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A CLASS OF INVERSE M–MATRICES*

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Abstract. Nonnegative matrices whose inverses are M–matrices are called inverse M–matrices. It is still an open problem to characterize all inverse M–matrices. In this note a new class of inverse M–matrices is established. This class of nonsymmetric matrices generalizes the class of strictly ultrametric matrices.

Key words. M-matrices, inverse M–matrices, nonnegative matrices, ultrametric matrices

AMS subject classifications. 15A48, 15A57

1. Introduction and Notations. One of the most beautiful properties of a nonsingular M–matrix is that its inverse is a nonnegative matrix. However, the converse of this result is not in general true, i.e., the inverse of a nonsingular nonnegative matrix is not in general an M–matrix. In 1977 Willoughby [15] called the problem of finding or characterizing nonnegative matrices whose inverses are M–matrices the inverse M–matrix problem. First, Markham [5] established in 1972 a sufficient condition for a nonnegative symmetric matrix to be an inverse of a Stieltjes matrix (a nonsingular symmetric M–matrix). However, since that time just a few classes of inverse M–matrices have been found; see [13] for a collection of inverse M–matrices.

In 1994 Martínez, Michon and San Martín [6] introduced so-called strictly ultrametric matrices whose entries satisfy the following inequalities:

$$a_{ij} = a_{ji} \geq 0 \quad \text{for all } i,j \tag{1}$$

$$a_{ij} \geq \min(a_{ik}, a_{kj}) \quad \text{for all } i,j,k \tag{2}$$

$$a_{ii} > a_{ij} \quad \text{for all } i,j,\ i \neq j. \tag{3}$$

They proved that the inverse of a strictly ultrametric matrix is a row and column diagonally dominant M–matrix. A characterization of strictly ultrametric matrices is established by Nabben and Varga in [11], which explains the structure of these matrices. Later on a number of papers considering ultrametric matrices and their relations to other classes of structured matrices appeared, e.g., [1], [2], [3], [7], [9], [10], [11], [12], [14].

Moreover, a large effort was made to generalize the above result to the nonsymmetric case. In [7] and [12] a class of nonsymmetric matrices were introduced consisting of matrices which satisfy the ultrametric inequality (2). These matrices are called generalized ultrametric matrices – or short GUMs. The inverse of a GUM is a row and column diagonally dominant M–matrix.

Recently, Fiedler found in [4] another class of nonsymmetric nonnegative matrices which satisfy the inequality (2) and whose inverses are row and column diagonally

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dominant M–matrices. This note is motivated by the results in [4]. Here we also present a new class of nonsymmetric nonnegative matrices satisfying the ultrametric inequality (2). We show under which conditions these matrices are nonsingular. Moreover we establish that these matrices are inverses of column diagonally dominant M–matrices.

2. Results. Our class of inverse M–matrices is constructed in a similar way as the class of GUMs which we will describe in the following (see [12]):

Assume that we have an arbitrary rooted tree $\Gamma = (V, E)$ consisting of a root 0, the set of vertices V and the set of edges $E \subset \{x, y \in V, x \neq y\}$. Let $L \subset V$ denote the set of the *leaves* of the tree with cardinality $|L| = n$. To each edge (x, y) of E, we assign two nonnegative numbers:

$$\ell(x,y) \geq 0 \ and \ r(x,y) \geq 0 \qquad (for \ \ all \ (x,y) \in E).$$

Then, for any $i \in L$, let $P_{i,0}$ denote the path connecting the leaf i to the root 0. For any i and j in L, set

$$\begin{array}{rcl} d_l(i,j) & := & \sum_{\{r,s\} \in P_{i,0} \cap P_{j,0}} l(r,s), \\ d_r(i,j) & := & \sum_{\{r,s\} \in P_{i,0} \cap P_{j,0}} r(r,s), \\ d(i,i) & := & max\{d_l(i,i); d_r(i,i)\}, \end{array} \tag{1}$$

where $\{r, s\} \in P_{i,0} \cap P_{j,0}$ denotes a common edge of the paths $P_{i,0}$ and $P_{j,0}$. If we number the leaves from 1 to n and define the matrix $A = [a_{i,j}] \in R^{n,n}$ by

$$a_{ij} := \begin{cases} d_r(i,j) & for \ i < j, \\ d(i,j) & for \ i = j, \\ d_l(i,j) & for \ i > j, \end{cases} \tag{2}$$

we obtain, for every rooted tree and for all weighting functions l and r defined on this rooted tree, a generalized ultrametric matrix. Conversely, for a given generalized ultrametric matrix A, there exists a rooted tree and weighting functions l and r such that the entries of A are given as indicated in (1) and (2). Moreover, there exists a permutation matrix P such that

$$PAP^T = \begin{bmatrix} A_{11} & \tau(A) e_s e_{n-s}^T \\ \omega(A) e_{n-s} e_s^T & A_{22} \end{bmatrix} \tag{3}$$

where $A_{11} \in R^{s \times s}$ and $A_{22} \in R^{n-s \times n-s}$, for some $s \in \{1, \ldots, n-1\}$, are also GUMs; $\tau(A)$ is the minimal element of the matrix A, $\omega(A) \in R^+$ and $e_t = (1, \ldots, 1)^T \in R^t$.

If A itself as well as all principal submatrices which are GUMs are in the form (3) we say that A is in nested block form; see [7].

Next we construct our new class of inverse M–matrices. Assume that we have a binary rooted tree Γ with nonnegative weights $l(i, j)$ and $r(i, j)$. Let the leaves of the tree are numbered such that the related GUM is in nested block form. Then we define the following nonnegative matrix $A = [a_{ij}]$: The upper triangular part of A is given by

$$a_{i,j} = \begin{cases} d_r(i,j) & for \ i < j \\ d(i,j) & for \ i = j. \end{cases} \tag{4}$$

The lower triangular part of A is defined as follows. Consider two disjoint branches Γ_1 and Γ_2 of Γ starting at the root or its successor such that $\Gamma_1 + \Gamma_2 = \Gamma$. Here we allow a void branch Γ_1. Let L_1 and L_2 be the sets of leaves of Γ_1 and Γ_2 respectively (where $i < j$ if $i \in \Gamma_1$ and $j \in \Gamma_2$). Then with $p = \max\{j| \in \Gamma_2\}$ let

$$a_{ij} = \begin{cases} d_r(i,p) & \textit{for } i \in L_2, j \in L\ j < i \\ d_l(i,j) & \textit{for } i,j \in L_1,\ j < i. \end{cases} \tag{5}$$

Definition 2.1. *The set of all matrices A for which there exists a rooted tree and weighting functions l and r such that A is given by* (4) *and* (5) *is denoted by $\mathcal{U}$.*

Before we state the main result of this note we mention some properties of matrices in $\mathcal{U}$.

First it follows immediately from the definition that an $n \times n$ matrix $A \in \mathcal{U}$ has the form

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \tag{6}$$

where $A_{11} \in R^{s\times s}$ and $A_{22} \in R^{n-s\times n-s}$ for some $s \in \{1,\ldots,n-1\}$, with

$$\begin{aligned} &A_{11} \quad \text{is a GUM,} \\ &A_{22} \in \mathcal{U}, \\ &A_{12} = \tau(A)e_s e_{n-s}^T, \\ &A_{21} = be_s^T \end{aligned}$$

where b is the last column of A_{22}. Note that A_{22} again has a similar 2×2 block structure, where the first diagonal block is a special GUM.

It is clear that matrices in $\mathcal{U}$ are in general not GUMs. However, the next proposition shows that the matrices in $\mathcal{U}$ satisfy the ultrametric inequality (2). Thus these matrices are generalization of strictly ultrametric matrices.

Proposition 2.2. *Let $A = [a_{i,j}] \in \mathcal{U}$. Then the entries of A satisfy*

$$a_{ij} \geq \min\{a_{ik}, a_{kj}\} \quad \textit{for all } i,j,k \in \{1,\ldots,n\} \tag{7}$$

Proof. If $i,j,k \leq s$ then (7) is fulfilled since A_{11} is a GUM (see [12]). If $i,j,k > s$ then (7) follows by induction. If $i,j \leq s, k > s$ we use the fact that $\tau(A_{11}) \geq \tau(A)$. If $i,k \leq s, j > s$ we have $a_{ij} = a_{kj}$. In all other cases we have $a_{ij} = a_{ik}$. ☐

We then obtain the following result.

Theorem 2.3. *Let $A = [a_{i,j}] \in \mathcal{U}$. Then A is nonsingular if and only if A does not contain a row or column of zeros, and no two rows or two columns are the same. If A is nonsingular then A^{-1} is a column diagonally dominant M–matrix. Moreover A_{11} and A_{22} in* (6) *are nonsingular and*

$$\begin{aligned} A/A_{11} &:= A_{22} - A_{21}A_{11}^{-1}A_{12} \in \mathcal{U} \\ A/A_{22} &:= A_{11} - A_{12}A_{22}^{-1}A_{21} \ \textit{is a GUM.} \end{aligned}$$

Proof. It is clear that A is singular if A does contain a row or column of zeros, or two rows or two columns are the same. For the other statements we use an induction on the dimension of A.

So assume that A does not contain a row or column of zeros, and no two rows or two columns are the same. Thus A_{11} and A_{22} satisfy also this property. Therefore, A_{11} is nonsingular (see [7]) and A_{22} is also nonsingular by the induction hypothesis. Moreover

$$A_{22} - A_{21}A_{11}^{-1}A_{12} = A_{22} - \tau e^T A_{11}^{-1} ebe^T.$$

Since $\tau e^T A_{11}^{-1} e < 1$ (see [12]) and b is the last column of A_{22} the Schur complement A/A_{11} is in $\mathcal{U}$. Moreover, A/A_{11} does not contain a row or column of zeros, and no two rows or two columns are the same. On the other hand we have

$$A_{11} - A_{12}A_{22}^{-1}A_{21} = A_{11} - \tau ee^T.$$

Thus A/A_{22} is a GUM and it does not contain a row or column of zeros, and no two rows or two columns are the same. Thus $detA = detA_{22}det(A/A_{22}) \neq 0$, i.e., A is nonsingular. The inverse of A is given in the form

$$A^{-1} = \begin{bmatrix} (A/A_{22})^{-1} & -A_{11}^{-1}A_{12}(A/A_{11})^{-1} \\ -A_{22}^{-1}A_{21}(A/A_{22})^{-1} & (A/A_{11})^{-1} \end{bmatrix}.$$

Since

$$\begin{aligned} -A_{22}^{-1}A_{21}(A/A_{22})^{-1} &= -\tilde{e}_{n-s}e_s^T((A/A_{22})^{-1} \leq 0 \\ -A_{11}^{-1}A_{12}(A/A_{11})^{-1} &= -\tau(A)A_{11}^{-1}e_s * e_{n-s}^T(A/A_{11})^{-1} \leq 0, \end{aligned}$$

where $\tilde{e}_{n-s} = (0, \ldots, 0, 1)^T \in R^{n-s}$, A^{-1} is an M–matrix. Moreover we have

$$\begin{aligned} &e_s^T(A/A_{22})^{-1} - e_{n-s}^T A_{22}^{-1}A_{21}(A/A_{22})^{-1} \\ &= e_s^T(A/A_{22})^{-1} - e_{n-s}^T\tilde{e}_{n-s}e_s^T((A/A_{22})^{-1} \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} &e_{n-s}^T(A/A_{11})^{-1} - e_s^T A_{11}^{-1}A_{12}(A/A_{11})^{-1} \\ &= e_{n-s}^T(A/A_{11})^{-1} - e_s^T\tau(A)A_{11}^{-1}e_s * e_{n-s}^T(A/A_{11})^{-1} \\ &\geq 0. \end{aligned}$$

Hence A^{-1} is a column diagonally dominant M–matrix. □

It was conjectured by Neumann in [13] that $A \circ A$, is an inverse M–matrix if A is an inverse M–matrix. Here $\circ$ denotes the Hadamard product of a matrix. Obviously this conjecture is true for matrices $A \in \mathcal{U}$ since $A \circ A \in \mathcal{U}$.

Example 2.4. Consider the weighted rooted tree in Figure 2.1. The related matrix $A \in \mathcal{U}$ is given by

$$A = \begin{bmatrix} 8 & 5 & 1 & 1 & 1 \\ 6 & 8 & 1 & 1 & 1 \\ 3 & 3 & 8 & 3 & 3 \\ 7 & 7 & 7 & 9 & 7 \\ 9 & 9 & 9 & 9 & 9 \end{bmatrix}.$$

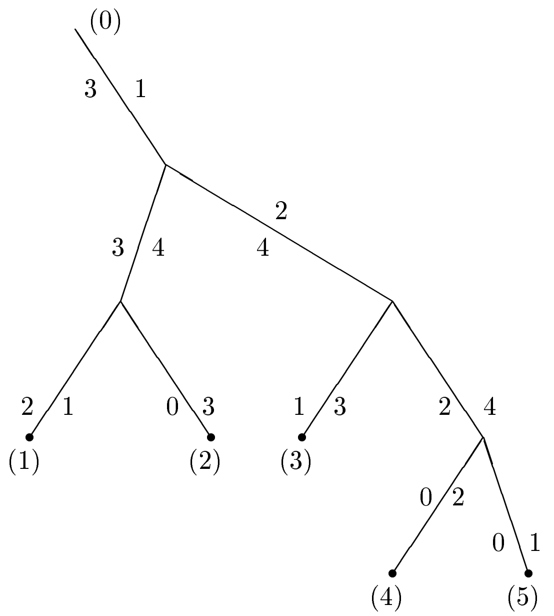

Figure 2.1

The inverse of A (rounded to four digits) is

$$A^{-1} = \begin{bmatrix} 0.2414 & -0.1379 & -0.0000 & 0.0000 & -0.0115 \\ -0.1724 & 0.2414 & 0.0000 & -0.0000 & -0.0077 \\ 0.0000 & 0.0000 & 0.2000 & -0.0000 & -0.0667 \\ 0.0000 & -0.0000 & 0 & 0.5000 & -0.3889 \\ -0.0690 & -0.1034 & -0.2000 & -0.5000 & 0.5858 \end{bmatrix}.$$

Moreover, we have

$$e_5^T A^{-1} = [0,\ 0,\ 0,\ 0,\ 0.1111]$$
$$A^{-1} e_5 = [0.0920,\ 0.0613,\ 0.1333,\ 0.1111, -0.2866]^T.$$

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