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THE ALGEBRAIC CONNECTIVITY OF TWO TREES CONNECTED BY AN EDGE OF INFINITE WEIGHT*

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Abstract. Let T_1 and T_2 be two weighted trees with algebraic connectivities $\mu(T_1)$ and $\mu(T_2)$, respectively. A vertex on one of the trees is connected to a vertex on the other by an edge of weight w to obtain a new tree $\hat{T}_w$. By interlacing properties of eigenvalues of symmetric matrices it is known that $\mu(\hat{T}_w) \leq \min\{\mu(T_1), \mu(T_2)\} =: m$. It is determined precisely when $\mu(\hat{T}_w) \to m$ as $w \to \infty$. Finally, a possible interpretation is given of this result to the theory of electrical circuits and Kirchoff's laws.

Key words. Weighted tree, Laplacian matrix, Algebraic connectivity, Electrical circuit.

AMS subject classifications. 5C50, 15A48

1. Introduction. A *weighted graph on n vertices* is an undirected graph $\mathcal{G}$ on n vertices such that with each edge e of $\mathcal{G}$, there is an associated positive number $w(e)$ which is called the *weight* of the edge e. The *Laplacian matrix $L = (\ell_{i,j})$ of a weighted graph on n vertices* is the $n \times n$ matrix obtained as follows:

$$\ell_{i,j} = \begin{cases} -w(e) & \text{if } i \neq j \text{ and } e \text{ is the edge joining } i \text{ and } j, \\ 0 & \text{if } i \neq j \text{ and } i \text{ is not adjacent to } j, \\ -\sum_{k \neq i} \ell_{i,k} & \text{if } i = j. \end{cases}$$

It is known that L is a singular M–matrix and hence positive semidefinite. Let $0 = \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$ be an arrangement of the eigenvalues of L in nondecreasing order. Fiedler [2] showed that $\lambda_2 > 0$ if and only if $\mathcal{G}$ is connected and he termed $\mu(\mathcal{G}) := \lambda_2$ the *algebraic connectivity of $\mathcal{G}$*.

In a graph $\mathcal{G}$, a *cycle* is a sequence of adjacent vertices $i_1, \ldots, i_{k+1}$, such that $i_{k+1} = i_1$ with $i_\ell \neq i_m$ for all $\ell \neq m$ and for all $1 \leq \ell, m \leq k$. A *tree* is a connected graph with no cycles. Thus, given any two vertices v_1 and v_2 in a tree, there is a unique path from v_1 to v_2.

In this paper we shall consider the following question:

PROBLEM 1.1. Let T_1 and T_2 be two trees. Join the trees at vertices $x \in T_1$ and $y \in T_2$ by an edge e of weight w to obtain a new tree $\hat{T}_w$. Then what is

$$\lim_{w \to \infty} \mu\left(\hat{T}_w\right) \ ? \tag{1}$$

To consider this question we need to use several results from the literature, some of which we will quote here for convenience. The first result from [3] discusses the

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eigenvector(s) of L which correspond to the algebraic connectivity. Fiedler has shown that there are two distinct possibilities for the structure of these eigenvectors.

THEOREM 1.2. (Fiedler [3]) *Let $\mathcal{T}$ be a weighted tree on n vertices, labeled $1, \ldots, n$, with Laplacian matrix L and algebraic connectivity μ. Let y be an eigenvector of L associated with μ. Then exactly one of the following cases occurs:*
(a) *Some entry of y is 0. In this case, the subgraph of $\mathcal{T}$ induced by the set of vertices corresponding to the $0'$s in y is connected. Moreover, there is a unique vertex k such that $y_k = 0$ and k is adjacent to a vertex m with $y_m \neq 0$. The entries of y are either increasing, decreasing, or identically 0 along any path in $\mathcal{T}$ which starts at k.*
(b) *No entry of y is 0. In this case, there is a unique pair of vertices i and j such that i and j are adjacent in $\mathcal{T}$ with $y_i > 0$ and $y_j < 0$. Furthermore, the entries of y are increasing along any path in $\mathcal{T}$ which starts at i and does not contain j, while the entries of y are decreasing along any path in $\mathcal{T}$ which starts at j and does not contain i.*

A weighted tree $\mathcal{T}$ is said to be of *Type I* if condition (a) holds and it is of *Type II* if condition (b) holds. If condition (a) holds, then k is called the *characteristic vertex* of $\mathcal{T}$. If (b) holds, then both i and j are called the *characteristic vertices* of $\mathcal{T}$. (This terminology was suggested by Merris [9].) It is known from [9] that if μ is not a simple eigenvalue, then all corresponding eigenvectors yield the same type of tree and the same characteristic vertex (or vertices). Since we are joining two trees T_1 and T_2 by an edge of weight w to create a new tree $\hat{T}_w$ and since we shall let $w \to \infty$, we now define the concept of the type of a tree at infinity.

DEFINITION 1.3. Let T_1 and T_2 be trees and join them by an edge e of weight w to obtain a new tree $\hat{T}_w$. Letting $w \to \infty$, we define the limit tree $\hat{T}_\infty$ *to be a Type I tree at infinity* with characteristic vertex v if there exists $w_0 > 0$ such that for all $w \in (w_0, \infty)$, $\hat{T}_w$ is a Type I tree with characteristic vertex v. Similarly, we define $\hat{T}_\infty$ *to be a Type II tree at infinity* with characteristic vertices i and j if there exists $w_0 > 0$ such that for all $w \in (w_0, \infty)$, $\hat{T}_w$ is a Type II tree with characteristic vertices i and j.

In the remainder of this paper we shall write $\hat{T}$ for $\hat{T}_\infty$.

Let v be a vertex in a tree $\mathcal{T}$. Let L_v be the matrix obtained by deleting the v–th row and v–th column of L. Since L is a singular and irreducible M–matrix, it follows that $M := L_v^{-1}$ exists and is a nonnegative matrix. M is called the *bottleneck matrix at v.* Let $P_{a,b}$ denote the set of edges on the path from a to b. Then, according to [8],

$$m_{i,j} = \sum_{e \in P_{i,v} \cap P_{j,v}} \frac{1}{w(e)}. \tag{2}$$

M is permutationally similar to a block diagonal matrix, where the number of blocks is the degree of vertex v and each such block is a positive matrix which corresponds to a unique branch at v. Each diagonal block will be referred to as the *bottleneck matrix for that branch at v.* Since the spectral radius of a positive matrix is necessarily an eigenvalue (called the *Perron value*) of the matrix, it follows that the spectral radius of M is equal to the Perron value of one (or several) of the diagonal blocks of M. The branch(es) at v corresponding to the block(s) of M with the largest Perron value are known as the *Perron branch(es) at vertex v.*

When bottleneck matrices are being used, the notation $M_{(a-b),T}$ will mean *the bottleneck matrix for the branch at vertex a containing vertex b in tree T*. Likewise, $M_{(a-e),T}$ will denote *the bottleneck matrix for the branch at vertex a containing edge e in tree T*. $M_{a,T}$ will mean the *bottleneck matrix for the Perron branch(es) at a in T*. If a has more that one Perron branch in T, then $M_{a,T}$ will be (permutationally similar to) the block diagonal matrix where each block is the bottleneck matrix for a Perron branch at a in T. Throughout the paper we shall also adopt the notation that if $A \in \mathbb{R}^{n,n}$ and α and β are ordered subsets of $\{1, \ldots, n\}$ of strictly increasing integers, then $A[\alpha, \beta]$ is the submatrix of A based on rows and columns determined by the sets of indices α and β, respectively. We shall also use the notation $A[\alpha] = A[\alpha, \alpha]$.

The next result from the literature that we shall need here comes from [7].

THEOREM 1.4. (Kirkland, Neumann, and Shader [7]) *Let $\mathcal{T}$ be a weighted tree. Then $\mathcal{T}$ is a Type I tree if and only if there is exactly one vertex at which there are two or more Perron branches. $\mathcal{T}$ is a Type II tree if and only if at each vertex there is a unique Perron branch. Moreover:*
(a) *If v is not a characteristic vertex of $\mathcal{T}$, then the unique Perron branch of $\mathcal{T}$ at v is the branch which contains the characteristic vertex (or vertices) of $\mathcal{T}$.*
(b) *If i is the characteristic vertex of a Type I tree, then i is the unique vertex with two or more Perron branches.*
(c) *If i and j are characteristic vertices of a Type II tree, then the Perron branch at i contains j, while the Perron branch at j contains i.*

From Definition 1.3, in order for $\hat{T}$ to be of Type I or of Type II at infinity, there must exist an appropriate value $w_0 > 0$ such that in (w_0, ∞), $\hat{T}_w$ does not change its type nor its characteristic vertices. In the next theorem we show that such a w_0 always exists:

THEOREM 1.5. *Let $\hat{T}_w$ be a tree obtained from joining two trees by T_1 and T_2 by an edge e of weight w. Then there exists a number $w_0 > 0$ such that in (w_0, ∞), $\hat{T}_w$ does not change its type nor its characteristic vertex (vertices).*

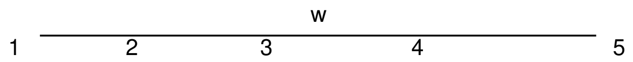
Proof. Let $x > 0$ and let v be a (not necessarily unique) characteristic vertex of $\hat{T}_x$. Suppose first that at v, $\hat{T}_x$ *has at least two Perron branches which do not contain* e. Then if $y > x$, it follows from (2) and from Corollary 2.1.5(b) of [1] that

$$\rho\left(M_{(v-e),\hat{T}_y}\right) < \rho\left(M_{(v-e),\hat{T}_x}\right), \tag{3}$$

where $M_{(v-e),\hat{T}_x}$ and $M_{(v-e),\hat{T}_y}$ are the bottleneck matrices for the branch at v containing e in $\hat{T}_x$ and $\hat{T}_y$ respectively. Therefore, for all $y \geq x$, at least two of the Perron branches at v in $\hat{T}_x$ are identical to at least two of the Perron branches at v in $\hat{T}_y$. Thus v has at least two Perron branches in $\hat{T}_y$ making $\hat{T}_y$ a Type I tree with characteristic vertex v for all $y \geq x$.

Suppose now that at v, $\hat{T}_x$ *has only one Perron branch which does not contain* e. Then by (3), v has only one Perron branch in $\hat{T}_y$ if $y > x$. Let S_w be the set of vertices such that $s \in S_w$ if and only if *any* characteristic vertex of $\hat{T}_w$ is on the path from s to e. As the weight w of e is increased, the spectral radii of the bottleneck matrices at all vertices $s \in S_w$ for the branch containing e decrease. Since the spectral radii of the bottleneck matrices at all vertices $s \in S_w$ for branches not containing e (and hence not containing a characteristic vertex of $\hat{T}_w$) remain constant as w increases, and since in $\hat{T}_w$ the unique Perron branch at *any non–characteristic* vertex $s \in S_w$ contains a characteristic vertex of $\hat{T}_w$ (see Theorem 1.4), it follows that if $b > a$, the characteristic vertex (or vertices) of $\hat{T}_b$ lie in S_a. Thus if $b > a$ then $S_b \subseteq S_a$. Therefore as w increases beyond x, the characteristic vertex (or vertices) of $\hat{T}_w$ move from v away from e. Since for each w there are only a finite number of vertices in S_w and since $S_b \subseteq S_a$ if $b > a$, it follows that for some w_0 large enough, S_w will equal S_{w_0} for all $w > w_0$. Hence by definition of S_w, the characteristic vertex (or vertices) will not change as w is increased beyond w_0. □

Before we continue let us mention that in Theorem 1.5, w_0 cannot, in general, be taken to be 0. For example, let T_1 be a path on 3 vertices and T_2 be a path on 2 vertices and consider $\hat{T}_w$, $w > 0$, to be the connection of T_1 and T_2 through their pendant vertices as follows:



Then $\hat{T}_w$ is a Type II tree with 3 and 4 as its characteristic vertices when $w \in (0, 1)$. $\hat{T}_1$ is a Type I tree with 3 as its characteristic vertex, while in $(1, \infty)$, $\hat{T}_w$ is a Type II tree with 2 and 3 as its characteristic vertices.

In our work here we shall need the following result:

THEOREM 1.6. (Kirkland, Neumann, and Shader [7]) *Let $\mathcal{T}$ be a weighted tree.*
(a) *If $\mathcal{T}$ is a Type I tree with characteristic vertex k, then the algebraic connectivity of $\mathcal{T}$ is $1/\rho(M_k)$, where M_k is the bottleneck matrix at vertex k.*
(b) *If $\mathcal{T}$ is a Type II tree with characteristic vertices i and j joined by an edge of weight θ, then there exists $0 < \gamma < 1$ such that $\rho(M_1 - (\gamma/\theta)J) = \rho(M_2 - ((1-\gamma)/\theta)J)$, where M_1 is the bottleneck matrix for the branch at j containing i, and M_2 is the bottleneck matrix for the branch at i containing j. Moreover, the algebraic connectivity of $\mathcal{T}$ is $1/\rho(M_1 - (\gamma/\theta)J)$ (which equals $1/\rho(M_2 - ((1-\gamma)/\theta)J)$).*

Since the main theorem of this paper concerns the algebraic connectivity of the tree $\hat{T} := \hat{T}_\infty$, it is important to elaborate on how $\mu(\hat{T})$ is computed. Suppose first that $\hat{T}_w$ is a tree of Type I with characteristic vertex v for all $w \in (w_0, \infty)$. Then for all $w \in (w_0, \infty)$, v has at least two Perron branches in $\hat{T}_w$. Since $\rho(M_{(v-e),\hat{T}_w})$ decreases as w increases, it follows that e is not on a Perron branch at v in $\hat{T}$. Therefore

$$\mu\left(\hat{T}\right) = \frac{1}{\rho\left(M_{v,\hat{T}_w}\right)}, \quad w \in (w_0, \infty).$$

Suppose next that $\hat{T}_w$ is of Type II with characteristic vertices i and j for all $w \in (w_0, \infty)$. If e joins i and j, then

$$\mu\left(\hat{T}\right) = \lim_{w\to\infty} \mu\left(\hat{T}_w\right) = \lim_{w\to\infty} \frac{1}{\rho\left(M_{(i-j),\hat{T}_w} - \frac{\gamma_w}{w}J\right)} = \frac{1}{\rho\left(M_{(i-j),\hat{T}}\right)},$$

where $0 < \gamma_w < 1$ according to Theorem 1.6. It should be noted that if $\hat{T}$ is of Type II and both its characteristic vertices i and j are incident to e, then it is possible for

i and j to have other Perron branches, respectively, besides the branch that contains the other characteristic vertex. This is due to the fact that since the Perron branch at i in $\hat{T}_w$ contains e when $w \in (w_0, \infty)$, $\rho(M_{(i-j),\hat{T}})$ decreases as w increases. Hence at infinity, it is possible for $\rho(M_{(i-j),\hat{T}})$ to equal $\rho(M_{(i-k),\hat{T}})$, where k is a vertex on another branch at i on $\hat{T}$. An example of this would be when T_1 and T_2 are each paths on 2 vertices. If we join vertex i of T_1 to vertex j of T_2 with an edge of weight w to create $\hat{T}_w$, we see that $\hat{T}_w$ is a Type II tree with characteristic vertices i and j for all $w > 0$. Letting w tend to infinity, we observe that $\hat{T}$ is of type II with both its characteristic vertices incident to e, yet each characteristic vertex has two Perron branches.

Suppose now $\hat{T}$ is of Type II and i is not joined to j by e. Then either the path from i to e contains j or the path from j to e contains i. Without loss of generality, let the path from i to e contain j. Then $\hat{T}_w$ is of Type II with characteristics vertices i and j for all $w \in (w_0, \infty)$. Therefore

$$\mu\left(\hat{T}\right) \;=\; \lim_{w\to\infty} \frac{1}{\rho\left(M_{(j-i),\hat{T}_w} - \frac{\gamma_w}{\theta}J\right)},$$

where θ is the weight of the edge joining i and j while $0 < \gamma_w < 1$ according to Theorem 1.6.

We need one final result from the literature:

THEOREM 1.7. (Kirkland and Neumann [6]) *If a tree $\mathcal{T}$ is created by joining a vertex of one tree T_1 with a vertex of another tree T_2, then the characteristic vertex* (or vertices) *of $\mathcal{T}$ lie on the path from the characteristic vertices of T_1 to the characteristic vertices of T_2.*

In this paper we shall consider the question formulated in Problem 1.1 and, specifically, in (1), for the joining of trees T_1 and T_2 when either are allowed to be any of the two types. The results which we shall prove in Section 3 will lead us to the following conclusions:

THEOREM 1.8. *Let T_1 and T_2 be trees and suppose that $\hat{T}_w$ is a tree formed from joining a vertex v in T_1 to a vertex u in T_2 by an edge e of weight w. Then for the tree $\hat{T}$ at infinity we have that $\mu(\hat{T}) = min\{\mu(T_1), \mu(T_2)\}$ if and only if the component tree, T_1 or T_2, with the lower algebraic connectivity, say T_1, is of Type I and one of the following conditions holds:*

(a) *The characteristic vertex of T_1 is the only characteristic vertex of $\hat{T}$.*

(b) *The characteristic vertex of T_1 is incident to e and $\rho(M_{u,T_2}) \leq 1/\mu(T_1)$.*

In Section 2 we offer a consideration which, in a natural way, leads us to consider the problem of an infinite edge. We also offer there illustrating examples. In Section 3, we prove Theorem 1.8. Finally, in Section 4 we shall offer an interpretation of our results to resistive electrical circuits and argue that the most reasonable interpretation is that edge weight should be regarded as the cross–sectional diameter of the wire (cable) joining the two circuits.

2. Motivation and Examples. As a motivation for the general question concerning joining two trees with an edge of infinite weight, let us look at the complete balanced binary tree T on k levels and hence with $n = 2^k - 1$ vertices. Label the vertices of the tree so that the root vertex is vertex 1, and the immediate successors of vertex j are always $2j$ and $2j+1$, $j = 1, ..., 2^{k-1} - 1$. Hence the Laplacian matrix for T is

$$L = \left[\begin{array}{cccccccccc} 2 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\ -1 & 3 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & \ldots \\ -1 & 0 & 3 & 0 & 0 & -1 & -1 & 0 & 0 & \ldots \\ 0 & -1 & 0 & & & & & & & \\ 0 & -1 & 0 & & & & & & & \\ 0 & 0 & -1 & & & C & & & & \\ 0 & 0 & -1 & & & & & & & \\ 0 & 0 & 0 & & & & & & & \\ 0 & 0 & 0 & & & & & & & \\ \vdots & \vdots & \vdots & & & & & & & \end{array}\right],$$

where C is the appropriate $(n-3) \times (n-3)$ matrix. The Schur complement of a matrix L with respect to $\ell_{1,1}$ is the $(n-1) \times (n-1)$ matrix given by

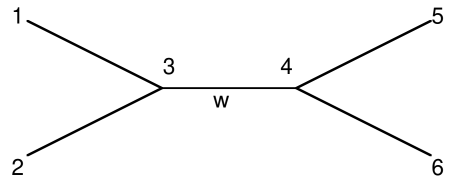
$$\begin{aligned} L/\ell_{1,1} &= L_1 - \frac{1}{\ell_{1,1}} L\left[\{1,\ldots,n-1\},\{1\}\right] L\left[\{1\},\{1,\ldots,n-1\}\right] \\ &= \left[\begin{array}{ccccccccc} 2.5 & -0.5 & -1 & -1 & 0 & 0 & 0 & 0 & \ldots \\ -0.5 & 2.5 & 0 & 0 & -1 & -1 & 0 & 0 & \ldots \\ -1 & 0 & & & & & & & \\ -1 & 0 & & & & & & & \\ 0 & -1 & & & C & & & & \\ 0 & -1 & & & & & & & \\ 0 & 0 & & & & & & & \\ 0 & 0 & & & & & & & \\ \vdots & \vdots & & & & & & & \end{array}\right]. \end{aligned}$$

It is easily observed that $L/\ell_{1,1}$ is, actually, the Laplacian matrix of two complete balanced binary trees on $k-1$ levels that are joined at the root vertices by an edge of weight $w = 1/2$. Call this tree T'.

Continuing, if x is an eigenvector of L such that $x_1 = 0$, then it follows that $x_2 = -x_3$. Moreover, $x' := [\, x_2 \; x_3 \; \ldots \; x_n \,]^T$ is an eigenvector of $L/\ell_{1,1}$ corresponding to the same eigenvalue. Since T is symmetric about the root vertex, by Theorems 1.2 and 1.4, T is a Type I tree with 1 as its characteristic vertex. Thus any eigenvector x corresponding to $\mu(T)$, the algebraic connectivity of T, has 0 as its first coordinate showing that x' is an eigenvector of $L/\ell_{1,1}$ corresponding to μ. By the interlacing theorem of eigenvalues, it follows that $\mu(T)$ is the second smallest eigenvalue of $L/\ell_{1,1}$ and so the algebraic connectivity of T is equal to the algebraic connectivity of T'.

Above we have shown the effect on algebraic connectivity when we join the root vertices of two complete, unweighted, balanced, binary trees on $k-1$ levels by an edge of weight $w = 1/2$. It is tempting to ask what happens to the algebraic connectivity of the two balanced binary trees on the $k-1$ levels if we modify the weight w and, in fact, let $w \to \infty$. This question served as a motivation for the more general question considered in this paper: *Given two arbitrary trees, T_1 and T_2 which do not necessarily have the same number of vertices, with algebraic connectivities $\mu(T_1)$ and $\mu(T_2)$, respectively, we join them to create a new tree $\hat{T}_w$ by connecting a vertex x of T_1 to a vertex y of T_2 by an edge e of weight $w > 0$. Then what happens to the algebraic connectivity of the new tree as $w \to \infty$?* We give two examples. In the first example, the algebraic connectivity of the new tree attains the minimum of the algebraic connectivity of the component trees as the weight tends to infinity, while in the second example, that minimum is never attained.

EXAMPLE 2.1. Suppose that $\hat{T}_w$ is the tree on 6 vertices which has been obtained by joining the *center vertices* of two unweighted star trees T_1 and T_2 on 3 vertices by an edge of of positive weight w:



The Laplacian of $\hat{T}_w$ is given by

$$L(\hat{T}_w) = \begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ -1 & -1 & 2+w & -w & 0 & 0 \\ 0 & 0 & -w & 2+w & -1 & -1 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \end{bmatrix}.$$

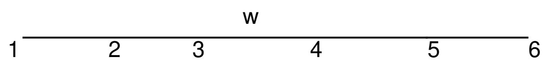
Upon computing the eigenvalues of $\hat{T}_w$ we find that

$$\mu\left(\hat{T}_w\right) = 3/2 + w - \frac{1}{2}\sqrt{9 + 4\,w + 4\,w^2}.$$

It is not difficult to ascertain that $\mu\left(\hat{T}_w\right)$ is a strictly increasing function of w and

$$\lim_{w\to\infty} \mu\left(\hat{T}_w\right) = 1 = \mu(T_1) = \mu(T_2).$$

EXAMPLE 2.2. This time suppose that $\hat{T}_w$ is obtained by connecting *pendant* vertices of two unweighted path trees T_1 and T_2 on 3 vertices by an edge of weight w:



The Laplacian of the new tree is

$$L\left(\hat{T}_w\right) = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 1+w & -w & 0 & 0 \\ 0 & 0 & -w & 1+w & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}.$$

We already know that $\mu(T_1) = \mu(T_2) = 1$. On the other hand, by the interlacing eigenvalues for symmetric matrices (see [4]), the second smallest eigenvalue of the 4×4 principal submatrix of $L\left(\hat{T}\right)$ determined by rows and columns $1, 2, 5$, and 6, which equals $(1/2)\left(3-\sqrt{5}\right) \approx .3820$, majorizes $\mu\left(\hat{T}\right)$ for all $w > 0$. Thus, in the present example, the algebraic connectivity of $\hat{T}_w$ never attains the algebraic connectivity of its building trees T_1 and T_2.

3. Proof of Theorem 1.8. We begin with two preliminary claims. The first may be regarded as essentially known, but whose proof is included here for the sake of completeness.

CLAIM 3.1. *Let T_1 and T_2 be trees with algebraic connectivities $\mu(T_1)$ and $\mu(T_2)$ respectively. Suppose that the tree $\hat{T}_w$ is obtained by joining a vertex in T_1 to a vertex in T_2 by an edge of weight $w > 0$. Then*

$$\mu(\hat{T}_w) \ \le \ \min\{\mu(T_1), \mu(T_2)\}. \tag{4}$$

Proof. Let L_1 and L_2 be the Laplacian matrices for T_1 and T_2, respectively. Form the 2×2 block matrix L with the $(1,1)$–block and the $(2,2)$–block being L_1 and L_2, respectively, and with the corner blocks being zero matrices of the appropriate sizes. Thus $\lambda_1(L) = \lambda_2(L) = 0$, while $\lambda_3(L) = \min\{\mu(T_1), \mu(T_2)\}$. Now let $\hat{L}_w$ be the Laplacian matrix of $\hat{T}$. As $(\hat{L}_w)$ is a rank 1 perturbation of L, by [5, Theorem 4.3.4(b)], we have that

$$\mu(\hat{T}_w) \ = \ \lambda_2\left(L\left(\hat{T}_w\right)\right) \ \le \ \lambda_3(L) \ = \ \min\{\mu(T_1), \mu(T_2)\}. \qquad \square$$

The next claim will be useful in proving the three subsequent lemmas.

CLAIM 3.2. *Let T be a tree. Let i and j be adjacent vertices in T joined by an edge of weight α_1; let p and q be adjacent vertices in T joined by an edge of weight α_2. If the path from i to q contains j and p, and if $0 < \gamma_1, \gamma_2 < 1$, then*

$$\rho\left(M_{(j-i),T} - \frac{\gamma_1}{\alpha_1}J\right) \ < \ \rho\left(M_{(j-i),T}\right) \ < \ \rho\left(M_{(q-p),T} - \frac{\gamma_2}{\alpha_2}J\right) \ < \ \rho\left(M_{(q-p),T}\right).$$

Proof. The first and third inequality follow from

$$M_{(j-i),T} - \frac{\gamma_1}{\alpha_1}J \ < \ M_{(j-i),T}\,, \quad M_{(q-p),T} - \frac{\gamma_2}{\alpha_2}J \ < \ M_{(q-p),T} \tag{5}$$

and the fact that all matrices in (5) are nonnegative and irreducible.

To prove the second inequality, let $(j-i)_T$ denote the subgraph of T that consists of j and the branch at j containing i. Likewise, let $(q-p)_T$ denote the subgraph of T consisting of q as well as its branch containing p. Let β be the set of indices of $M_{(q-p),T}$ that correspond to the vertices that determine $M_{(j-i),T}$ according to (2). Since $(j-i)_T$ is a subgraph of $(q-p)_T$ and since p and q are joined by an edge of weight α_2, it is clear from (2) that

$$M_{(j-i),T} \ \leq \ M_{(q-p),T}\,[\beta] - \frac{1}{\alpha_2}J.$$

Since all the remaining entries of $M_{(q-p),T} - (1/\alpha_2)J$ are positive, it follows that

$$\rho\left(M_{(j-i),T}\right) \ \leq \ \rho\left(M_{(q-p),T} - \frac{1}{\alpha_2}J\right).$$

But as

$$\rho\left(M_{(q-p),T} - \frac{1}{\alpha_2}J\right) \ < \ \rho\left(M_{(q-p),T} - \frac{\gamma_2}{\alpha_2}J\right),$$

the second inequality in this claim thus follows. □

We are now ready to begin examining three possible cases of joining two trees by an edge whose weight tends to infinity. Recall that our aim is to determine precisely when equality in (4) holds. We start with three useful lemmas. The first lemma examines the case when one of the building trees, say T_1, is of Type II. In this lemma, θ_1 will denote the weight of the edge joining the characteristic vertices of T_1, and $\gamma_1 \in (0,1)$ will denote the value of γ referred to in Theorem 1.6(b).

LEMMA 3.3. *Let T_1 be of Type II and let $\hat{T}_w$ be obtained from T_1 and T_2 by joining a vertex x of T_1 to a vertex y of T_2 by an edge of weight w. Then $\mu(\hat{T}) < \mu(T_1)$.*

Proof. Let i and j be the characteristic vertices of T_1 such that the path from i to x contains j. Suppose $\hat{T}$ is of Type I with v as its characteristic vertex. Then, by Theorem 1.7, v lies on the path from i to any characteristic vertex of T_2. Therefore,

$$\begin{aligned}\mu(\hat{T}) \ &= \ \lim_{w\to\infty}\mu(\hat{T}_w) \ = \ \lim_{w\to\infty}\frac{1}{\rho(M_{(v-i),\hat{T}_w})}\\ &\leq \ \frac{1}{\rho(M_{(j-i),T_1})} \ < \ \frac{1}{\rho(M_{(j-i),T_1} - \frac{\gamma_1}{\theta_1}J)} \ = \ \mu(T_1).\end{aligned}$$

Suppose now that $\hat{T}$ is of Type II. Let p and q be the characteristic vertices of $\hat{T}$ and let p be on the path from q to i. Then

$$\begin{aligned}\mu(\hat{T}) \ &= \ \lim_{w\to\infty}\mu(\hat{T}_w) \ = \ \lim_{w\to\infty}\frac{1}{\rho(M_{(q-p),\hat{T}_w} - \frac{\gamma_w}{\theta}J)}\\ &\leq \ \frac{1}{\rho(M_{(j-i),T_1} - \frac{\gamma_1}{\theta_1}J)} \ = \ \mu(T_1), \end{aligned} \tag{6}$$

where the inequality follows from the fact that $\mu(\hat{T}) \le \mu(T_1)$. However, according to Claim 3.2 the inequality in (6) is strict. Thus $\mu(\hat{T}) < \mu(T_1)$ whenever T_1 is a Type II tree. □

The next two lemmas examine the relationship between $\mu(\hat{T})$ and $\mu(T_1)$ when T_1 is of Type I. The first of these lemmas concerns the case when T_1 and T_2 are joined by an edge e of weight w, where e is not incident to the characteristic vertex of T_1.

LEMMA 3.4. *Suppose $\mu(T_1) \le \mu(T_2)$ and T_1 is of Type I with characteristic vertex i. Let $\hat{T}_w$ be obtained from T_1 and T_2 by joining a vertex x of T_1 to a vertex y of T_2 by an edge e of weight w. If e is not incident to i, then $\mu(\hat{T}) = \mu(T_1)$ if and only if i is the only characteristic vertex of $\hat{T}$.*

Proof. Suppose $\hat{T}$ is of Type I with characteristic vertex v. It follows from Theorem 1.7 that v lies on the path from i to any characteristic vertex of T_2. Thus

$$\mu(\hat{T}) \;=\; \lim_{w\to\infty} \mu(T_w) \;=\; \lim_{w\to\infty} \frac{1}{\rho(M_{v,\hat{T}_w})} \;=\; \frac{1}{\rho(M_{v,\hat{T}})} \;\le\; \frac{1}{\rho(M_{i,T_1})} \;=\; \mu(T_1),$$

where the inequality follows from the fact that $\mu(\hat{T}) \le \mu(T_1)$. Since e is not incident to i equality holds if and only if $v = i$.

Now suppose that $\hat{T}$ is of Type II with characteristic vertices p and q with p on the path from q to i. Therefore,

$$\begin{aligned} \mu(\hat{T}) \;&=\; \lim_{w\to\infty} \mu(\hat{T}_w) \;=\; \lim_{w\to\infty} \frac{1}{\rho(M_{(q-p),\hat{T}_w} - \frac{\gamma_w}{\theta} J)} \\ &=\; \frac{1}{\rho(M_{(q-p),\hat{T}} - \frac{\gamma}{\theta} J)} \;\le\; \frac{1}{\rho(M_{i,T_1})} \;=\; \mu(T_1), \end{aligned} \tag{7}$$

where, again, the inequality follows from $\mu(\hat{T}) \le \mu(T_1)$. However, by Claim 3.2,

$$\rho\left(M_{i,T_1}\right) \;<\; \rho\left(M_{(q-p),\hat{T}} - \frac{\gamma}{\theta} J\right).$$

Thus the inequality in (7) is strict. Hence $\mu(\hat{T}) < \mu(T_1)$ if $\hat{T}$ has more than one characteristic vertex. □

We see from Lemma 3.4 that in order for $\mu(\hat{T})$ to equal $\mu(T_1)$ when T_1 is a Type I tree, the characteristic vertex of T_1 must be the only characteristic vertex of $\hat{T}$. Since y is the vertex in T_2 that is incident to e, we see that $\rho(M_{y,T_2})$ must be small.

The following lemma examines the relationship between $\mu(\hat{T})$ and $\mu(T_1)$ when e is incident to the characteristic vertex of T_1.

LEMMA 3.5. *Suppose $\mu(T_1) \le \mu(T_2)$ and T_1 is of Type I with characteristic vertex i. Let $\hat{T}_w$ be obtained from T_1 and T_2 by joining i to a vertex u of T_2 by an edge e of weight w. Then $\mu(\hat{T}) = \mu(T_1)$ if and only if $\rho(M_{u,T_2}) \le 1/\mu(T_1)$.*

Proof. We shall consider three cases.

Case I: If $\rho(M_{u,T_2}) < 1/\mu(T_1)$, then i is the unique vertex in $\hat{T}$ that has more than one Perron branch and so $\hat{T}$ is of Type I and $\mu(\hat{T}) = 1/\mu(T_1) = \mu(T_1)$.

Case II: If $\rho(M_{u,T_2}) = 1/\mu(T_1)$, then as $w \to \infty$, it follows from (2) that

$$\rho(M_{(i-u),\hat{T}}) \;=\; \rho(M_{(u-i),\hat{T}}) \;=\; 1/\mu(T_1).$$

Hence $\hat{T}$ is of Type II with i and u as its characteristic vertices. Thus

$$\mu(\hat{T}) \;=\; \lim_{w\to\infty}\mu(\hat{T}_w) \;=\; \lim_{w\to\infty}\frac{1}{\rho(M_{(i-u),\hat{T}_w}-\frac{\gamma_w}{w}J)} \;=\; \frac{1}{\rho(M_{(i-u),\hat{T}})} \;=\; \mu(T_1).$$

Case III: If $\rho(M_{u,T_2}) > 1/\mu(T_1)$, then by Theorem 1.7 the characteristic vertex (or vertices) of $\hat{T}$ lies in T_2. If $\hat{T}$ is of Type I with characteristic vertex v, then

$$\mu(\hat{T}) \;=\; \frac{1}{\rho(M_{v,\hat{T}})} \;<\; \frac{1}{\rho(M_{i,T_1})} \;=\; \mu(T_1).$$

Now if $\hat{T}$ is of Type II with characteristic vertices p and q where p lies on the path from q to i, then

$$\begin{aligned}\mu(\hat{T}) \;&=\; \lim_{w\to\infty}\mu(\hat{T}_w) \;=\; \lim_{w\to\infty}\frac{1}{\rho(M_{(q-p),\hat{T}_w}-\frac{\gamma_w}{\theta}J)}\\ &=\; \frac{1}{\rho(M_{(q-p),\hat{T}}-\frac{\gamma}{\theta}J)} \;<\; \frac{1}{\rho(M_{i,T_1})} \;=\; \mu(T_1),\end{aligned} \tag{8}$$

where the strict inequality is due to Claim 3.2. The conclusion of this lemma now follows. □

The results we have proved in this section permit us now to proceed to the proof of the main result of this paper.

Proof of Theorem 1.8. Without loss of generality, let $\mu(T_1) \le \mu(T_2)$. (Hence $\min\{\mu(T_1),\mu(T_2)\} = \mu(T_1)$.) The necessity of T_1 being a Type I tree in order for $\mu(\hat{T})$ to equal $\mu(T_1)$ follows from Lemma 3.3. Let i be the characteristic vertex of T_1. If e is not incident to i, then by Lemma 3.4, $\mu(\hat{T}) = \mu(T_1)$ if and only if i is the characteristic vertex of $\hat{T}$. If e is incident to i, then by Lemma 3.5 $\mu(\hat{T}) = \mu(T_1)$ if and only if $\rho(M_{u,T_2}) \le 1/\mu(T_1)$. □

REMARK 3.6. As a closing remark suppose that equality holds in (4), namely, the algebraic connectivity of the tree $\hat{T}$ at infinity is equal to the minimum of the algebraic connectivities of T_1 and T_2. Then it is possible to determine precisely what type of tree we obtain for $\hat{T}$. Assume without loss of generality that $\mu(\hat{T}) = \mu(T_1) \le \mu(T_2)$. Then by Lemma 3.3, T_1 is a Type I tree. Let i be the characteristic vertex of T_1. If i is incident to e, then by Lemma 3.5, $\hat{T}$ is of Type I if $\rho(M_{u,T_2}) < 1/\mu(T_1)$, while $\hat{T}$ is of Type II if $\rho(M_{u,T_2}) = 1/\mu(T_1)$, where u is the vertex in T_2 which is incident to e. Finally, if i is not incident to e, then by Lemma 3.4, i is the only characteristic vertex of $\hat{T}$ and thus $\hat{T}$ is of Type I.

4. An Application to Electrical Circuits. The amount of energy per unit charge moving between two surfaces is a measure of the *difference of potential* between the two surfaces. This is measured in *volts.* Let the surfaces be two adjacent vertices in our tree. Let V denote the voltage measured in volts, I denote the current measured in amperes, and R denote the resistance measured in ohms. According to Ohm's Law, the voltage required to send a current through the resistance between two surfaces (in our case, vertices) through a conductor (in our case, an edge) is equal to the product of the current and the resistance. In other words,

$$V = IR.$$

Therefore, if the resistance is infinite, then it is impossible to get enough voltage to send any amount of current through the conductor. The *voltage drop* is the voltage required to send current through a conductor in which no effect other than resistance is present. According to Kirchoff, around any continuous path in a closed electrical circuit, the algebraic sum of the voltage drop is zero. For references see [10, 11].

Taking a closer look at resistance, suppose we have a conductor and we increase its length. Then it will take a larger voltage drop to send the current through the conductor. Hence the resistance increases as the length of our conductor increases. By similar reasoning, if we take our conductor and increase the cross–sectional area, then we have more cross–sectional area for the current to flow. Thus we need less voltage drop in order to push our current through the conductor. This means that resistance lessens as our cross–sectional increases. Finally, some materials are more resistant to conducting electricity than others. Let ρ represent the constant of how resistant the material is to conducting electricity. The greater ρ is, the more resistant to conducting electricity the particular conductor. Therefore, resistance is proportional to both the length of the conductor and the constant ρ associated with the conductor, while it is inversely proportional to the cross–sectional area of the conductor. Hence

$$R = \rho \frac{\ell}{a}, \tag{9}$$

where ℓ is the length and a is the cross–sectional area of the conductor.

From both a graph–theoretical and resistive networks points of view, we can say the following: Suppose we have two trees, T_1 and T_2, and we create a new tree, $\hat{T}$ by joining a vertex $x \in T_1$ to a vertex $y \in T_2$ by an edge e whose weight tends to infinity. Immediately, one can think of two possible interpretations for e. If the weight of an edge is considered to be the length of the conductor, then current will be unable to flow from T_1 to T_2 as the weight of e approaches infinity. However, if the weight of an edge is considered to be the cross–sectional area of the conductor, then it would take zero voltage drop to push the current through e as its weight approaches infinity. It is as if vertex x of T_1 was being identified with vertex y of T_2. We shall next see that interpreting edge weight as the cross–sectional area is the more reasonable interpretation.

Recall that if $\mathcal{G}$ is a graph on n vertices and if $\mu(\mathcal{G}) = 0$, then the graph is not connected and therefore is the union of more than one connected component. Hence, current cannot necessarily flow from one vertex to another in $\mathcal{G}$. This implies that there is infinite resistance in current flow between some pairs of vertices, namely pairs of vertices that come from different components of $\mathcal{G}$. Likewise, if $\mathcal{G}$ is the complete graph on n vertices, then $\mathcal{G}$ has more edges than any other graph on n vertices thus enabling current to flow more easily in $\mathcal{G}$ than in any other graph on n vertices. Consequently, it appears that if the algebraic connectivity of a graph is decreased, the more resistant the graph is to conducting electricity.

Recalling (9), we need to determine if we should regard edge weight as ℓ, the length of the conductor, or a, the cross–sectional area of the conductor. Supposing that $\mu(T_k) > 0,\ k = \{1, 2\}$, we know by Lemma 3.1 that $\mu(\hat{T}) \leq \min\{\mu(T_1), \mu(T_2)\}$. Since increasing the weight of e without bound only increases the algebraic connectivity of $\hat{T}$, we see that $\mu(\hat{T}) > 0$. Therefore, since increasing the weight of e without bound does not cause $\hat{T}$ to be disconnected, it follows that we do not want to assign edge weight to a variable in (9) that will cause the resistance R to equal infinity. Therefore, we should not regard the edge weight as ℓ, for otherwise the resistance in $\hat{T}$ would be infinite, which is clearly not the case since $\mu(\hat{T}) > 0$. If we interpret edge weight to be a, the cross–sectional area of the conductor, then the resistance R of the current flowing in $\hat{T}$ from vertex x of T_1 to vertex y of T_2 is 0. Hence the current is flowing completely freely from x to y in that it takes zero voltage drop to send the current through e. Therefore, in terms of current flow, it is as if we identify vertex x with vertex y. But since the entries of bottleneck matrices are determined by the reciprocals of edge weights, identifying vertex x with vertex y is precisely what we do when we construct bottleneck matrices. Since the algebraic connectivity of $\hat{T}$ is determined by bottleneck matrices, it appears that interpreting edge weight as the cross–sectional area a is the more reasonable point of view.

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