The Electronic Journal of Linear Algebra.
A publication of the International Linear Algebra Society.
Volume 7, pp. 59-72, July 2000.
ISSN 1081-3810. http://math.technion.ac.il/iic/ela

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ON THE GROUP $GL(2, R[X])$*

VALERIĬ FAĬZIEV†

Abstract. Suppose that G is an arbitrary group and S is its subset such that $S^{-1} = S$. Let $gr(S)$ be the subgroup of G generated by S. Denote by $l_S(g)$ the length of element $g \in gr(S)$ relative to the set S. Let V be a finite subset of a free group F of countable rank and let the verbal subgroup $V(F)$ be a proper subgroup of F. For an arbitrary group G, denote by $\overline{V}(G)$ the set of values in the group G of all the words from the set V. The present paper establishes the infinity of the set $\{l_S(g),\ g \in V(G)\}$, where $G = GL(2, R[x])$, $S = \overline{V}(G) \cup \overline{V}(G)^{-1}$ for an arbitrary field R.

Key words. Verbal subgroup, Width of verbal subgroup, Pseudocharacter.

AMS subject classifications. 20E06, 20F22

1. Introduction. In 1940, Ulam [26, 27] posed the following problem. Given a group G_1, a metric group (G_2, d), and a positive number ε, does there exist a $\delta > 0$ such that if $f : G_1 \to G_2$ satisfies $d(f(xy), f(x)f(y)) < \delta$ for all $x, y \in G_1$, then a homomorphism $T : G_1 \to G_2$ exists with $d(f(x), T(x)) < \varepsilon$ for all $x, y \in G_1$?

The first affirmative answer was given by Hyers [12] in 1941.

THEOREM 1.1. *Let E_1, E_2 be Banach spaces and let $f : E_1 \to E_2$ satisfy the following condition: there is an $\varepsilon > 0$ such that*

$$|| f(x + y) - f(x) - f(y) || < \varepsilon \text{ for all } x, y \in E_1.$$

Then there exists $T : E_1 \to E_2$ such that

$$T(x + y) - T(x) - T(y) = 0 \text{ for all } x, y \in E_1 \tag{1}$$

and

$$|| f(x) - T(x) || < \varepsilon \text{ for all } x \in E_1. \tag{2}$$

The subject rested there until Rassias [21] considered a generalized version of the previous result which permitted the Cauchy difference to become unbounded. That is, he assumed that

$$|| f(x + y) - f(x) - f(y) || < \varepsilon \cdot (||x||^p + ||y||^p) \text{ for all } x, y \in E_1,$$

where $0 \le p < 1$.

By making use of a direct method, Rassias proved in this case too that there is an additive function T from E_1 into E_2 given by the formula

$$T(x) = \lim_{n\to\infty} \frac{1}{2^n} f(2^n x)$$

*Received by the editors on 19 October 1999. Accepted for publication on 28 May 2000. Handling editor: Daniel Hershkowitz.

†Tver State Agricultural Academy, Sakharovo Tver, Russia. *Current address*: Tereshkova Sq. 47/27-23, Tver, 170037, Russia (p001655@tversu.ru)

such that

$$||T(x) - f(x)|| \leq k \cdot \varepsilon \cdot ||x||^p,$$

where k depends on p as well as ε.

Rassias [22], during the 27th International Symposium on Functional Equations, asked whether such a theorem can also be proved for $p \geq 1$.

Gajda [10], following the same approach as [21], gave an affirmative solution to this question for $p > 1$. Several papers were devoted to the generalization of these results; see [13, 14, 15, 16, 17, 21, 22, 23]. In connection with these results, the following question arises.

Let S be an arbitrary semigroup or group and let a mapping $f : S \to \mathbb{R}$ satisfy the following condition: the set $\{f(xy) - f(x) - f(y), x, y \in S\}$ is bounded. Is it true that there is $T : S \to \mathbb{R}$ satisfying the following conditions?

(1) $T(xy) - T(x) - T(y) = 0,\ x,\ y \in S$.

(2) The set $\{T(x) - f(x),\ x \in S\}$ is bounded.

The negative answer was given by Forti [9] by means of the following example. Let $F(\alpha, \beta)$ be the free group generated by the two elements α, β. Let each word $x \in F(\alpha, \beta)$ be written in reduced form, i.e., x does not contain pairs of the forms $\alpha\alpha^{-1}$, $\alpha^{-1}\alpha$, $\beta\beta^{-1}$, $\beta^{-1}\beta$ and has no exponents different from 1 and -1. Define the function $f : F(\alpha, \beta) \to \mathbb{R}$ as follows. If $r(x)$ is the number of pairs of the form $\alpha\beta$ in x and $s(x)$ is the number of pairs of the form $\beta^{-1}\alpha^{-1}$ in x, put $f(x) = r(x) - s(x)$.

It is easily shown that for all $x, y \in F(\alpha, \beta)$ we have $f(xy) - f(x) - f(y) \in \{-1, 0, 1\}$. Now, assume that there is $T : F(\alpha, \beta) \to \mathbb{R}$ such that the relations (1), (2) hold.

But T is completely determined by its values $T(\alpha)$ and $T(\beta)$, while f is identically zero on the subgroups A and B generated by α and β, respectively. For $\alpha \in A$ we have $T(\alpha^n) = nT(\alpha)$ and $f(\alpha^n) = 0$ for $n \in N$. Since $T(\alpha^n) - f(\alpha^n) = nT(\alpha)$ for $n \in N$, it follows that $T(\alpha) = 0$. Similarly we have $T(\beta) = 0$, so that T is identically zero on $F(\alpha, \beta)$. Hence, $f - T = f$ on $F(\alpha, \beta)$, where f is unbounded. This contradiction proves that there is not a homomorphism $T : F(\alpha, \beta) \to \mathbb{R}$ such that the relation (2) holds.

It turns out that the existence of mappings that are "almost homomorphisms" but are not small perturbations of homomorphisms has an algebraic nature.

Definition 1.2. A *quasicharacter* of a semigroup S is a real-valued function f on S satisfying the condition that the set $\{f(xy) - f(x) - f(y) \mid x,\ y \in S\}$ is bounded.

Definition 1.3. By a *pseudocharacter* on a semigroup S (group S) we mean its quasicharacter f that satisfies the following condition: $f(x^n) = nf(x)\ \forall\ x \in S$ and $\forall\ n \in \mathbb{N}$ (and $\forall\ n \in Z$, if S is group).

The set of quasicharacters of semigroup S is a vector space (with respect to the usual operations of addition of functions and their multiplication by numbers), which will be denoted by $KX(S)$. The subspace of $KX(S)$ consisting of pseudocharacters will be denoted by $PX(S)$ and the subspace consisting of real additive characters of the semigroup S will be denoted by $X(S)$.

We say that a pseudocharacter φ of the group G is *nontrivial* if $\varphi \notin X(G)$.

In connection with the example of Forti, note that in [5, 6] the set of all pseudocharacters of free groups was described.

Let H be a Hilbert space and let $U(H)$ be the group of unitary operators of H endowed by operator-norm topology. If H is n-dimensional, $n \in N$, then denote the group $U(H)$ by $U(n)$.

DEFINITION 1.4. Let $0 < \varepsilon < 2$. Let T be a mapping of a group G into $U(H)$. We say that T is an *ε-representation* if for any x, y from group G the relation

$$\|T(xy) - T(x)T(y)\| < \varepsilon$$

holds.

V. Milman raised this question: Let $\rho : G \to U(H)$ be an ε-representation with small ε. Is it true that ρ is near to an actual representation π of the group G in H, i.e., does there exist some small $\delta > 0$ such that $\|\rho(x) - \pi(x)\| < \delta$ for all $x \in G$? In answer to this question Kazhdan, in [18], obtained the following result.

THEOREM 1.5. *There is a group Γ with the following property. For any $0 < \varepsilon < 1$ and any natural number $n > \frac{3}{\varepsilon}$ there exists an ε-representation ρ such that for any homomorphism $\pi : G \to U(n)$ the relation*

$$\|\rho - \pi\| = \sup\{\|\rho(x) - \pi(x)\|,\ x \in \Gamma\} > \frac{1}{10}$$

holds.

Note that the group Γ has the following presentation in terms of generations and defining relations: $\Gamma = \langle x, y, a, b \,\|\, x^{-1}y^{-1}xya^{-1}b^{-1}ab\rangle$.

In [7], by using pseudocharacters, a stronger version of Kazhdan's theorem was established as follows. We say that a group G belongs to the class $\mathcal{K}$ if every nonunit quotient group of G has an element of order two.

THEOREM 1.6. *Let H be a Hilbert space and let $U(H)$ be its group of unitary operators. Suppose that groups A and B belong to the class $\mathcal{K}$ and the order of B is more than two. Then the free product $G = A * B$ has the following property. For any $\varepsilon > 0$ there exists a mapping $T : G \to U(H)$ satisfying the following conditions:*

(1) $\|T(xy) - T(x) \cdot T(y)\| \leq \varepsilon\ \forall\ x, \forall\ y \in G$,

(2) for any representation $\pi : G \to U(H)$ the relation

$$\sup\{\|T(x) - \pi(x)\|,\ x \in G\} = 2$$

holds.

In the present paper we consider an application of pseudocharacters to the problem of expressibility in groups.

2. The problem of expressibility in the group $GL(2, R[x])$. Let G be an arbitrary group and let S be its subset such that $S^{-1} = S$. Denote by $gr(S)$ the subgroup of G generated by S. We say that the width of the set S is finite if there is $k \in N$ such that any element g of $gr(S)$ is representable in the form

$$(3) \qquad g = s_1 s_2 \cdots s_n, \quad \text{where } s_i \in S \cup S^{-1},\ n \leq k.$$

The minimal k with this property we call the *width* of the set S in G and denote it by $wid(S,G)$.

We say that the width of the set S in the group G is infinite if for any $k \in N$ there is an element $g_k \in gr(S)$ which does not have a presentation of the form (3). Many papers were devoted to the problem of the width of different subsets; see [1, 2, 3, 11, 20, 24, 25].

In this paper we consider the problem of *the width of verbal subgroups.* Namely, let V be a finite subset of the free group F of countable rank. We say that V is proper if the verbal subgroup $V(F)$ is a proper subgroup of F.

Let G be an arbitrary group. Denote by $\overline{V}(G)$ the set of values in the group G of all the words from the set V. By the width of verbal subgroup $V(G)$ we mean the width of the set $\overline{V}(G) \cup \overline{V}(G)^{-1}$ in the group G.

Numerous papers devoted to the problem of the width of verbal subgroups have been written (see [2, 11, 24] and references therein).

The present paper establishes that if V is a proper finite subset of F, then the width of $V(GL(2, R[x]))$ is infinite for an arbitrary field R.

In [4] the following result was obtained.

THEOREM 2.1. *Let f be a quasicharacter of semigroup S such that $|f(xy) - f(x) - f(y)| < c\ \forall\ x,\ y \in S$. Then the function*

$$(4) \qquad \widehat{f}(x) = \lim_{n\to\infty} \frac{1}{2^n} f(x^{2^n})$$

is well defined and is the pseudocharacter of S such that $|\widehat{f}(xy) - \widehat{f}(x) - \widehat{f}(y)| < 4c\ \forall\ x,\ y \in S$.

COROLLARY 2.2. *Let f be a quasicharacter of group G such that $|f(xy) - f(x) - f(y)| < c\ \forall\ x,\ y \in G$. Then the function*

$$\widehat{f}(x) = \lim_{n\to\infty} \frac{1}{2^n} f(x^{2^n})$$

is well defined and is the pseudocharacter of G such that $|\widehat{f}(xy) - \widehat{f}(x) - \widehat{f}(y)| < 4c\ \forall\ x,\ y \in G$.

Proof. Theorem 2.1 implies that in order to prove that $\widehat{f}$ is a pseudocharacter of group G it remains to verify that for each $x \in G$ the equality $\widehat{f}(x^{-1}) = -\widehat{f}(x)$ holds.

From the relation $\widehat{f}(x^n) = n\widehat{f}(x)\ \forall\ x \in G,\ \forall\ n \in N$ we obtain $\widehat{f}(1^n) = n\widehat{f}(1)$. Hence, $\widehat{f}(1) = 0$ and for each x from G we have $|\widehat{f}(1) - \widehat{f}(x) - \widehat{f}(x^{-1})| < 4c$ and $|\widehat{f}(x) + \widehat{f}(x^{-1})| < 4c$.

Therefore, $n|\widehat{f}(x) + \widehat{f}(x^{-1})| = |\widehat{f}(x^n) + \widehat{f}((x^{-1})^n)| < 4c\ \forall\ x \in G,\ \forall\ n \in N$. This is possible only if $\widehat{f}(x^{-1}) = -\widehat{f}(x)$. Now let $k > 0$. Then we have $\widehat{f}(x^{-k}) = \widehat{f}((x^k)^{-1}) = -\widehat{f}(x^k) = -k\widehat{f}(x)$. The corollary is proved. □

Let R be an arbitrary field and let $R[z]$ be the ring of polynomials over R. Let H be the subgroup of the group $A=GL(2,R)$ consisting of matrices

$$\begin{bmatrix} \alpha & t \\ 0 & \beta \end{bmatrix}, \quad \text{where } \alpha,\ \beta\in R^*,\ t\in R.$$

Let B be the subgroup of $G=GL(2,R[z])$ consisting of matrices

$$\begin{bmatrix} k_1 & f(z) \\ 0 & k_2 \end{bmatrix}, \text{ where } k_1,\ k_2\in R^*,\ f(z)\in R[z].$$

It is clear that $H\subset B$. It is well known that the group $G=L(2,R[z])$ is an amalgamated product $G=A*_H B$. See [19].

LEMMA 2.3.

(1) Let

$$Q=\left\{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ x & 1 \end{bmatrix},\ x\in R\right\}.$$

Then Q is a system of representatives of left and right cosets of the group A by its subgroup H.

(2) Elements

$$P=\left\{\begin{bmatrix} 1 & \varphi(z) \\ 0 & 1 \end{bmatrix},\ x\in R\right\}$$

form a left and a right system of representatives of the group B by subgroup H and $P^{-1}=P$.

(3) $P\triangleleft B$ and B is the semidirect product $B=H\cdot P$.

Proof. The proof is obtained by direct calculations. □

DEFINITION 2.4. By the *reduced* form of element $g\in G\setminus H$ we mean its presentation in the form

$$g=c_1c_2\cdots c_k,\ \text{where } c_i\in(A\cup B)\setminus H,\ c_ic_{i+1}\notin(A\cup B).$$

For this reduced form of element g we set $\dot g=c_1,\ \ddot g=c_k$.

DEFINITION 2.5. By the *regular subdivision* of g we mean a presentation in the form $g=g_1g_2\cdots g_k$, where $\ddot g_i\dot g_{i+1}\notin H$.

Let $X=\{x_n,\ n\in N\}$. Denote by D a free semigroup over alphabet X. To each element g of G we assign a word $\sigma(g)$ in alphabet X as follows. If $g\in A$, then we set $\sigma(g)=\wedge$, where $\wedge$ denotes the empty word. For any $f(z)\in R[z]$ denote by $\sigma(f(z))$ the degree of $f(z)$. Now if

$$b=\begin{bmatrix} k_1 & f(z) \\ 0 & k_2 \end{bmatrix}\in T(2,R[z]),$$

we denote by $\sigma(b)$ the degree of polynomial $f(z)$.

If $\sigma(f(z)) = 0$, then we set $\sigma(b) = \wedge$. If $\sigma(f(z)) > 0$, then we set $\sigma(b) = x_{\sigma(f(z))}$. Let $v = c_1 c_2 \cdots c_k$ be a reduced form of the element v from G. Then we set $\sigma(v) = x_{\sigma(c_1)} x_{\sigma(c_2)} \cdots x_{\sigma(c_k)}$. Hence, to each element from G we assign a word from D. It is obvious that the mapping σ is well defined.

Definition 2.6. We say that two words from D are *conjugate* if one of them is obtained from another by cyclic permutation of letters.

The relation of conjugacy in D we denote by $\sim_D$. It is evident that if elements $u, v \in G$ are conjugate, then the words $\sigma(u)$ and $\sigma(u)$ are conjugate to each other.

Now for each word v from the semigroup D we introduce the set of "beginnings" $H(v)$ and the set of "ends" $K(v)$ as follows. If $v \in X$, we put $H(v) = K(v) = \emptyset$. If $v = x_{i_1} \cdot x_{i_2} \cdot \ldots \cdot x_{i_n}$, $n > 1$, where $x_{ij} \in X$, we set

$$\begin{aligned} H(v) &= \{x_{i_1}, x_{i_1} x_{i_2}, \ldots, x_{i_1} x_{i_2} \ldots x_{i_{n-1}}\}, \\ K(v) &= \{x_{i_2} \cdots x_{i_n}, x_{i_3} \cdots x_{i_n}, \ldots, x_{i_{n-1}} x_{i_n}, x_{i_n}\}. \end{aligned}$$

Let $v \in D$. We set $\overline{H}(v) = H(v) \cup \{v\}$ and $\overline{K}(v) = K(v) \cup \{v\}$.

For $v = x_{i_1} x_{i_2} \cdots x_{i_n}$, we set $v^* = x_{i_n} x_{i_{n-1}} \cdots x_{i_1}$.

Denote by $|v|$ the length of a word v in the alphabet X. It is clear that $H(w) \cap K(w) = \emptyset$ if and only if $H(w^*) \cap K(w^*) = \emptyset$.

Denote by P the set of words w in alphabet X such that $H(w) \cap K(w) = \emptyset$ and $w \not\sim_D w^*$. It is clear that if $w \in P$, then $|w| > 1$.

For each pair of elements x, y from D we define measures $\mu_{x,y}$ on P as follows.

We set $\mu_{x,y}(w) = 1$ if there exist a and b such that $a \in \overline{K}(x)$, $b \in \overline{H}(y)$, and $w = ab$; otherwise we set $\mu_{x,y}(w) = 0$.

Let $w \in P$ and $v \in D$ and denote by $\eta_w(v)$ the number of occurrences of w in the word v. It is easy to see that the function $v \to \eta_w(v)$ is a quasicharacter of semigroup D such that for any u, v from D the relation

$$\eta_w(uv) - \eta_w(u) - \eta_w(v) = \mu_{u,v}(w) \tag{5}$$

holds.

Let us set

$$\psi_w(v) = \eta_w(v) - \eta_{w^*}(v).$$

From (5) we get

$$\psi_w(uv) - \psi_w(u) - \psi_w(v) = \mu_{u,v}(w) - \mu_{u,v}(w^*).$$

For any $u, v \in D$ we set $\Delta_{u,v}(w) = \mu_{u,v}(w) - \mu_{u,v}(w^*)$.

Hence,

$$\psi_w(uv) - \psi_w(u) - \psi_w(v) = \Delta_{u,v}(w). \tag{6}$$

Whence for any $u_1, u_2, \ldots, u_k \in D$ and any $w \in P$ the relation

$$\psi_w(u_1u_2\cdots u_k) - \sum_{i=1}^{k} \psi_w(u_i) = \Delta_{u_1,u_2\cdots u_k}(w) + \Delta_{u_2,u_3\cdots u_k}(w) + \ldots + \Delta_{u_{k-1},u_k}(w)$$

holds. It is easy to see that $\Delta_{u,v}(w) \in \{-1,0,1\}\ \forall\ u,v \in D,\ \forall\ w \in P$.

Hence the function $v \to \psi_w(v)$ is a quasicharacter of D such that for any u, v from D the following relations hold:

$$|\psi_w(uv) - \psi_w(u) - \psi_w(v)| \leq 1,\ \psi_w(v^*) = -\psi_w(v).$$

Now define a function ρ_w on the group G as follows. For any $g \in G$ we set $\rho_w(g) = \psi_w(\sigma(g))$. It is clear that $\rho_w(g^{-1}) = -\rho_w(g)$.

LEMMA 2.7. *Suppose that the words g, t are reduced and $\ddot{g}\dot{t} \notin H$. One can choose (at most six) pairs of elements $u_i, v_i \in D$ such that for any $w \in P$ the relation*

$$\rho_w(gt) = \rho_w(g) + \rho_w(t) + \sum_i \varepsilon_i \Delta_{u_i,v_i}(w), \quad \textit{where } \varepsilon_i \in \{-1,1\},$$

holds.

Proof. Consider two cases: (a) $\ddot{g}\dot{t} \notin B$ and (b) $\ddot{g}\dot{t} \in B$.

It is clear that the relations (a) and (b) do not depend on the reduced forms of elements g and t.

It is obvious that in case (a) the equality $\sigma(gt) = \sigma(g)\sigma(t)$ holds. Hence, $\rho_w(gt) = \rho_w(g) + \rho_w(t) + \Delta_{\sigma(g),\sigma(t)}(w)$.

Consider case (b). Let $\sigma(\ddot{g}) = b_1,\ \sigma(\dot{t}) = b_2$, and $g = g_1b_1$, $t = b_2t_1$, be a regular subdivision.

It is clear that $\sigma(g) = \sigma(g_1)\sigma(b_1)$, $\sigma(t) = \sigma(b_2)\sigma(t_1)$.

Hence, $\sigma(gt) = \sigma(g_1)\sigma(b_1b_2)\sigma(t_1)$, $\sigma(g)\sigma(t) = \sigma(g_1)\sigma(b_1)\sigma(b_2)\sigma(t_1)$.

Therefore, we obtain

$$\begin{aligned}\psi_w(\sigma(gt)) &= \psi_w(\sigma(g_1)\sigma(b_1b_2)\sigma(t_1)) \\ &= \psi_w(\sigma(g_1)) + \psi_w(\sigma(b_1b_2)) + \psi_w(\sigma(t_1)) \\ &\quad +\Delta_{\sigma(g_1),\sigma(b_1b_2)\sigma(t_1)}(w) + \Delta_{\sigma(b_1b_2),\sigma(t_1)}(w),\end{aligned}$$

$$\begin{aligned}\psi_w(\sigma(g)\sigma(t)) &= \psi_w(\sigma(g_1)\sigma(b_1)\sigma(b_2)\sigma(t_1)) \\ &= \psi_w(\sigma(g_1)) + \psi_w(\sigma(b_1)) + \psi_w(\sigma(b_2)) \\ &\quad +\psi_w(\sigma(t_1)) + \Delta_{\sigma(g_1),\sigma(b_1)\sigma(b_2)\sigma(t_1)}(w) \\ &\quad +\Delta_{\sigma(b_1),\sigma(b_2)\sigma(t_1)}(w) + \Delta_{\sigma(b_2),\sigma(t_1)}(w),\end{aligned}$$

$$\begin{aligned}\psi_w(\sigma(gt)) - \psi_w(\sigma(g)\sigma(t)) &= \psi_w(\sigma(b_1b_2)) - \psi_w(\sigma(b_1)) - \psi_w(\sigma(b_2)) \\ &\quad +\Delta_{\sigma(g_1),\sigma(b_1b_2)\sigma(t_1)}(w) + \Delta_{\sigma(b_1b_2),\sigma(t_1)}(w) \\ &\quad -\Delta_{\sigma(g_1),\sigma(b_1)\sigma(b_2)\sigma(t_1)}(w) \\ &\quad -\Delta_{\sigma(b_1),\sigma(b_2)\sigma(t_1)}(w) - \Delta_{\sigma(b_2),\sigma(t_1)}(w).\end{aligned}$$

Now taking into account that the length of each element w from P is at least two we have $\psi_w(\sigma(b)) = 0$ for any $b \in B$.

Hence, we get

$$\begin{aligned}\psi_w(\sigma(gt)) - \psi_w(\sigma(g)\sigma(t)) &= \Delta_{\sigma(g_1),\sigma(b_1b_2)\sigma(t_1)}(w) + \Delta_{\sigma(b_1b_2),\sigma(t_1)}(w)\\ &\quad -\Delta_{\sigma(g_1),\sigma(b_1)\sigma(b_2)\sigma(t_1)}(w)\\ &\quad -\Delta_{\sigma(b_1),\sigma(b_2)\sigma(t_1)}(w) - \Delta_{\sigma(b_2),\sigma(t_1)}(w).\end{aligned}$$

Now, using (6), we obtain

$$\begin{aligned}\psi_w(\sigma(gt)) - \psi_w(\sigma(g)) - \psi_w(\sigma(t)) &= \Delta_{\sigma(g_1),\sigma(b_1b_2)\sigma(t_1)}(w)\\ &\quad +\Delta_{\sigma(b_1b_2),\sigma(t_1)}(w)\\ &\quad -\Delta_{\sigma(g_1),\sigma(b_1)\sigma(b_2)\sigma(t_1)}(w)\\ &\quad -\Delta_{\sigma(b_1),\sigma(b_2)\sigma(t_1)}(w)\\ &\quad -\Delta_{\sigma(b_2),\sigma(t_1)}(w) + \Delta_{\sigma(g),\sigma(t)}(w).\end{aligned}$$

The lemma is proved. □

LEMMA 2.8. *For any x, y from G one can choose (at most eight) pairs of elements $u_i, v_i \in D$ such that for any $w \in P$ the relation*

$$\rho_w(xy) = \rho_w(x) + \rho_w(y) + \sum_i \varepsilon_i \Delta_{u_i,v_i}(w), \ \textit{where } \varepsilon_i \in \{-1, 1\},$$

holds. Hence,

$$|\rho_w(xy) - \rho_w(x) - \rho_w(y)| \le 8$$

and the function ρ_w is a quasicharacter of G.

Proof. Suppose that $x = g \cdot z_1$, $y = z_2 \cdot t$ are regular subdivisions such that $z_1 z_2 \in H$, $\ddot{g}\dot{t} \in (A \cup B) \setminus H$. Then we have $\sigma(x) = \sigma(g)\sigma(z_1)$, $\sigma(y) = \sigma(z_2)\sigma(t)$, $\sigma(xy) = \sigma(gt)$. It is easy to see that $\sigma(z_2) = \sigma(z_1)^*$. Hence, we have

$$\psi_w(\sigma(x)) = \psi_w(\sigma(g)\sigma(z_1)) = \psi_w(\sigma(g)) + \psi_w(\sigma(z_1)) + \Delta_{\sigma(g),\sigma(z_1)}(w),$$

$$\psi_w(\sigma(y)) = \psi_w(\sigma(z_2)\sigma(t)) = \psi_w(\sigma(t)) + \psi_w(\sigma(z_2)) + \Delta_{\sigma(z_2),\sigma(t)}(w),$$

$$\psi_w(\sigma(x)) + \psi_w(\sigma(y)) = \psi_w(\sigma(g)) + \psi_w(\sigma(t)) + \Delta_{\sigma(g),\sigma(z_1)}(w) + \Delta_{\sigma(z_2),\sigma(t)}(w),$$

$$\psi_w(\sigma(xy)) = \psi_w(\sigma(gt)),$$

$$\begin{aligned}\psi_w(\sigma(xy)) - \psi_w(\sigma(x)) - \psi_w(\sigma(y)) &= \psi_w(\sigma(gt)) - \psi_w(\sigma(g)) - \psi_w(\sigma(t))\\ &\quad -\Delta_{\sigma(g),\sigma(z_1)}(w) - \Delta_{\sigma(z_2),\sigma(t)}(w).\end{aligned}$$

It was established in Lemma 2.7 that one can choose (at most six) pairs of elements $u_i, v_i \in D$ such that

$$\psi_w(\sigma(gt)) = \psi_w(\sigma(g)) + \psi_w(\sigma(t)) + \sum_i \varepsilon_i \Delta_{u_i,v_i}(w),$$

where $\varepsilon_i \in \{-1, 1\}$.

Hence, one can choose (at most eight) pairs of elements $u_i, v_i \in D$ such that

$$\psi_w(\sigma(xy)) - \psi_w(\sigma(x)) - \psi_w(\sigma(y)) = \sum_i \varepsilon_i \Delta_{u_i,v_i}(w),$$

where $\varepsilon_i \in \{-1, 1\}$. The lemma is proved. □

COROLLARY 2.9. *For any $g_1, g_2, \ldots, g_n$ from G, one can choose (at most $8(n-1)$) pairs of elements $u_i, v_i \in D$ such that for each $w \in P$ the relation*

$$\rho_w(g_1 g_2 \cdots g_n) = \sum_{j=1}^{n} \rho_w(g_j) + \sum_i \varepsilon_i \Delta_{u_i,v_i}(w), \tag{7}$$

where $\varepsilon_i \in \{-1, 1\}$, holds.

Let

$$a = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \; b_n = \begin{bmatrix} 1 & z^n \\ 0 & 1 \end{bmatrix}, \; x = \sigma(ab_1), \; y = \sigma(ab_2).$$

Consider the set $\mathcal{M} = \{w_k = x^{3k} y^{2k} x^k y^k, \; k \in N\}$. Let $\mathcal{M}^* = \{w_k^* \mid w_k \in \mathcal{M}\}$.

It can easily be checked that w_l is not a subword of w_k for $k \neq l$ and also that

$$H(w_k) \cap K(w_l) = \emptyset \text{ for all } k, l \in N,$$

$$H(w_k) \cap K(w_p^*) = \emptyset \text{ if } k \neq p,$$

$$H(w_p) \cap K(w_k) = \emptyset \text{ if } k \neq p.$$

It is easily shown that for any k and q, w_k^* is not a subword of w_q.

Hence, $\mathcal{M} \subset P$, $\mathcal{M}^* \subset P$, and for any $u, v \in D$ the relations

$$|\mathcal{M} \cap supp \; \mu_{u,v}| \leq 1, \; |\mathcal{M}^* \cap supp \; \mu_{u,v}| \leq 1$$

hold. Furthermore, we have the following estimation:

$$|\mathcal{M} \cap supp \; \Delta_{u,v}| \leq 2 \; \forall \, u, \; v \in D. \tag{8}$$

COROLLARY 2.10. *For any $g_1, g_2, \ldots, g_n$ from G the following assertions are true:*

(1) there are at most $16(n-1)$ *elements* $w \in \mathcal{M}$ *such that*

$$\rho_w(g_1g_2\cdots g_n) \neq \sum_{j=1}^{n} \rho_w(g_j),$$

(2) $|\rho_w(g_1g_2\cdots g_n) - \sum_{j=1}^{n} \rho_w(g_j)| \leq 16(n-1)\ \forall\ w \in \mathcal{M}$.

Proof. (1) From (7) it follows that if $w \in \mathcal{M}$, then there are at most $8(n-1)$ pairs of elements u_i, v_i such that $w \in \cup_i supp\,\Delta_{u_i,v_i}$. Now from (8), we get that there are at most $16(n-1)$ elements $w \in \mathcal{M}$ such that $\rho_w(g_1g_2\cdots g_n) \neq \sum_{j=1}^{n} \rho_w(g_j)$.

(2) This assertion follows from (7) and (8). □

Let $m \geq 2$. Then for any $g \in G$, the set

$$O_m(g) = \{w \mid w \in \mathcal{M},\ \rho_w(g) \not\equiv 0 \bmod (m)\}$$

is finite. Denote by $\gamma_m(g)$ the cardinality of $O_m(g)$. It is clear that for each $g \in G$ the relation $O_m(g) = O_m(g^{-1})$ holds. Hence, $\gamma_m(g) = \gamma_m(g^{-1})$.

From Corollary 2.9 we have that for any $g_1, g_2, \ldots, g_n$ from G one can choose (at most $8(n-1)$) pairs of elements $u_i, v_i \in D$ such that the relation

$$O_m(g_1g_2\cdots g_n) \subseteq \cup_{j=1}^{n} O_m(g_j)\ \cup\ \cup_i\, supp\ \Delta_{u_i,v_i} \tag{9}$$

holds. From (9) we obtain

$$\gamma_m(g_1g_2\cdots g_n) \leq \sum_{j=1}^{n} \gamma_m(g_j)\ +\ \sum_i |supp\ \Delta_{u_i,v_i}|, \tag{10}$$

where $|supp\ \Delta_{u_i,v_i}|$ denotes the cardinality of the set $supp\ \Delta_{u_i,v_i}$.

PROPOSITION 2.11. *For any* $x, y \in G$ *the following relations hold:*

(1) $\gamma_m(xy) \leq \gamma_m(x) + \gamma_m(y) + 16$,

(2) $|\gamma_m(x^{-1}yx) - \gamma_m(y)| \leq 32$,

(3) $\gamma_m(x^{-1}y^{-1}xy) \leq 48$,

(4) $\gamma_m(x^m) \leq 16(m-1)$.

Proof. Assertion (1) follows from (10). Let us prove assertion (2). We have

$$O_m(x^{-1}yx) = \{w \mid w \in \mathcal{M},\ \rho_w(x^{-1}yx) \not\equiv 0 \bmod (m)\}.$$

From (7) we have that one can choose at most 16 pairs of elements u_i, v_i from D such that

$$\rho_w(x^{-1}yx) = \rho_w(x^{-1}) + \rho_w(y) + \rho_w(x) + \sum_i \varepsilon_i \Delta_{u_i,v_i}(w).$$

Since $\rho_w(x^{-1}) + \rho_w(x) = 0$, we have

$$\rho_w(x^{-1}yx) = \rho_w(y) + \sum_i \varepsilon_i \Delta_{u_i,v_i}(w).$$

Hence,

$$\begin{aligned} O_m(x^{-1}yx) &= \{w \mid w \in \mathcal{M},\ \rho_w(x^{-1}yx) \not\equiv 0 \bmod (m)\} \\ &= \{w \mid w \in \mathcal{M},\ \rho_w(y) + \textstyle\sum_i \varepsilon_i \Delta_{u_i,v_i}(w) \not\equiv 0 \bmod (m)\} \end{aligned}$$

and

$$O_m(x^{-1}yx) \subseteq O_m(y)\ \cup\ \cup_i\, supp\ \Delta_{u_i,v_i}.$$

From the latter and (8) we get

$$\gamma_m(x^{-1}yx) \leq \gamma_m(y) + \sum_i |\, supp\ \Delta_{u_i,v_i}\,|.$$

Hence, for any $x, y \in G$ we have

$$\gamma_m(x^{-1}yx) \leq \gamma_m(y) + 32. \tag{11}$$

Replacing y by xyx^{-1} we obtain

$$\gamma_m(y) \leq \gamma_m(xyx^{-1}) + 32.$$

Now replacing x by x^{-1} we obtain

$$\gamma_m(y) \leq \gamma_m(x^{-1}yx) + 32. \tag{12}$$

From (11), (12) we get

$$|\, \gamma_m(x^{-1}yx) - \gamma_m(y)\,| \leq 32.$$

Similarly, we verify that assertion (3) is true. Now let us prove assertion (4). From (7) we get that for any $x \in G$ one can choose (at most $8(m-1)$) pairs of elements $u_i, v_i \in D$ such that for each $w \in \mathcal{M}$ the relation

$$\rho_w(x^m) = m\rho_w(x) + \sum_i \varepsilon_i \Delta_{u_i,v_i}(w), \quad \text{where } \varepsilon_i \in \{-1, 1\},$$

holds.

Hence,

$$\begin{aligned} O_m(x^m) &= \{w \mid w \in \mathcal{M},\ \rho_w(x^m) \not\equiv 0 \bmod (m)\} \\ &= \{w \mid w \in \mathcal{M},\ m\rho_w(x) + \textstyle\sum_i \varepsilon_i \Delta_{u_i,v_i}(w) \not\equiv 0 \bmod (m)\} \\ &= \{w \mid w \in \mathcal{M},\ \textstyle\sum_i \varepsilon_i \Delta_{u_i,v_i}(w) \not\equiv 0 \bmod (m)\} \\ &\subseteq \cup_i \big(\mathcal{M}\ \cap\ supp\Delta_{u_i,v_i}\big). \end{aligned}$$

Taking into account (8), we get

$$\gamma_m(x^m) \leq \sum_i |\, supp\, \Delta_{u_i,v_i}\,| \leq 16(m-1).$$

The proposition is proved. □

PROPOSITION 2.12. *Let $C \triangleleft G$. Then there is a pseudocharacter φ of the group G such that $\varphi\big|_C \not\equiv 0$.*

Proof. There is an element $g \in C$ such that $g = a_1 b_1 \cdots a_k b_k$, where $k \geq 1$, $a_i \in A \setminus H$, $b_i \in B \setminus H$.

Since for every g from C and every t from G the element $tgt^{-1}g$ and every cyclic permutation of $g = a_1 b_1 \cdots a_k b_k$ belong to the subgroup C, we can assume that $\sigma(b_k) \geq \sigma(b_i)$, $\sigma(b_k) > \sigma(b_1)$.

Let $v = \sigma(g) = x_{\sigma(b_1)} = \cdots = x_{\sigma(b_k)} = x_{i_1} \cdots x_{i_k}$, $x_{i_j} \in X$. Suppose that $v \sim v^*$.

Consider the element

$$\begin{aligned} t \quad = \quad & \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & z^{2m} \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & z^{2m-1} \\ 0 & 1 \end{bmatrix} \cdot \cdots \\ & \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & z^{m-1} \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & \gamma_1 z^m + \gamma_2 z^{m-1} \\ 0 & 1 \end{bmatrix}, \end{aligned}$$

where $m = \sigma(b_k)$.

Let us choose γ_1, γ_2 such that the relation

$$\sigma\left(b_k \cdot \begin{bmatrix} 1 & \gamma_1 z^m + \gamma_2 z^{m-1} \\ 0 & 1 \end{bmatrix}^{-1}\right) = m - 1$$

holds. It is easy to verify that $\sigma(t) \in P$ and that there is no cyclic permutation of the word $\sigma(tgt^{-1}g)$ containing $\sigma(t^{-1}) = \sigma(t)^*$ as subword. This implies the following equalities:

$$\rho_{\sigma(t)}(tgt^{-1}g) = \psi_{\sigma(t)}(\sigma(tgt^{-1}g)) = \eta_{\sigma(t)}(\sigma(tgt^{-1}g)) = 1.$$

Let $\widehat{\rho}_{\sigma(t)}$ be a pseudocharacter of G defined by (4). Since for each $n \in N$ the word $(\sigma(tgt^{-1}g))^n$ has no subword which is equal to $\sigma(t^{-1})$, we obtain $\widehat{\rho}_{\sigma(t)}(tgt^{-1}g) = 1$. Now consider the case when $v \not\sim v^*$. The Lemma 8 from [8] implies that there is $w \in P$ and $m \in N$ such that $v \sim w^n$.

Hence $\rho_w(g) = \psi_w(\sigma(g)) = \eta_w(v) - \eta_{w^*}(v)$. Taking into account the relations $m - 1 \leq \eta_w(v) \leq m$, $\eta_{w^*}(v) = 0$ we get $m - 1 \leq \rho_w(g) \leq m$.

Therefore for any $k \in N$ the relation $km - 1 \leq \rho_w(g^k) \leq km$ holds. This implies the equality $\widehat{\rho}_w(b) = m$. The proposition is proved. □

LEMMA 2.13. *Let $\varphi \in PX(G)$. Suppose that for any $x, y \in G$ we have $|\varphi(x \cdot y) - \varphi(x) - \varphi(y)| < \varepsilon$. Then*

(1) the inequality $|\varphi(x_1 \cdot x_2 \cdot \ldots \cdot x_{n+1}) - \sum_{i=1}^{n+1} \varphi(x_i)| < n \cdot \varepsilon$ holds for any positive integer n and any $x_1, x_2, \ldots, x_n \in G$;

(2) if φ is a bounded function, then $\varphi \equiv 0$;

(3) $\varphi(a^{-1}ba) = \varphi(b)$ for any $a, b \in G$.

Proof. Assertion (1) is easily proved by induction on n. Let us prove assertion (2). If δ is a positive number such that $|\varphi(x)| < \delta$ for any $x \in G$, then for any positive integer n we have $n|\varphi(x)| = |\varphi(x^n)| < \delta$. Therefore, $\varphi(x) = 0$.

Let us prove assertion (3). From assertion (1) it follows that $|\varphi((a^{-1}ba)^n) - \varphi(a^{-1}) - \varphi(b^n) - \varphi(a)| < 2\varepsilon$. Hence, $|\varphi(a^{-1}b^na) - \varphi(b^n)| < 2\cdot\varepsilon$, or $n|\varphi(a^{-1}ba) - \varphi(b)| < 2\cdot\varepsilon$. Since the latter inequality holds for all $n > 1$, we obtain $\varphi(a^{-1}ba) = \varphi(b)$.

The lemma is proved. □

THEOREM 2.14. *Let V be a finite subset of the free group F of countable rank and let $V(F)$ be a proper verbal subgroup of F. Then the width of verbal subgroup $V(G)$ is infinite relative to the set $\overline{V}(G) \cup \overline{V}(G)^{-1}$.*

Proof. Suppose that $V(F) \subset F'$. Let $\varphi \in PX(G)$ and for any x, y from G let the relation $|\varphi(xy) - \varphi(x) - \varphi(y)| \leq r$ hold. From Lemma 2.13 it follows that for any $x, y \in G$ we have $\varphi(x^{-1}) + \varphi(y^{-1}xy) = 0$. Hence,

$$|\varphi(x^{-1}y^{-1}xy)| = |\varphi(x^{-1}y^{-1}xy) - \varphi(x^{-1}) - \varphi(y^{-1}xy)| \leq r.$$

From the latter inequality it follows that if $wid(\overline{V}(G) \cup \overline{V}(G)^{-1}, G) < \infty$, then pseudocharacter φ is bounded on $V(G)$. Indeed, since V is finite there is an integer l such that each element of V is a product of at most l commutators, and we deduce that $\varphi(b) \leq (l-1)r$ for all $g \in \overline{V}(G)$. Hence, if $wid(\overline{V}(G) \cup \overline{V}(G)^{-1}, G) < \infty$, it follows that the pseudocharacter φ is bounded on $V(G)$. By Lemma 2.13 we obtain $\varphi \equiv 0$ on $V(G)$. This contradicts Proposition 2.12. Now suppose that $V(F) \not\subseteq F'$. Let $V = \{v_1, v_2, \ldots, v_k\}$ and let $X = \{z_1, z_2, \ldots\}$ be the set of free generators of the group F.

Then there is a positive integer n such that every element v_i is uniquely representable in the form

$$v_i = z_1^{l_{i_1}} z_2^{l_{i_2}} \cdots z_n^{l_{i_n}} \cdot u_i, l_{i_j} \geq 0,\ u_i \in F'.$$

Let m be the maximal common factor of the numbers $\{|l_{i_j}|,\ i = 1, \ldots, k,\ j = 1, \ldots, n\}$. Since $V(F)$ is a proper verbal subgroup of F, it follows that $m \geq 2$. Proposition 2.12 implies that there is $l \in \mathbb{N}$ such that for any $u \in \overline{V}(G)$ the relation $\gamma_m(u) \leq l$ holds. This implies that if the width of verbal subgroup $V(G)$ is finite, then the function γ_m is bounded on $V(G)$. Now consider elements $g_{km} = (ab_1)^{3km}(ab_2)^{2km}(ab_1)^{km}(ab_2)^{km},\ k \in N$.

It is evident that $g_{km} \in V(G)\ \forall\ k \in N$ and for any $n \in N$ and $i \leq n$ the relation $\rho_{w_{mi}}(g_m g_{2m} \cdots g_{nm}) = 1$ holds.

Hence, $\gamma_w(g_m g_{2m} \cdots g_{nm}) \geq n$. This contradicts the assumption that $wid(\overline{V}(G) \cup \overline{V}(G)^{-1}, G) < \infty$.

This completes the proof. □

REFERENCES

[1] S.I. Adjan and J. Mennicke. On bounded generation of $SL(n, Z)$. *Internat. J. Algebra Comput.*, 2:357–365, 1992.
[2] V.G. Bardakov. On the theory of braid groups. *Mat. Sb.*, 183:3–43, 1992.
[3] E.W. Ellers. Products of transvections in one conjugacy class in the simplectic group over $GF(3)$. *Linear Algebra Appl.*, 202:1–23, 1994.
[4] V.A. Faĭziev. Pseudocharacters on semidirect product of semigroups. *Mat. Zametki*, 53:132–139, 1993.

[5] V.A. Faĭziev. Pseudocharacters on free groups and some group constructions. *Uspekhi Mat. Nauk*, 43:225–226, 1988.
[6] V.A. Faĭziev. Pseudocharacters on free groups. *Izv. Ross. Akad. Nauk Ser. Mat.*, 58:121–143, 1994.
[7] V.A. Faĭziev. On almost representations of groups. *Proc. Amer. Math. Soc.*, 127:57–61, 1999.
[8] V.A. Faĭziev. Pseudocharacters on free semigroups. *Russian J. Math. Physic*, 3:191–206, 1995.
[9] G.L. Forti. Remark 11 in Report of the 22nd Internat. Symp. on Functional Equations. *Aequationes Math.*, 29:90–91, 1985.
[10] Z. Gajda. On stability of additive mappings. *Internat. J. Math. Math. Sci.*, 14:431–434, 1991.
[11] H.B. Griffiths. A note on commutators in free products. *Proc. Cambridge Phil. Soc.*, 50:178–188, 1954.
[12] D.H. Hyers. On the stability of the linear functional equation. *Proc. Nat. Acad. Sci. U.S.A.*, 27:222–224, 1941.
[13] D.H. Hyers, G. Isac, and Th.M. Rassias. *Topics in Nonlinear Analysis and Applications.* World Scientific, Singapore, 1997.
[14] D.H. Hyers, G. Isac, and Th.M. Rassias. *Stability of Functional Equations in Several Variables.* Birkhäuser, Boston, Basel, Berlin, 1998.
[15] D.H. Hyers and Th.M. Rassias. Approximate homomorphisms. *Aequationes Math.*, 44:125–153, 1992.
[16] G. Isac and Th.M. Rassias. Stability of ψ-additive mappings: Applications to nonlinear analysis. *Internat. J. Math. Math. Sci.*, 19:219–228, 1996.
[17] G. Isac and Th.M. Rassias. On the Hyers–Ulam stability of ψ-additive mappings. *J. Approx. Theory*, 72:131–137, 1993.
[18] D. Kazhdan. On ε-representations. *Israel J. Math.*, 43:315–323, 1982.
[19] R.S. Lyndon and P.E. Shupp. *Combinatorial Group Theory.* Springer-Verlag, Berlin, 1977.
[20] M. Newman. Unimodular commutators. *Proc. Amer. Math. Soc.*, 101:605–609, 1987.
[21] Th.M. Rassias. On the stability of the linear mapping in Banach spaces. *Proc. Amer. Math. Soc.*, 72:3297–3300, 1978.
[22] Th.M. Rassias. Problem 16, in Report of the 27th Internat. Symp. on Functional Equations. *Aequationes Math.*, 39:309, 1990.
[23] Th.M. Rassias and J. Tabor (eds). *Stability of Mappings of Hyers–Ulam Type.* Hadronic Press, Palm Harbor, Florida, 1994.
[24] A.H. Rhemtulla. A problem of expressibility in free products. *Proc. Cambridge Phil. Soc.*, 64:573–584, 1968.
[25] A.H. Rhemtulla. Commutators of certain finitely generated solvable groups. *Canad. J. Math.*, 21:1160–1164, 1969.
[26] S.M. Ulam. *A Collection of Mathematical Problems.* Interscience, New York, 1960.
[27] S.M. Ulam. *Sets, Numbers and Universes.* MIT Press, Cambridge, Massachusetts, 1974.