maximal, any primitive subdigraph has exponent at least as large as $exp(D)$. □

THEOREM 2.3. *Suppose that $n \geq 3$ is odd and $j = (n-1)/2$. Then D is a primitive digraph on n vertices with $exp(D) \geq \lfloor \omega_n/2 \rfloor + 2$ and cycle lengths n and j iff D is isomorphic to a (primitive) subdigraph of the digraph formed by taking the cycle $1 \to n \to n-1 \to \cdots \to 2 \to 1$, and adding in the arcs $i \to i+j-1$ for $1 \leq i \leq (n-1)/2 = j$.*

Proof. First assume that $exp(D) \geq \lfloor \omega_n/2 \rfloor + 2 = (n-1)^2/2 + 2$. Observe that if vertex i is on a j cycle, then (by Frobenius-Schur) there is a walk of length

$\leq (n-1)+(n-1)(j-1) = j(n-1) = (n-1)^2/2$ from i to each vertex of D. It follows that there must be a vertex with distance 2 to the nearest j cycle. W.l.o.g. that vertex is n, with vertex $n-2$ on a j cycle. In fact that j cycle is $n-2 \rightarrow n-3 \rightarrow \cdots \rightarrow (n-1)/2 = j \rightarrow n-2$, otherwise $n-1$ or n is on a j cycle. None of the vertices $j+1, j+2, \cdots, n$ can have outdegree 2 (otherwise one of $n-1$ or n is on a j cycle). However, the $j-1$ additional arcs $i \rightarrow i+j-1$ for $i = 1, 2, \ldots, j-1$ may be included in D. Thus it follows that D is a subdigraph of the digraph that has the $n-1$ cycle and the additional j arcs as in the theorem statement.

For the converse, note that if D is isomorphic to a subdigraph of the specified digraph, then a walk from n to $n-1$ of length greater than 1 must traverse the entire Hamilton cycle, so walks from n to $n-1$ have length 1 or $n+1+c_1n+c_2j$ where c_1 and c_2 are nonnegative integers. Thus (by Frobenius-Schur) there is no walk from n to $n-1$ of length $n+1+(n-1)(n-3)/2-1 = (n-1)^2/2+1$, so that $exp(D) \geq (n-1)^2/2+2$, as desired. □

Using the structures of Hamiltonian digraphs D with large exponents given in Theorems 2.2 and 2.3, we determine the exact value of $exp(D)$ in terms of a parameter a that depends on which j cycles occur in D.

COROLLARY 2.4. *Suppose that D is a primitive digraph on $n \geq 3$ vertices with $exp(D) \geq \lfloor \omega_n/2 \rfloor + 2$, a Hamilton cycle and all other cycles of length j, where $n > j \geq \lceil (n-1)/2 \rceil$. Suppose that the Hamilton cycle is $1 \rightarrow n \rightarrow n-1 \rightarrow \cdots \rightarrow 2 \rightarrow 1$. Let $1 \leq a \leq n-j+1$ if $j \geq n/2$, and $1 \leq a \leq j$ if $j = (n-1)/2$. Suppose that D also contains the arc(s) $1 \rightarrow j$ and $a \rightarrow a+j-1$, and that if i is a vertex of outdegree 2, then $1 \leq i \leq a$. Then $exp(D) = n-a+1+(n-2)j$.*

Proof. The shortest walk from n to $a+j$ that passes through a vertex on a j cycle has length $n-a-j+n$, so it follows (by Frobenius Schur) that there is no walk from n to $a+j$ of length $n-a-j+n+(n-1)(j-1)-1$. Thus $exp(D) \geq n-a+1+(n-2)j$. Further, since there is a walk between any two vertices of length at most $n-a-j+n$ that goes through a vertex on a j cycle, it follows that $exp(D) \leq n-a+1+(n-2)j$, and thus $exp(D) = n-a+1+(n-2)j$. □

If $j \geq n/2$, note that $exp(D) = n-a+1+(n-2)j \geq j(n-1)$ for $1 \leq a \leq n-j+1$, giving the result of [6, Corollary 3.1] when $k = n$ without the additional assumption. Also note that if $j = n-1$ and $a = 1$, then $exp(D)$ achieves its maximum value of ω_n, and $D = D(W_n)$, as described in Section 1. It is interesting to note that in the above corollary, it is only the value of a that influences the value of the exponent; if $2 \leq i \leq a-1$, the presence or absence of the arc $i \rightarrow i+j-1$ does not affect the exponent. For fixed n and j, this result gives a range of values of $exp(D)$ in which there are no gaps; see [6].

3. The Case $k = n-1$. If D on n vertices has large exponent with cycle lengths $n-1$ and $j < n-1$, then Theorem 1.2 shows that $j \geq \lceil n/2 \rceil$ provided that $n \geq 5$. (There are no such digraphs for $n \leq 4$.) Our next two theorems characterize these digraphs for $n \geq 6$. As in the Hamiltonian case, most digraphs are covered by the first result (Theorem 3.3), but the case $j = n/2$ (when n is even) is different, and is given by the second result (Theorem 3.4). Before proving our main results, we give a definition and a preliminary Lemma. Note that since there is a cycle of length $n-1$, indices are taken mod $(n-1)$. Vertex n *replicates vertex* $v \in \{1,\ldots,n-1\}$ in a digraph D on n vertices if for all $a, b \in \{1,\ldots,n-1\}, a \to n$ iff $a \to v$ and $n \to b$ iff $v \to b$. Thus in the adjacency matrix A with $D = D(A)$, the rows (and columns) corresponding to vertices n and v are the same.

LEMMA 3.1. *Let D be a strongly connected digraph on $n \geq 5$ vertices, with cycle lengths $n-1$ and j, where $n-1 > j \geq 3$. Suppose that $1 \to n-1 \to \cdots \to 2 \to 1$ is an $n-1$ cycle, and that $c \to n$. Then n has outdegree at most 2, with either $n \to c-2$ or $n \to c+j-2$ or both. Furthermore, if the outdegree of n is 2, then the indegree of n is 1.*

Proof. First suppose that there is an arc $n \to a$. Then there is a cycle $n \to a \to a-1 \to \cdots \to c \to n$ of length $a-c+2$ if $a > c$, or length $n+1+a-c$ if $c > a$. In the former case, $a-c+2 = j$ or $n-1$, from which it follows that $a = c+j-2$ or $c-2$; in the latter case similarly $a = c+j-2$ or $c-2$. This establishes the possible outarcs from n. Finally, assume that $n \to c-2$ and $n \to c+j-2$. Suppose that $d \to n$ for some $d \neq c$. As above the two outarcs from n can be written as $d-2$ and $d+j-2$. As $d \neq c$, it follows that $d-2 = c+j-2$ and $c-2 = d+j-2$. Hence $d-c = j$ and $c-d = j$, giving a contradiction. Thus the indegree of n is 1. □

COROLLARY 3.2. *Let D be as in Lemma 3.1. If $n \to c$, then either $c+2 \to n$ or $c+2-j \to n$ or both. Furthermore, if the indegree of n is 2, then the outdegree of n is 1.*

Proof. Form D' by reversing the orientation of each arc in D. Then Lemma 3.1 applies to D', and the result follows. □

THEOREM 3.3. *Suppose that $n \geq 6$ and $n-1 > j > n/2$. Then D is a primitive digraph on n vertices with $exp(D) \geq \lfloor \omega_n/2 \rfloor + 2$ and cycle lengths $n-1$ and j iff (up to relabeling of vertices and reversal of each arc) D is a (primitive) subdigraph of a digraph formed by taking an $n-1$ cycle $1 \to n-1 \to n-2 \to \cdots \to 2 \to 1$, adding in the arcs $a \to a+j-1$ for $1 \leq a \leq n-j$, and one of the following:*

(a) arcs so that n replicates i_0 for a fixed $i_0 \in \{1,\ldots,n-1\}$,

(b) arcs $1 \to n, n \to n-2$ and $n \to j-1$.

Proof. First suppose that D is primitive with $exp(D) \geq \lfloor \omega_n/2 \rfloor + 2$ and cycle lengths $n-1$ and j. By relabeling the vertices and/or reversing each arc in D if necessary, we may assume that the $n-1$ cycle is as above, and that vertex n has indegree 1 (Lemma 3.1 and Corollary 3.2). If the subdigraph induced by $\{1,\ldots,n-1\}$ is not primitive, then this subdigraph is just the $n-1$ cycle, and without loss of generality $1 \to n$, so by Lemma 3.1 the outarcs of n are a subset of those given in (b). So suppose that the subdigraph induced by $\{1,\ldots,n-1\}$ is primitive. It follows from a result of Beasley and Kirkland [1, Theorem 1], that the exponent of this induced subdigraph is at least $\lfloor \omega_n/2 \rfloor$, which in turn is at least $\lfloor \omega_{n-1}/2 \rfloor + 2$. Hence without loss of generality, take the subdigraph to contain the arc $1 \to j$, and (by Theorem 2.2) to have the property that if $a \to a+j-1$, then $1 \leq a \leq n-j$. Let a_0 be the maximum such a. Suppose that $i \to n$ and note from Lemma 3.1 that the only possible outarcs from n are $n \to i-2$ and $n \to i+j-2$. Consider the two cases: (i) $n \not\to i+j-2$, (ii) $n \to i+j-2$.

Case (i) $n \not\to i+j-2$: Vertex n has outdegree 1 with $n \to i-2$ (and indegree 1 with $i \to n$). From the structure of the subgraph induced by $\{1,\ldots,n-1\}$ (described above), D is a subdigraph of one constructed as in (a) (with $i_0 = i-1$).

Case (ii): $n \to i+j-2$: If $1 \le i-1 \le n-j$ or $n-1 \ge i-1 \ge a_0+j-1$, then D is a subdigraph of one of the ones constructed in (a) (if $i \ne 1$, with $i_0 = i+j-1$) or in (b) (if $i=1$). Suppose now that $n-j+1 \le i-1 \le a_0+j-2$. Then $n \le i+j-2 \le a_0+2j-3$, so that $1 \le i+j-2-(n-1) \le a_0+2j-3-(n-1) < a_0-2$. Note that D contains the closed walk $a_0 \to a_0+j-1 \to a_0+j-2 \to ... \to i \to n \to i+j-2-(n-1) \to i+j-3-(n-1) \to ... \to 1 \to j \to j-1 \to ... \to a_0$, which has length $3j-(n-1)$. Any closed walk can be decomposed into cycles, thus $3j-(n-1) = c_1 j + c_2(n-1)$ for some nonnegative integers c_1, c_2. Since $j < 3j-(n-1) < 2(n-1)$, the only possible cases are that $3j-(n-1)$ is one of $n-1$ (with $c_1 = 0, c_2 = 1$), $2j$ (with $c_1 = 2, c_2 = 0$) and $j+n-1$ (with $c_1 = 1, c_2 = 1$). The last two of these imply that $j = n-1$ (a contradiction). The first of these three can only occur if $3j = 2(n-1)$, and since j and $n-1$ are relatively prime, this is also impossible. Consequently, it must be the case that $1 \le i-1 \le n-j$ or $n-1 \ge i-1 \ge a_0+j-1$, so that D is a subgraph of one of the ones constructed in (a) or (b).

For the converse, consider a maximal digraph H constructed as in (a). Since n replicates i_0, $exp(H) = exp(H')$ where H' is formed from H by deleting n and its incident arcs. Now H' is Hamiltonian on $n-1$ vertices and has the digraph structure of Theorem 2.2, thus $exp(H') \ge \lfloor \omega_{n-1}/2 \rfloor + 2$. Applying Corollary 2.4 to H' with n replaced by $n-1$ and $a = n-j$, $exp(H') = j(n-2) \ge \lfloor \omega_n/2 \rfloor + 2$, since $j > n/2$ and $n \ge 6$. For case (b), observe that there is no walk from $n-1$ to 1 of length $(n-2)j-1$ (by the usual Frobenius- Schur argument), so that the exponent is at least $(n-2)j$, giving the required result as in (a). ☐

Note that the result of Theorem 3.3 does not hold for small values of n. For example, if $n=5$ a digraph as in (a) of Theorem 3.3 with exponent equal to $9 < 10 = \lfloor \omega_5/2 \rfloor + 2$ can be constructed by taking a Hamiltonian digraph on 4 vertices with two additional arcs from consecutive vertices forming two 3−cycles (see, e.g., [2, pp. 82-83]) and vertex 5 replicating vertex 1.

THEOREM 3.4. *Suppose that $n \ge 6$ is even and $j = n/2$. Then D is a primitive digraph on n vertices with $exp(D) \ge \lfloor \omega_n/2 \rfloor + 2$ and cycle lengths $n-1$ and j iff (up to relabeling of vertices and reversal of each arc) D is a (primitive) subdigraph of a digraph formed by taking an $n-1$ cycle $1 \to n-1 \to n-2 \to \cdots \to 2 \to 1$, adding in the arcs $i \to i+j-1$ for $1 \le i \le n/2-3$, and one of the constructions (a) or (b) in Theorem 3.3.*

Proof. First suppose that D is primitive with $exp(D) \ge \lfloor \omega_n/2 \rfloor + 2$ and cycle lengths $n-1$ and j. As in the proof of Theorem 3.3, assume that the $n-1$ cycle is as above, that the subdigraph induced by $\{1, \ldots, n-1\}$ is primitive, with $1 \to j$, and with the property that if $a \to a+j-1$, then $1 \le a \le n-j$. Finally, also suppose that $i \to n$. By Lemma 3.1 and Corollary 3.2 there are two cases to consider: (i) D contains exactly one of the arcs $n \to i+j-2$ and $i-j \to n$, (ii) D contains neither the arc $n \to i+j-2$ nor the arc $i-j \to n$.

Case (i): We claim that we may assume that $n \to i+j-2$. To see the claim, observe that if instead we have the arc $i-j \to n$ (and thus, by Lemma 3.1, $n \to i-2$), we can reverse every arc in D and relabel vertices $1, \ldots, n-1$ by sending t to $n-t$ for each such t. With this relabeling, it follows from Lemma 3.1 that $n-i+2 \to n$ and

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$n \to n-i+j$. With $n-i+2$ replaced by i, this digraph contains the arc $n \to i+j-2$. So without loss of generality, we assume that the arc $n \to i+j-2$ is in D. Since vertex n is on a j-cycle and since D has diameter at most $n-1$, it follows that there is a walk from n to any vertex of length $n-1+(n-2)(n/2-1) = (n^2-2n+2)/2$, and similarly that from any vertex in D there is a walk to n of length $(n^2-2n+2)/2$. Since $exp(D) \geq \lfloor \omega_n/2 \rfloor + 2 = (n^2-2n+6)/2$, it must be the case that there are vertices u and $v \in \{1,\ldots,n-1\}$ such that there is no walk from u to v of length $(n^2-2n+4)/2$. Observe that for any vertex $w \in \{1,\ldots,n-1\}$ that is on a j-cycle, there is a walk from w to every vertex in $\{1,\ldots,n-1\}$ of length $n-2+(n-2)(n/2-1) = (n^2-2n)/2$. As a result, the shortest walk from u to a vertex in $\{1,\ldots,n-1\}$ that is on a j-cycle must have length at least 3. It follows from this that in fact vertex $n-1$ must be at least 3 steps from the nearest j-cycle, so that in particular, none of $n-1$, $n-2$ and $n-3$ can be on a j-cycle. Thus in D, $n-j \not\to n-1$, $n-j-1 \not\to n-2$ and $n-j-2 \not\to n-3$, and so if $a \to a+j-1$, then $a \leq n-j-3 = n/2-3$. Further, it must be the case that $1 \leq i \leq n-j-2$, otherwise one of vertices $n-1$, $n-2$ and $n-3$ is on a j-cycle (involving vertices i and n). Consequently, D can be relabeled to yield a subdigraph of one of those constructed in (a) with $i_0 = i-1$ (if $2 \leq i \leq n-j-2$), or (b) (if $i=1$).

Case (ii): If D contains neither the arc $n \to i+j-2$ nor the arc $i-j \to n$, then n has both indegree and outdegree 1, with $i \to n \to i-2$. Now if D contains either of the arcs $i-1 \to i+j-2$ or $i-j \to i-1$, then the labels of vertices $i-1$ and n can be exchanged and case (i) above applies. On the other hand if D contains neither of those two arcs, then $i-1$ has indegree and outdegree 1, with $i \to i-1 \to i-2$, so that vertex n replicates vertex $i-1$. Thus $exp(D) = exp(D')$ where D' is formed from D by deleting vertex n and the arcs incident with it. From Corollary 2.4 with n replaced by $n-1$, $exp(D') = n-1-a+1+(n-3)j$ where $a = max\{b$ is a vertex in D' : the arc $b \to b+j-1$ is in $D'\}$. Thus $exp(D') = exp(D) = n-a+(n-3)n/2 \geq (n^2-2n+6)/2$, which implies that $a \leq n/2-3$. Consequently D is a subdigraph of one of those constructed in (a) with $i_0 = i-1$.

For the converse, consider a digraph H constructed as in (a). Since n replicates i_0, $exp(H) = exp(H')$, where H' is formed from H by deleting n and its incident arcs. Appealing to Corollary 2.4 with n replaced by $n-1$, $a = n/2-3$, and $j = n/2$, $exp(H') = (n^2-2n+6)/2 = \lfloor \omega_n/2 \rfloor + 2$ if n is even. Finally, consider the digraph H constructed in (b). Evidently the walks from vertex $n-1$ to $n-3$ can only have lengths equal to 2, or to $2+n-1+c_1(n-1)+c_2 j$ for nonnegative integers c_1 and c_2. It follows that there is no walk from $n-1$ to $n-3$ of length $(n^2-2n+4)/2$, so that $exp(H) \geq (n^2-2n+6)/2$. ☐

4. Eigenvalue Results. In this section we explore results on the multiplicities of eigenvalues of primitive stochastic matrices having large exponent. These complement eigenvalue results in [4]. Our first theorem gives conditions for a stochastic matrix with large exponent to have a multiple nonzero eigenvalue. This result, which is not restricted to $k = n$ or $k = n-1$, shows that a multiple nonzero eigenvalue must be negative with algebraic multiplicity 2.

THEOREM 4.1. *Let A be a primitive, row stochastic n-by-n matrix with $n \geq 3$ and $exp(A) \geq \lfloor \omega_n/2 \rfloor + 2$. Let k and j be the two cycle lengths in $D(A)$ with $n \geq k > j$. Then A has a multiple nonzero eigenvalue λ iff $\lambda = -r$, where r is the unique positive root of $kx^j + jx^k = k - j$. When this is the case, k is odd and j is even.*

Proof. By Theorem 1 in [4], the characteristic equation of A is $z^n - \alpha z^{n-j} - (1-\alpha)z^{n-k} = 0$, for some $\alpha \in (0,1)$. Thus a nonzero eigenvalue satisfies

$$z^k - \alpha z^{k-j} - (1-\alpha) = 0. \tag{2}$$

Note that 1 is always an eigenvalue, and (by Descartes' rule of signs) there is no other positive eigenvalue. Let $\lambda = \rho e^{i\theta}$ be an eigenvalue with $\rho > 0$ and $0 < \theta < 2\pi$. By differentiating, if λ is a multiple eigenvalue, then it also satisfies $\lambda^j = \alpha(k-j)/k$, giving $\rho^j = \alpha(k-j)/k$ and $\theta = 2\pi l/j$ for some positive integer $l < j$. Further differentiation shows that the algebraic multiplicity of λ is 2. By taking imaginary parts of the characteristic equation, $\rho^k sin(k\theta) = \alpha\rho^{k-j} sin((k-j)\theta)$. On substituting for ρ^j, this gives $(k-j)sin(k\theta) = ksin((k-j)\theta) = ksin((k-j)2\pi l/j) = ksin(k\theta)$. Thus $sin(k\theta) = 0$, so that $\theta = \pi m/k$ for some positive integer m. Hence $2lk = mj$, and since $gcd(k,j) = 1$ and j divides $2l$, it must be that $j = 2l$. As a result $\theta = \pi, \lambda = -\rho$, j is even, k is odd and $\alpha = k\rho^j/(k-j)$. Substituting into (2) gives $k\rho^j + j\rho^k = k - j$. The converse is straightforward. □

From the characteristic equation, a matrix satisfying the conditions of Theorem 4.1 has zero as an eigenvalue iff $k < n$, and its algebraic multiplicity is $n - k$.

The digraph characterizations in Sections 2 and 3 lead to results about the geometric multiplicities of eigenvalues of primitive, stochastic matrices with large exponent.

THEOREM 4.2. *Let A be a primitive, row stochastic n-by-n matrix with $n \geq 3$ and $exp(A) \geq \lfloor \omega_n/2 \rfloor + 2$. If $D(A)$ is Hamiltonian, then each eigenvalue of A is geometrically simple.*

Proof. Let the length of the shorter cycle(s) in $D(A)$ be $j \geq \lceil (n-1)/2 \rceil$ by Lemma 2.1. For $j \geq n/2$ take $p = n - j + 1$, and for $j = (n-1)/2$ take $p = (n-1)/2$. Then by Theorems 2.2 and 2.3, without loss of generality by permutation similarity $A = [a_{ij}]$ has the following form: $a_{1,n} = 1 - \alpha_1$; $a_{i,i-1} = 1 - \alpha_i$ for $2 \leq i \leq p$; $a_{i,i-1} = 1$ for $p+1 \leq i \leq n$; $a_{i,i+j-1} = \alpha_i$ for $1 \leq i \leq p$; and all other $a_{ij} = 0$. Here α_i satisfy $0 < \alpha_1 < 1$ and $0 \leq \alpha_i < 1$ for $2 \leq i \leq p$. Thus for all $j \geq \lceil (n-1)/2 \rceil$, A is an unreduced Hessenberg matrix. By deleting row 1 and column n, it can be seen that rank $A \geq (n-1)$ [7, Exercise 22, p. 274]. Similarly, rank $(A - \lambda I) = n - 1$ for each eigenvalue λ of A. This implies that each eigenvalue has geometric multiplicity one. □

As an example of the above eigenvalue results, consider the 3-by-3 row stochastic matrix A having $k = 3$ and $j = 2$ as in the proof of Theorem 4.2 with $\alpha_1 = \alpha_2 = 1/2$. Note that $exp(A) = 4$. The characteristic equation of A is $z^3 - \alpha z - (1-\alpha) = 0$, with $\alpha = 3/4$; thus A has eigenvalues $1, -1/2, -1/2$. Here $-1/2$ is an eigenvalue of algebraic multiplicity 2 (as predicted by Theorem 4.1), but geometric multiplicity 1 (as predicted by Theorem 4.2).

THEOREM 4.3. *Let A be a primitive, row stochastic n-by-n matrix with $n \geq 6$ and $exp(A) \geq \lfloor \omega_n/2 \rfloor + 2$. If the maximal cycle length in $D(A)$ is $n-1$, then each* *eigenvalue of A is geometrically simple.*

Proof. Since $k = n-1$, $\lambda = 0$ is a simple eigenvalue of A. Let the length of the shorter cycle(s) in $D(A)$ be $j \geq \lceil n/2 \rceil$ by Theorem 1.2. For simplicity, only the proof for the case $j > n/2$ is given, the case $j = n/2$ is essentially the same. For $j > n/2$, by Theorem 3.3, without loss of generality by permutation similarity $A = [a_{ij}]$, or its transpose, must have one of two forms corresponding to (a) or (b).

In case (a), without loss of generality n can be taken to replicate a vertex with outdegree 1. (This is because, by Lemma 3.1, n has either indegree or outdegree 1, so, if necessary, take A^T.) Let vertex n replicate vertex i where $n-1 \geq i > n-j$. Consider the matrix $A - \lambda I$, where $\lambda \neq 0$ and the digraph of A is as in Theorem 3.3(a). Form B from $A - \lambda I$ by deleting the first row and the last column. Then B is block upper triangular with a $(1,1)$ block of order $i-2$ and a $(2,2)$ block of order $n-i+1$. Since the $(1,1)$ block is upper triangular with positive diagonal entries, it is nonsingular. The $(2,2)$ block has the first $n-i$ diagonal entries positive, $-\lambda$ in each superdiagonal entry, and a 1 in the last row first column. Every other entry in the $(2,2)$ block is zero. By expanding about the first row, the determinant of the $(2,2)$ block has magnitude λ^{n-i}. As a result, B is nonsingular, so that $A - \lambda I$ has a submatrix of rank $n-1$.

In case (b), $a_{i,i+j-1} = \alpha_i$ for $1 \leq i \leq n-j$; $a_{1,n-1} = \beta_1$; $a_{1,n} = 1 - \alpha_1 - \beta_1$; $a_{i,i-1} = 1 - \alpha_i$ for $2 \leq i \leq n-j$; $a_{i,i-1} = 1$ for $n-j+1 \leq i \leq n-1$; $a_{n,j-1} = \gamma_n$; $a_{n,n-2} = 1 - \gamma_n$; and all other $a_{ij} = 0$. Here the parameters satisfy: $0 \leq \alpha_1 < 1$; $0 < \beta_1 < 1$; $1 - \alpha_1 - \beta_1 > 0$; $0 \leq \alpha_i < 1$ for $2 \leq i \leq n-j$; and $0 < \gamma_n \leq 1$, such that A is primitive. Deleting row n and column $n-1$, the remaining submatrix of $A - \lambda I$ is upper Hessenberg, and has rank $n-1$ for all values of λ, because it has a unique nonzero transversal of length $n-1$ (from the subdiagonal and $(1,n)$ entries of $A - \lambda I$).

Thus rank $(A - \lambda I) = n-1$ for every eigenvalue λ of A, and the geometric multiplicity of each eigenvalue is one. ◻

We close the paper with a class of examples to show that for $k \leq n-2$, a row stochastic matrix with large exponent can have an eigenvalue of large geometric multiplicity.

EXAMPLE 4.4. For a fixed n, take $k \leq n-2$ so that $\omega_k \geq \lfloor \omega_n/2 \rfloor + 2$. Select α such that $0 < \alpha < 1$, and form the primitive row stochastic n-by-n matrix A with nonzero entries as follows: $a_{1,k-1} = \alpha$, $a_{1k} = 1-\alpha$, $a_{i,i-1} = 1$ for $i \in \{2,3\} \cup \{5,\ldots,k\}$, $a_{4i} = 1/(n-k+1)$ for $i \in \{3\} \cup \{k+1,\ldots,n\}$, and $a_{i2} = 1$ for $i \in \{k+1,\ldots n\}$. The digraph of A can be formed by starting from $D(W_k)$ and taking each of the vertices $k+1,\ldots,n$ replicating vertex 3. Since vertex 3 is replicated $n-k$ times, there is a walk involving any of the vertices $k+1,\ldots,n$ in $D(A)$ iff there is a corresponding walk involving vertex 3 in $D(W_k)$. Thus $exp(A) = exp(D(W_k)) = \omega_k \geq \lfloor \omega_n/2 \rfloor + 2$. Observe that since each of rows $k+1$ through n is a copy of row 3, A has nullity at least $n-k$. Further, from the statement after Theorem 4.1, the algebraic multiplicity of 0 as an eigenvalue of A is equal to $n-k$. Thus the algebraic and geometric multiplicities of 0 coincide, with common value $n-k \geq 2$. The smallest example in this class has $n = 9, k = 7$ with other cycles of length 6. In this case, 0 is an eigenvalue of (algebraic and geometric) multiplicity 2.

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