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HOW TO ESTABLISH UNIVERSAL BLOCK-MATRIX FACTORIZATIONS*

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Abstract. A general method is presented for establishing universal factorization equalities for 2×2 and 4×4 block matrices. As applications, some universal factorization equalities for matrices over four-dimensional algebras are established, in particular, over the Hamiltonian quaternion algebra.

Key words. Block matrix, Determinant, Quaternion matrices, Universal factorization, Involutory matrices, Idempotent matrices.

AMS subject classifications. 15A23, 15A33

1. Introduction. We consider in this article how to establish universal factorization equalities for block matrices. This work is motivated by the following two well-known factorizations of 2×2 block matrices (see, e.g., [4, 6])

$$\begin{bmatrix} A & B \\ B & A \end{bmatrix} = Q_m^{(2)} \begin{bmatrix} A+B & 0 \\ 0 & A-B \end{bmatrix} Q_n^{(2)}, \ Q_t^{(2)} = \frac{1}{\sqrt{2}} \begin{bmatrix} I_t & I_t \\ I_t & -I_t \end{bmatrix}, \ t = m,\, n, \tag{1.1}$$

$$\begin{bmatrix} A & -B \\ B & A \end{bmatrix} = P_m^{(2)} \begin{bmatrix} A+iB & 0 \\ 0 & A-iB \end{bmatrix} P_n^{(2)}, \ P_t^{(2)} = \frac{1}{\sqrt{2}} \begin{bmatrix} I_t & iI_t \\ -iI_t & -I_t \end{bmatrix}, \ t = m,\, n. \tag{1.2}$$

Obviously $P_t^{(2)} = (P_t^{(2)})^{-1}$ and $Q_t^{(2)} = (Q_t^{(2)})^{-1}$ $(t = m,\, n)$, and they are independent of A and B. Thus (1.1) and (1.2) could be called universal factorization equalities for block matrices. From these two equalities we find many useful consequences, such as

$$\operatorname{rank}\begin{bmatrix} A & B \\ B & A \end{bmatrix} = \operatorname{rank}(A+B) + \operatorname{rank}(A-B),$$

$$\det\begin{bmatrix} A & -B \\ B & A \end{bmatrix} = \det(A+iB)\det(A-iB),$$

$$(A+iB)^\dagger = \frac{1}{2}[\, I_n,\ iI_n \,] \begin{bmatrix} A & -B \\ B & A \end{bmatrix}^\dagger \begin{bmatrix} I_m \\ -iI_m \end{bmatrix},$$

and so on. Here $(\cdot)^\dagger$ denotes the Moore–Penrose inverse.

In this article we shall present a general method for establishing such universal factorizations, and then establish some useful factorization equalities for block matrices.

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2. The 2×2 case. We begin with a simple but essential problem on similarity of a matrix. Suppose $P=\begin{bmatrix} p_1 & p_2 \\ p_3 & p_4 \end{bmatrix}$ is a given nonsingular matrix over an arbitrary field $\mathbb{F}$, i.e., $|P| = p_1p_4 - p_2p_3 \neq 0$. Then find the general expression of X such that PXP^{-1} is a diagonal matrix. The answer is trivially obvious: $X = P^{-1}\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}P$, where $\lambda_1,\ \lambda_2 \in \mathbb{F}$ are arbitrary, that is,

$$X = P^{-1}\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}P = \frac{1}{|P|}\begin{bmatrix} p_4 & -p_2 \\ -p_3 & p_1 \end{bmatrix}\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}\begin{bmatrix} p_1 & p_2 \\ p_3 & p_4 \end{bmatrix}. \tag{2.1}$$

The motivation for us to consider (2.1) is that if we replace λ_1 and λ_2 by some particular expressions, or if P in (2.1) has some special form, then X in (2.1) will have some nice expressions. All of our subsequent results are in fact derived from this consideration. From (2.1), we can derive the following theorem.

THEOREM 2.1. *Let $p_1, \ldots, p_4 \in \mathbb{F}$ with $p_1p_4 \neq p_2p_3$ be given. Then for any $a,\ b \in \mathbb{F}$, the following two equalities hold*

$$\begin{bmatrix} a & -p_2p_4b \\ p_1p_3b & a + (\,p_1p_4 + p_2p_3\,)b \end{bmatrix} = \begin{bmatrix} p_1 & p_2 \\ p_3 & p_4 \end{bmatrix}^{-1}\begin{bmatrix} a + p_2p_3b & 0 \\ 0 & a + p_1p_4b \end{bmatrix}\begin{bmatrix} p_1 & p_2 \\ p_3 & p_4 \end{bmatrix}, \tag{2.2}$$

$$\begin{bmatrix} a - p_2p_3b & -p_2p_4b \\ p_1p_3b & a + p_1p_4b \end{bmatrix} = \begin{bmatrix} p_1 & p_2 \\ p_3 & p_4 \end{bmatrix}^{-1}\begin{bmatrix} a & 0 \\ 0 & a - (\,p_2p_3 - p_1p_4\,)b \end{bmatrix}\begin{bmatrix} p_1 & p_2 \\ p_3 & p_4 \end{bmatrix}. \tag{2.3}$$

In particular, if $p_1p_4 = -p_2p_3 \neq 0$, then

$$\begin{bmatrix} a & -p_2p_4b \\ p_1p_3b & a \end{bmatrix} = \begin{bmatrix} p_1 & p_2 \\ p_3 & p_4 \end{bmatrix}^{-1}\begin{bmatrix} a + p_2p_3b & 0 \\ 0 & a - p_2p_3b \end{bmatrix}\begin{bmatrix} p_1 & p_2 \\ p_3 & p_4 \end{bmatrix}. \tag{2.4}$$

If $p \neq 0$, then

$$\begin{bmatrix} a & p^2b \\ b & a \end{bmatrix} = \begin{bmatrix} 1 & p \\ p^{-1} & -1 \end{bmatrix}^{-1}\begin{bmatrix} a + pb & 0 \\ 0 & a - pb \end{bmatrix}\begin{bmatrix} 1 & p \\ p^{-1} & -1 \end{bmatrix}. \tag{2.5}$$

If $p,\ q \in \mathbb{C}$, the field of complex numbers, with $pq \neq 0$, then

$$\begin{bmatrix} a & pb \\ qb & a \end{bmatrix} = \begin{bmatrix} 1 & \sqrt{\frac{p}{q}} \\ \sqrt{\frac{q}{p}} & -1 \end{bmatrix}^{-1}\begin{bmatrix} a + \sqrt{pq}\,b & 0 \\ 0 & a - \sqrt{pq}\,b \end{bmatrix}\begin{bmatrix} 1 & \sqrt{\frac{p}{q}} \\ \sqrt{\frac{q}{p}} & -1 \end{bmatrix}. \tag{2.6}$$

Proof. Substituting $\lambda_1 = a + p_2p_3b$ and $\lambda_2 = a + p_1p_4b$ in (2.1) yields (2.2) and then substituting $\lambda_1 = a$ and $\lambda_2 = a - (\,p_2p_3 - p_1p_4b\,)$ in (2.1) yields (2.3). The factorizations (2.4)–(2.6) are direct consequences of (2.2). $\square$

3. The 2×2 and 4×4 block cases. The factorization equalities in Theorem 2.1 can be easily extended to the 2×2 block case. In fact, replacing a and b in (2.2) with two $m\times n$ matrices A and B, p_i with p_iI_m and $p_iI_n (i = 1,\ 2,\ 3,\ 4)$, respectively, we immediately have the following theorem.

THEOREM 3.1. *Let $p_1, \ldots, p_4 \in \mathbb{F}$ be given with $p_1p_4 \neq p_2p_3$, and $A,\ B \in \mathbb{F}^{m\times n}$. Then*

$$\begin{bmatrix} A & -p_2p_4B \\ p_1p_3B & A+(p_1p_4+p_2p_3)B \end{bmatrix} = \begin{bmatrix} p_1I_m & p_2I_m \\ p_3I_m & p_4I_m \end{bmatrix}^{-1} \begin{bmatrix} A+p_2p_3B & 0 \\ 0 & A+p_1p_4B \end{bmatrix} \begin{bmatrix} p_1I_n & p_2I_n \\ p_3I_n & p_4I_n \end{bmatrix}. \tag{3.1}$$

If $p_1p_4 = -p_2p_3 \neq 0$, then

$$\begin{bmatrix} A & -p_2p_4B \\ p_1p_3B & A \end{bmatrix} = \begin{bmatrix} p_1I_m & p_2I_m \\ p_3I_m & p_4I_m \end{bmatrix}^{-1} \begin{bmatrix} A+p_2p_3B & 0 \\ 0 & A-p_2p_3B \end{bmatrix} \begin{bmatrix} p_1I_n & p_2I_n \\ p_3I_n & p_4I_n \end{bmatrix}. \tag{3.2}$$

If $p \neq 0$, then

$$\begin{bmatrix} A & p^2B \\ B & A \end{bmatrix} = \begin{bmatrix} I_m & pI_m \\ p^{-1}I_m & -I_m \end{bmatrix}^{-1} \begin{bmatrix} A+pB & 0 \\ 0 & A-pB \end{bmatrix} \begin{bmatrix} I_n & pI_n \\ p^{-1}I_n & -I_n \end{bmatrix}. \tag{3.3}$$

If $p,\ q \in \mathbb{C}$ with $pq \neq 0$, then

$$\begin{bmatrix} A & pB \\ qB & A \end{bmatrix} = \begin{bmatrix} I_m & \sqrt{\frac{p}{q}}\,I_m \\ \sqrt{\frac{q}{p}}\,I_m & -I_m \end{bmatrix}^{-1} \begin{bmatrix} A+\sqrt{pq}\,B & 0 \\ 0 & A-\sqrt{pq}\,B \end{bmatrix} \begin{bmatrix} I_n & \sqrt{\frac{p}{q}}\,I_n \\ \sqrt{\frac{q}{p}}\,I_n & -I_n \end{bmatrix}. \tag{3.4}$$

Clearly, the two equalities in (1.1) and (1.2) are special cases of (3.3) when $p = 1$ and $p^2 = -1$, respectively. Without much effort, we can extend the universal block matrix factorizations in (3.1)–(3.4) to 4×4 block matrices. For simplicity, we only give the extension of (3.3) and its consequences.

THEOREM 3.2. *Let $A_0, \ldots, A_3 \in \mathbb{F}^{m\times n}$ be given, and $\lambda,\ \mu \in \mathbb{F}$ with $\lambda\mu \neq 0$. Then they satisfy the universal factorization equality*

$$A := \begin{bmatrix} A_0 & \lambda^2A_1 & \mu^2A_2 & \mu^2\lambda^2A_3 \\ A_1 & A_0 & \mu^2A_3 & \mu^2A_2 \\ A_2 & \lambda^2A_3 & A_0 & \lambda^2A_1 \\ A_3 & A_2 & A_1 & A_0 \end{bmatrix} = Q_m^{(4)} \begin{bmatrix} N_1 & & & \\ & N_2 & & \\ & & N_3 & \\ & & & N_4 \end{bmatrix} Q_n^{(4)}, \tag{3.5}$$

where

$$N_1 = A_0 + \lambda A_1 + \mu A_2 + \mu\lambda A_3, \qquad N_2 = A_0 - \lambda A_1 + \mu A_2 - \mu\lambda A_3,$$
$$N_3 = A_0 + \lambda A_1 - \mu A_2 - \mu\lambda A_3, \qquad N_4 = A_0 - \lambda A_1 - \mu A_2 + \mu\lambda A_3,$$
$$Q_t^{(4)} = (Q_t^{(4)})^{-1} = \frac{1}{2}\begin{bmatrix} I_t & \lambda I_t & \mu I_t & \lambda\mu I_t \\ \lambda^{-1}I_t & -I_t & \lambda^{-1}\mu I_t & -\mu I_t \\ \mu^{-1}I_t & \lambda\mu^{-1}I_t & -I_t & -\lambda I_t \\ (\lambda\mu)^{-1}I_t & -\mu^{-1}I_t & -\lambda^{-1}I_t & I_t \end{bmatrix}, \quad t = m,\ n.$$

In particular, when $m = n$, (3.5) *becomes a universal similarity factorization equality over $\mathbb{F}$.*

Proof. Let

$$E = \begin{bmatrix} A_0 & \lambda^2 A_1 \\ A_1 & A_0 \end{bmatrix}, \qquad F = \begin{bmatrix} A_2 & \lambda^2 A_3 \\ A_3 & A_2 \end{bmatrix}.$$

Then by (3.3), we first obtain

$$\begin{bmatrix} E & \mu^2 F \\ F & E \end{bmatrix} = S_m^{(4)} \begin{bmatrix} E + \mu F & 0 \\ 0 & E - \mu F \end{bmatrix} S_n^{(4)}, \tag{3.6}$$

where

$$S_t^{(4)} = (S_t^{(4)})^{-1} = \frac{1}{\sqrt{2}} \begin{bmatrix} I_{2t} & \mu I_{2t} \\ \mu^{-1} I_{2t} & -I_{2t} \end{bmatrix}, \qquad t = m,\ n,$$

and

$$E + \mu F = \begin{bmatrix} A_0 + \mu A_2 & \lambda^2 (\, A_1 + \mu A_3\,) \\ A_1 + \mu A_3 & A_0 + \mu A_2 \end{bmatrix}, \ E - \mu F = \begin{bmatrix} A_0 - \mu A_2 & \lambda^2 (\, A_1 - \mu A_3\,) \\ A_1 - \mu A_3 & A_0 - \mu A_2 \end{bmatrix}.$$

Applying (3.3) to $E \pm \mu F$, we further have

$$\begin{aligned} E + \mu F &= P_m^{(2)} \begin{bmatrix} A_0 + \lambda A_1 + \mu A_2 + \mu\lambda A_3 & 0 \\ 0 & A_0 - \lambda A_1 + \mu A_2 - \mu\lambda A_3 \end{bmatrix} P_n^{(2)} \\ &= P_m^{(2)} \begin{bmatrix} N_1 & 0 \\ 0 & N_2 \end{bmatrix} P_n^{(2)}, \end{aligned}$$

$$\begin{aligned} E - \mu F &= P_m^{(2)} \begin{bmatrix} A_0 + \lambda A_1 - \mu A_2 - \mu\lambda A_3 & 0 \\ 0 & A_0 - \lambda A_1 - \mu A_2 + \mu\lambda A_3 \end{bmatrix} P_n^{(2)} \\ &= P_m^{(2)} \begin{bmatrix} N_3 & 0 \\ 0 & N_4 \end{bmatrix} P_n^{(2)}, \end{aligned}$$

where

$$P_t^{(2)} = (P_t^{(2)})^{-1} = \frac{1}{\sqrt{2}} \begin{bmatrix} I_t & \lambda I_t \\ \lambda^{-1} I_t & -I_t \end{bmatrix}, \qquad t = m,\ n.$$

Thus (3.6) becomes

$$\begin{bmatrix} E & \mu^2 F \\ F & E \end{bmatrix} = S_m^{(4)} \begin{bmatrix} P_m^{(2)} & 0 \\ 0 & P_m^{(2)} \end{bmatrix} \begin{bmatrix} N_1 & & & \\ & N_2 & & \\ & & N_3 & \\ & & & N_4 \end{bmatrix} \begin{bmatrix} P_n^{(2)} & 0 \\ 0 & P_n^{(2)} \end{bmatrix} S_n^{(4)}. \tag{3.7}$$

Now let

$$Q_m^{(4)} = S_m^{(4)} \begin{bmatrix} P_m^{(2)} & 0 \\ 0 & P_m^{(2)} \end{bmatrix} \quad \text{and} \quad Q_n^{(4)} = \begin{bmatrix} P_n^{(2)} & 0 \\ 0 & P_n^{(2)} \end{bmatrix} S_n^{(4)}.$$

Then it is easy to verify that $(Q_m^{(4)})^{-1} = Q_m^{(4)}$ and $(Q_n^{(4)})^{-1} = Q_n^{(4)}$. Thus (3.7) is exactly (3.5). □

We now let $\lambda = \mu = 1$ in (3.5) and permute the second and third block rows and block columns of $\mathrm{diag}(\, N_1,\ N_2,\ N_3,\ N_4\,)$ in (3.5). Then we immediately obtain the following corollary.

COROLLARY 3.3. *Let $A_0,\ \ldots,\ A_3 \in \mathbb{F}^{m\times n}$ be given. Then they satisfy the universal factorization equality*

$$\begin{bmatrix} A_0 & A_1 & A_2 & A_3 \\ A_1 & A_0 & A_3 & A_2 \\ A_2 & A_3 & A_0 & A_1 \\ A_3 & A_2 & A_1 & A_0 \end{bmatrix} = Q_m^{(4)} \begin{bmatrix} N_1 & & & \\ & N_2 & & \\ & & N_3 & \\ & & & N_4 \end{bmatrix} Q_n^{(4)}, \tag{3.8}$$

where

$$N_1 = A_0 + A_1 + A_2 + A_3, \qquad N_2 = A_0 + A_1 - A_2 - A_3,$$
$$N_3 = A_0 - A_1 + A_2 - A_3, \qquad N_4 = A_0 - A_1 - A_2 + A_3,$$

$$Q_t^{(4)} = (Q_t^{(4)})^T = (Q_t^{(4)})^{-1} = \frac{1}{2}\begin{bmatrix} I_t & I_t & I_t & I_t \\ I_t & I_t & -I_t & -I_t \\ I_t & -I_t & I_t & -I_t \\ I_t & -I_t & -I_t & I_t \end{bmatrix}, \quad t = m,\ n. \tag{3.9}$$

In particular, when $m = n$, (3.8) *becomes a universal similarity factorization equality over $\mathbb{F}$.*

The equalities (3.5) and (3.8) can be used to establish various universal factorization equalities over quaternion algebras. We present the corresponding results in the next section.

4. Applications. We now present one of the most valuable applications of universal block-matrix factorizations, one which substantially reveals the relationship between real quaternion matrices and real block matrices.

THEOREM 4.1. *Let $A = A_0 + iA_1 + jA_2 + kA_3 \in \mathbb{H}^{m\times n}$, where $\mathbb{H}$ is the Hamilton real quaternion algebra, $A_0,\ \ldots,\ A_3 \in \mathbb{R}^{m\times n}$, $i^2 = j^2 = k^2 = -1$ and $ijk = -1$. Then A satisfies the universal factorization equality*

$$U_m^{(4)} \begin{bmatrix} A & & & \\ & A & & \\ & & A & \\ & & & A \end{bmatrix} U_n^{(4)} = \begin{bmatrix} A_0 & -A_1 & -A_2 & -A_3 \\ A_1 & A_0 & -A_3 & A_2 \\ A_2 & A_3 & A_0 & -A_1 \\ A_3 & -A_2 & A_1 & A_0 \end{bmatrix} := \phi(A), \tag{4.1}$$

where

$$U_t^{(4)} = (U_t^{(4)})^* = (U_t^{(4)})^{-1} = \frac{1}{2}\begin{bmatrix} I_t & iI_t & jI_t & kI_t \\ -iI_t & I_t & kI_t & -jI_t \\ -jI_t & -kI_t & I_t & iI_t \\ -kI_t & jI_t & -iI_t & I_t \end{bmatrix}, \quad t = m,\ n. \tag{4.2}$$

In particular, if $m = n$, *then* (4.1) *becomes a universal similarity factorization equality over* $\mathbb{H}$.

Proof. Observe that $Q_m^{(4)}$ and $Q_n^{(4)}$ in (3.8) only include identity blocks. Thus the equality (3.8) is also true for matrices over an arbitrary ring with identity. Now replacing $A_0,\ A_1,\ A_2,\ A_3$ in (3.8) by $A_0,\ iA_1,\ jA_2,\ kA_3$, respectively, we obtain the following

$$M := \begin{bmatrix} A_0 & iA_1 & jA_2 & kA_3 \\ iA_1 & A_0 & kA_3 & jA_2 \\ jA_2 & kA_3 & A_0 & iA_1 \\ kA_3 & jA_2 & iA_1 & A_0 \end{bmatrix} = Q_m^{(4)} \begin{bmatrix} N_1 & & & \\ & N_2 & & \\ & & N_3 & \\ & & & N_4 \end{bmatrix} Q_n^{(4)}, \tag{4.3}$$

where $Q_m^{(4)}$ and $Q_n^{(4)}$ are as in (3.9), and

$$N_1 = A_0 + iA_1 + jA_2 + kA_3, \qquad N_2 = A_0 + iA_1 - jA_2 - kA_3,$$
$$N_3 = A_0 - iA_1 + jA_2 - kA_3, \qquad N_4 = A_0 - iA_1 - jA_2 + kA_3.$$

It is easy to verify that $N_2 = iAi^{-1},\ N_3 = jAj^{-1}$ and $N_3 = kAk^{-1}$. Thus

$$\operatorname{diag}(\,N_1,\ N_2,\ N_3,\ N_4\,) = J_m^{(4)} \operatorname{diag}(\,A,\ A,\ A,\ A\,)(J_n^{(4)})^{-1}, \tag{4.4}$$

where $J_t^{(4)} = \operatorname{diag}(\,I_t,\ iI_t,\ jI_t,\ kI_t\,),\ t = m,\ n$. On the other hand, it is easy to verify that

$$(J_m^{(4)})^{-1} \begin{bmatrix} A_0 & iA_1 & jA_2 & kA_3 \\ iA_1 & A_0 & kA_3 & jA_2 \\ jA_2 & kA_3 & A_0 & iA_1 \\ kA_3 & jA_2 & iA_1 & A_0 \end{bmatrix} J_n^{(4)} = \begin{bmatrix} A_0 & -A_1 & -A_2 & -A_3 \\ A_1 & A_0 & -A_3 & A_2 \\ A_2 & A_3 & A_0 & -A_1 \\ A_3 & -A_2 & A_1 & A_0 \end{bmatrix} = \phi(A). \tag{4.5}$$

Now putting (4.3) and (4.4) in (4.5), we find

$$\begin{aligned}\phi(A) = (J_m^{(4)})^{-1} M J_n^{(4)} &= (J_m^{(4)})^{-1} Q_m^{(4)} \operatorname{diag}(\,N_1,\ N_2,\ N_3,\ N_4\,) Q_n^{(4)} J_n^{(4)} \\ &= (J_m^{(4)})^{-1} Q_m^{(4)} J_m^{(4)} \operatorname{diag}(\,A,\ A,\ A,\ A\,)(J_n^{(4)})^{-1} Q_n^{(4)} J_n^{(4)}.\end{aligned}$$

Let $U_t^{(4)} = (J_t^{(4)})^{-1} Q_t^{(4)} J_t^{(4)},\ t = m,\ n$. Then we have

$$\begin{aligned} U_t^{(4)} &= \frac{1}{2} \begin{bmatrix} I_t & & & \\ & -iI_t & & \\ & & -jI_t & \\ & & & -kI_t \end{bmatrix} \begin{bmatrix} I_t & I_t & I_t & I_t \\ I_t & I_t & -I_t & -I_t \\ I_t & -I_t & I_t & -I_t \\ I_t & -I_t & -I_t & I_t \end{bmatrix} \begin{bmatrix} I_t & & & \\ & iI_t & & \\ & & jI_t & \\ & & & kI_t \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} I_t & iI_t & jI_t & kI_t \\ -iI_t & I_t & kI_t & -jI_t \\ -jI_t & -kI_t & I_t & iI_t \\ -kI_t & jI_t & -iI_t & I_t \end{bmatrix}, \end{aligned}$$

which proves (4.1). $\square$

The equality (4.1) implies that for any quaternion matrix A, the corresponding block-diagonal matrix $\mathrm{diag}(\,A,\ A,\ A,\ A\,)$ is uniformly equivalent to its real representation $\phi(A)$. In particular, if A is square, then $\mathrm{diag}(\,A,\ A,\ A,\ A\,)$ is uniformly and unitarily similar to its real representation $\phi(A)$. Hence (4.1) can be effectively used for developing matrix analysis over the real quaternion algebra $\mathbb{H}$. For example, concerning the determinants of quaternion matrices, a fundamental topic related to quaternion matrices that has been examined for many years (see, e.g., [1, 2]), if we wish to define the determinant of a quaternion matrix satisfying only two axioms:

(i) $\mathrm{D}(I_n) = 1$;
(ii) $\mathrm{D}(AB) = \mathrm{D}(A)\mathrm{D}(B)$,

then we see immediately from (4.1) and (4.2) that

$$1 = \mathrm{D}(I_n) = \mathrm{D}(U_n^{(4)}U_n^{(4)}) = \mathrm{D}(U_n^{(4)})\mathrm{D}(U_n^{(4)}),$$

and then

$$\mathrm{D}[\phi(A)] = \mathrm{D}(U_n^{(4)})\mathrm{D}[\mathrm{diag}(\,A,\ A,\ A,\ A\,)]\mathrm{D}(U_n^{(4)}) = \mathrm{D}[\mathrm{diag}(\,A,\ A,\ A,\ A\,)].$$

This fact implies that if the above two axioms are satisfied, then the determinant of the diagonal block matrix $\mathrm{diag}(\,A,\ A,\ A,\ A\,)$ must be equal to the determinant of the real matrix $\phi(A)$. Since the conventional determinant of a real matrix is well defined, the equality (4.1) naturally suggests that we define the determinant of a quaternion matrix through the determinant of its real representation matrix $\phi(A)$, that is, for any square quaternion matrix A, we can define its determinant as

$$\mathrm{Rdet}(A) := \det[\phi(A)], \tag{4.6}$$

where $\det[\phi(A)]$ is the conventional determinant of the real matrix $\phi(A)$. In that case, it is easy to verify that this definition, up to some trivial power factors, is also identical to the determinants of a quaternion matrix examined in [2]. However, the merit of the definition (4.6) is in that the evaluation of $\det[\phi(A)]$ can be trivially realized in any case.

As an application of (4.1), suppose now two real matrices A and B satisfy $A^2 = B^2 = -I_n$ and $AB + BA = 0$, and let $M = aI_n + bA + cB + dAB$. In that case, we see from (4.1) that M satisfies the matrix factorization

$$P\begin{bmatrix} M & & & \\ & M & & \\ & & M & \\ & & & M \end{bmatrix}P^{-1} = \begin{bmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{bmatrix} \otimes I_n, \tag{4.7}$$

where

$$P = P^{-1} = \frac{1}{2}\begin{bmatrix} I_n & A & B & AB \\ -A & I_n & AB & -B \\ -B & -AB & I_n & A \\ -AB & B & -A & I_n \end{bmatrix},$$

and $\otimes$ stands for the Kronecker product of matrices. This similarity factorization reveals that the properties of M can all be determined by the matrix

$$\begin{bmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{bmatrix}.$$

We may find the determinant, rank, inverse, eigenvalues and eigenvectors, and so on, for the matrix M from (4.7). Some other applications of (4.1) to quaternion matrices were presented in Tian [12]. From (3.8) we can also derive another universal factorization equality for matrices over the four-dimensional commutative algebra over $\mathbb{R}$ generated by i and j with

$$i^2 = j^2 = 1, \qquad ij = ji.$$

The element of this algebra has the form $a = a_0 + ia_1 + ja_2 + ka_3$, where $k = ij$. In that case, we have the following theorem.

THEOREM 4.2. *Let $A = A_0 + iA_1 + jA_2 + kA_3$ be a matrix with $A_0, \ ..., \ A_3 \in \mathbb{R}^{m\times n}$, $i^2 = j^2 = 1$ and $k = ij = ji$. Then A satisfies the universal factorization equality*

$$(4.8) \qquad V_m^{(4)} \begin{bmatrix} D_1 & & & \\ & D_2 & & \\ & & D_3 & \\ & & & D_4 \end{bmatrix} V_n^{(4)} = \begin{bmatrix} A_0 & A_1 & A_2 & A_3 \\ A_1 & A_0 & A_3 & A_2 \\ A_2 & A_3 & A_0 & A_1 \\ A_3 & A_2 & A_1 & A_0 \end{bmatrix} := \psi(A),$$

where

$$D_1 = A_0 + iA_1 + jA_2 + kA_3, \qquad D_2 = A_0 + iA_1 - jA_2 - kA_3,$$
$$D_3 = A_0 - iA_1 + jA_2 - kA_3, \qquad D_4 = A_0 - iA_1 - jA_2 + kA_3,$$

and

$$V_4^{(4)} = (V_4^{(4)})^{-1} = \frac{1}{2}\begin{bmatrix} I_t & iI_t & jI_t & kI_t \\ iI_t & I_t & -kI_t & -jI_t \\ jI_t & -kI_t & I_t & -iI_t \\ kI_t & -jI_t & -iI_t & I_t \end{bmatrix}, \quad t = m, \ n.$$

In particular, if $m = n$, then (4.8) *becomes a universal similarity factorization equality.*

Proof. Replacing $A_0, \ A_1, \ A_2, \ A_3$ in (3.8) by $A_0, \ iA_1, \ jA_2, \ kA_3$, respectively, we obtain

$$(4.9) \qquad M := \begin{bmatrix} A_0 & iA_1 & jA_2 & kA_3 \\ iA_1 & A_0 & kA_3 & jA_2 \\ jA_2 & kA_3 & A_0 & iA_1 \\ kA_3 & jA_2 & iA_1 & A_0 \end{bmatrix} = Q_m^{(4)} \begin{bmatrix} D_1 & & & \\ & D_2 & & \\ & & D_3 & \\ & & & D_4 \end{bmatrix} Q_n^{(4)},$$

where $Q_m^{(4)}$ and $Q_n^{(4)}$ are as in (3.9). It is easy to verify that $D_1 = iD_1i^{-1}$, $D_2 = iD_2i^{-1}$, $D_3 = iD_3i^{-1}$, and $D_4 = iD_4i^{-1}$. Thus

$$\operatorname{diag}(D_1,\ D_2,\ D_3,\ D_4) = J_4^{(4)}\operatorname{diag}(D_1,\ D_2,\ D_3,\ D_4)J_n^{(4)}, \tag{4.10}$$

where $J_t^{(4)} = (J_t^{(4)})^{-1} = \operatorname{diag}(I_t,\ iI_t,\ jI_t,\ kI_t),\ t = m,\ n$. On the other hand, it is easy to verify that

$$J_m^{(4)}\begin{bmatrix} A_0 & iA_1 & jA_2 & kA_3 \\ iA_1 & A_0 & kA_3 & jA_2 \\ jA_2 & kA_3 & A_0 & iA_1 \\ kA_3 & jA_2 & iA_1 & A_0 \end{bmatrix}J_n^{(4)} = \begin{bmatrix} A_0 & A_1 & A_2 & A_3 \\ A_1 & A_0 & A_3 & A_2 \\ A_2 & A_3 & A_0 & A_1 \\ A_3 & A_2 & A_1 & A_0 \end{bmatrix}. \tag{4.11}$$

Now putting (4.9) and (4.10) in (4.11), we find

$$\begin{aligned}\psi(A) = J_m^{(4)}MJ_n^{(4)} &= J_m^{(4)}Q_m^{(4)}\operatorname{diag}(D_1,\ D_2,\ D_3,\ D_4)Q_n^{(4)}J_n^{(4)} \\ &= J_m^{(4)}Q_m^{(4)}J_m^{(4)}\operatorname{diag}(D_1,\ D_2,\ D_3,\ D_4)J_n^{(4)}Q_n^{(4)}J_n^{(4)}.\end{aligned}$$

Let $V_t^{(4)} = J_t^{(4)}Q_tJ_t^{(4)},\ t = m,\ n$. Then we have

$$\begin{aligned}V_t^{(4)} &= \frac{1}{2}\begin{bmatrix} I_t & & & \\ & iI_t & & \\ & & jI_t & \\ & & & kI_t \end{bmatrix}\begin{bmatrix} I_t & I_t & I_t & I_t \\ I_t & I_t & -I_t & -I_t \\ I_t & -I_t & I_t & -I_t \\ I_t & -I_t & -I_t & I_t \end{bmatrix}\begin{bmatrix} I_t & & & \\ & iI_t & & \\ & & jI_t & \\ & & & kI_t \end{bmatrix} \\ &= \frac{1}{2}\begin{bmatrix} I_t & iI_t & jI_t & kI_t \\ iI_t & I_t & -kI_t & -jI_t \\ jI_t & -kI_t & I_t & -iI_t \\ kI_t & -jI_t & -iI_t & I_t \end{bmatrix},\end{aligned}$$

which proves (4.8). □

Substituting (3.8) into the right-hand side of (4.8), we also get

$$L_{4m}\begin{bmatrix} D_1 & & & \\ & D_2 & & \\ & & D_3 & \\ & & & D_4 \end{bmatrix}L_{4n} = \begin{bmatrix} N_1 & & & \\ & N_2 & & \\ & & N_3 & \\ & & & N_4 \end{bmatrix}, \tag{4.12}$$

where

$$L_{4t} = L_{4t}^{-1} = \begin{bmatrix} l_1I_t & l_2I_t & l_3I_t & l_4I_t \\ l_2I_t & l_1I_t & -l_4I_t & -l_3I_t \\ l_3I_t & -l_4I_t & l_1I_t & -l_2I_t \\ l_4I_t & -l_3I_t & -l_2I_t & l_1I_t \end{bmatrix}$$

with

$$l_1 = \tfrac{1}{4}(1 + i + j + k), \qquad l_2 = \tfrac{1}{4}(1 + i - j - k),$$
$$l_3 = \tfrac{1}{4}(1 - i + j - k), \qquad l_4 = \tfrac{1}{4}(1 - i - j + k).$$

In particular, when $m = n$, (4.12) becomes a universal similarity factorization equality over $\mathbb{F}$.

From (4.12), we can establish a useful factorization equality for a linear combination of involutory matrices. Suppose that A and B are two involutory matrices, i.e., $A^2 = B^2 = I_n$, and suppose further that $AB = BA$. Then for any $M = aI_n + bA + cB + dAB$, where $a,\ b,\ c,\ d$ are scalars, the following factorization equality holds

$$L \begin{bmatrix} M_1 & & & \\ & M_2 & & \\ & & M_3 & \\ & & & M_4 \end{bmatrix} L^{-1} = \begin{bmatrix} m_1 I_n & 0 & 0 & 0 \\ 0 & m_2 I_n & 0 & 0 \\ 0 & 0 & m_3 I_n & 0 \\ 0 & 0 & 0 & m_4 I_n \end{bmatrix}, \tag{4.13}$$

where

$$\begin{aligned} M_1 &= aI_n + bA + cB + dAB, & M_2 &= aI_n + bA - cB - dAB, \\ M_3 &= aI_n - bA + cB - dAB, & M_4 &= aI_n - bA - cB + dAB, \\ m_1 &= a + b + c + d, & m_2 &= a + b - c - d, \\ m_3 &= a - b + c - d, & m_4 &= a - b - c + d, \end{aligned}$$

and

$$L = L^{-1} = \begin{bmatrix} L_1 & L_2 & L_3 & L_4 \\ L_2 & L_1 & -L_4 & -L_3 \\ L_3 & -L_4 & L_1 & -L_2 \\ L_4 & -L_3 & -L_2 & L_1 \end{bmatrix}$$

with

$$L_1 = \tfrac{1}{4}(I_n + A + B + AB), \qquad L_2 = \tfrac{1}{4}(I_n + A - B - AB), \tag{4.14}$$

$$L_3 = \tfrac{1}{4}(I_n - A + B - AB), \qquad L_4 = \tfrac{1}{4}(I_n - A - B + AB). \tag{4.15}$$

Many properties of $M_1,\ ...,\ M_4$ can be derived from (4.13). For example, $M_1,\ ...,\ M_4$ are all nonsingular if and only if $m_1,\ ...,\ m_4$ are nonzero. All $M_1,\ ...,\ M_4$ are diagonalizable, and $m_1,\ ...,\ m_4$ are eigenvalues of $M_1,\ ...,\ M_4$. Moreover, it is easy to find from (4.13) that M can be written as

$$M_1 = m_1 L_1 + m_2 L_2 + m_3 L_3 + m_4 L_4,$$

where $L_1,\ ...,\ L_4$ are defined in (4.14) and (4.15) and they satisfy $L_i^2 = L_i$ and $L_i L_j = 0$ for $i \neq j,\ i,\ j = 1,\ ...,\ 4$. Thus it follows that

$$M_1^k = m_1^k L_1 + m_2^k L_2 + m_3^k L_3 + m_4^k L_4$$

for any positive integer k.

We may also apply (4.13) to idempotent matrices. For any two idempotent matrices A and B with $AB = BA$ it follows that $(I - 2A)^2 = (I - 2B)^2 = I$, and

$(I-2A)(I-2B) = (I-2B)(I-2A)$. From (4.13) we obtain the following valuable factorization equality for idempotent matrices.

THEOREM 4.3. *Suppose $A^2 = A$, $B^2 = B$ and $AB = BA$. Then*

$$L\begin{bmatrix} M_1 & & & \\ & M_2 & & \\ & & M_3 & \\ & & & M_4 \end{bmatrix} L^{-1} = \begin{bmatrix} m_1 I_n & 0 & 0 & 0 \\ 0 & m_2 I_n & 0 & 0 \\ 0 & 0 & m_3 I_n & 0 \\ 0 & 0 & 0 & m_4 I_n \end{bmatrix}, \tag{4.16}$$

where

$$\begin{aligned} M_1 &= a_0 I_n + a_1 A + a_2 B + a_3 AB, \\ M_2 &= (a_0 + a_2) I_n + (a_1 + a_3) A - a_2 B - a_3 AB, \\ M_3 &= (a_0 + a_1) I_n - a_1 A + (a_2 + a_3) B - a_3 AB, \\ M_4 &= (a_0 + a_1 + a_2 + a_3) I_n - (a_1 + a_3) A - (a_2 + a_3) B + a_3 AB, \end{aligned}$$

$$m_1 = a_0, \qquad m_2 = a_0 + a_2, \qquad m_3 = a_0 + a_1, \qquad m_4 = a_0 + a_1 + a_2 + a_3,$$

and

$$L = L^{-1} = \begin{bmatrix} L_1 & L_2 & L_3 & L_4 \\ L_2 & L_1 & -L_4 & -L_3 \\ L_3 & -L_4 & L_1 & -L_2 \\ L_4 & -L_3 & -L_2 & L_1 \end{bmatrix}$$

with

$$L_1 = I_n - A - B + AB, \quad L_2 = B - AB, \quad L_3 = A - AB, \quad L_4 = AB.$$

These four matrices satisfy $L_i^2 = L_i$ and $L_i L_j = 0$ for $i \neq j$; $i, \ j = 1, \ldots, 4$.

From (4.16) we see that the matrix M_1 in (4.16) can be written as

$$M_1 = m_1 L_1 + m_2 L_2 + m_3 L_3 + m_4 L_4,$$

with

$$M_1^k = m_1^k L_1 + m_2^k L_2 + m_3^k L_3 + m_4^k L_4,$$

for any positive integer k. From these results we may investigate various properties of the matrix M_1, such as idempotency, r-potency, involution, and so on.

The universal factorization equality (4.1) can also be extended to a generalized quaternion algebra $\left(\frac{u,v}{\mathbb{F}}\right)$ (for more details on generalized quaternion algebras, see, e.g., [7, 8, 9]). Here we only present the main result without proof.

THEOREM 4.4. *Let $A = A_0 + iA_1 + jA_2 + kA_3 \in \left(\frac{u,v}{\mathbb{F}}\right)^{m\times n}$, where $\mathbb{F}$ is an arbitrary field of characteristic not equal to two, $u, \ v \in \mathbb{F}$, and $A_0, \ldots, A_3 \in \mathbb{F}^{m\times n}$,*

$$i^2 = u \neq 0, \qquad j^2 = v \neq 0, \qquad k = ij = -ji.$$

Then A satisfies the universal factorization equality

$$(4.17) \qquad V_m^{(4)} \begin{bmatrix} A & & & \\ & A & & \\ & & A & \\ & & & A \end{bmatrix} V_n^{(4)} = \begin{bmatrix} A_0 & uA_1 & vA_2 & -uvA_3 \\ A_1 & A_0 & vA_3 & -vA_2 \\ A_2 & -uA_3 & A_0 & uA_1 \\ A_3 & -A_2 & A_1 & A_0 \end{bmatrix},$$

where

$$V_t^{(4)} = (V_t^{(4)})^{-1} = \frac{1}{2} \begin{bmatrix} I_t & iI_t & jI_t & kI_t \\ u^{-1}iI_t & I_t & -u^{-1}kI_t & -jI_t \\ v^{-1}jI_t & v^{-1}kI_t & I_t & iI_t \\ -(uv)^{-1}kI_t & -v^{-1}jI_t & u^{-1}iI_t & I_t \end{bmatrix}, \quad t = m,\ n.$$

In particular, when $m = n, the factorization$ (4.17) becomes a universal similarity factorization equality over $\left(\frac{u,v}{\mathbb{F}}\right)$.

Finally we point out that the two fundamental equalities in (4.1) and (4.17) can also be extended to all 2^n-dimensional Clifford algebras (real, complex, generalized). The corresponding results can serve as a powerful tool for examining these kinds of algebras; see [10, 11]. For more details on Clifford algebras see, e.g., [5, 9].

5. Conclusions. In this article, we have presented a simple method for establishing universal factorization equalities for block matrices. Using this method, we can derive various useful factorization equalities for 2×2 block matrices, as well as for 4×4 block matrices. These results can be further used to establish universal factorization equalities for matrices over high-dimensional algebras. We also find a variety of interesting results for determinants, ranks, inverses, generalized inverses, eigenvalues and eigenvectors, and so on, for matrices over high-dimensional algebras.

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