The Electronic Journal of Linear Algebra.
A publication of the International Linear Algebra Society.
Volume 8, pp. 14-25, February 2001.
ISSN 1081-3810. http://math.technion.ac.il/iic/ela

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ALGEBRAIC CURVE SOLUTION FOR SECOND-ORDER POLYNOMIAL AUTONOMOUS SYSTEMS*

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Abstract. In this paper, by the method of the analysis of algebraic curves and the division theorem for two variables in $\mathbb{C}$, the nonexistence of the algebraic curve solution of the second-order polynomial autonomous systems in $\mathbb{C}$ is obtained. The results are of important significance in the qualitative theory of polynomial autonomous systems and the integrability of nonlinear ordinary differential equations.

Key words. Autonomous system, algebraic curve, division theorem, Van der Pol equation.

AMS subject classifications. 13P10, 34A05

1. Introduction. Consider the following polynomial autonomous system:

$$\left\{\begin{array}{l} \dot{x}=y, \\ \dot{y}=f(x)y+g(x), \end{array}\right. \tag{1}$$

where $f(x)=\sum_{i=0}^{l} c_i x^i$, $g(x)=\sum_{i=0}^{\xi} g_i x^i$, $c_i, g_i \in \mathbb{C}$, $l, \xi \in \mathbb{N}$, and $c_l \cdot g_\xi \neq 0$.

It is well known that the polynomial autonomous system (1) plays an important role in the qualitative theory of ordinary differential equations, because many practical problems can be converted to (1). It can also be widely applied in many scientific fields, such as engineering, control theory, fluid mechanics, and so on; see [1]. For example, when $f(x)=\varepsilon(1-x^2)$ and $g(x)=-x$, (1) is equivalent to the famous Van der Pol equation

$$\frac{d^2x}{dt^2}+\varepsilon(x^2-1)\frac{dx}{dt}+x=0. \tag{2}$$

However, for a given polynomial autonomous system that describes a physical phenomenon, the basic problem of seeking its solutions is yet unsolved. Eight years ago, in [2], it was shown that a polynomial autonomous system is not integrable if it does not have any algebraic curve solution in $\mathbb{C}$. Therefore, the problem of finding under what special conditions a polynomial autonomous system has the algebraic curve solution in $\mathbb{C}$ has become a very interesting research topic during the past years; see [3–11].

In this paper, we give a new approach, which we currently call the method of the analysis of algebraic curves and the division theorem for two variables in $\mathbb{C}$. Using this technique, we obtain the theorems for the nonexistence of the algebraic curve solution for the second-order polynomial autonomous system (1) in $\mathbb{C}$. These results are not only very important in the qualitative theory of polynomial autonomous systems,

*Received by the editors on 8 October 2000. Accepted for publication on 4 February 2001. Handling Editor: Daniel Hershkowitz.

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but also very useful in investigating the integrability of nonlinear ordinary differential equations such as Lienard equations; see [2].

This paper is organized as follows. In section 2, we prove the division theorem for two variables in $\mathbb{C}$. In section 3, we obtain that when $\deg f(x) > \deg g(x)$ or $\deg f(x) = \deg g(x)$ and $f(x) \not\equiv cg(x)$ $(c \neq 0)$, there is no algebraic curve solution for autonomous system (1) in $\mathbb{C}$. In section 4, we give a brief conclusion.

2. Division Theorem. In this section, we prove the division theorem for two variables in $\mathbb{C}$.

THEOREM 2.1. *Suppose that $P(\omega, z)$ and $Q(\omega, z)$ are polynomials in $\mathbb{C}[\omega, z]$, and that $P(\omega, z)$ is irreducible in $\mathbb{C}[\omega, z]$. If $Q(\omega, z)$ vanishes at any zero point of $P(\omega, z)$, then there exists a polynomial $G(\omega, z)$ in $\mathbb{C}[\omega, z]$ such that*

$$Q(\omega, z) = P(\omega, z) \cdot G(\omega, z). \tag{3}$$

Proof. For convenience, we first give the following lemmas.

LEMMA 2.2. *Suppose that $U(\omega, z)$ and $V(\omega, z)$ are polynomials in $\mathbb{C}[\omega, z]$, and $U(\omega, z)$ is irreducible in $\mathbb{C}[\omega, z]$. Suppose that $R(\omega, z)$ is a nonconstant polynomial and a factor of $U(\omega, z) \cdot V(\omega, z)$, and $\deg R(\omega, z) < \deg U(\omega, z)$ with respect to ω. Then $R(\omega, z) | V(\omega, z)$.*

LEMMA 2.3. *Suppose that $P(\omega, z)$ is an irreducible polynomial in $\mathbb{C}[\omega, z]$ and that $P_\omega(\omega, z)$ is the partial derivative with respect to ω. Then there exist two polynomials $A(\omega, z), B(\omega, z)$, and nonzero polynomial $D(z)$ in $\mathbb{C}[\omega, z]$, such that*

$$A(\omega, z) \cdot P(\omega, z) + B(\omega, z) \cdot P_\omega(\omega, z) = D(z). \tag{4}$$

The proofs of Lemma 2.2 and Lemma 2.3 can be seen in [12].

Notice that any polynomial $P(\omega, z)$ in $\mathbb{C}[\omega, z]$ can be written as

$$P(\omega, z) = \sum_{k=0}^{n} p_k(z)\omega^k, \tag{5}$$

where $p_k(z)$ $(k = 0, 1, \ldots, n)$ are polynomials in z and $p_n(z) \not\equiv 0$. If $P(\omega, z)$ is an irreducible polynomial in $\mathbb{C}[\omega, z]$, then $p_k(z)$ $(k = 0, 1, \ldots, n)$ are all relatively prime. For any fixed $z_0 \in \mathbb{C}$, $P(\omega, z_0)$ is a polynomial in ω. By the fundamental theorem of algebra, it has n zeros in $\mathbb{C}$.

DEFINITION 2.4. If z_0 is a complex number such that the polynomial $P(\omega, z_0)$ does not have n distinct zeros in $\mathbb{C}$, then z_0 is called a *special zero point* of the polynomial $P(\omega, z)$.

LEMMA 2.5. *If $P(\omega, z)$ is an irreducible polynomial in $\mathbb{C}[\omega, z]$, then $P(\omega, z)$ has at most finitely many special zero points.*

Proof. Write $P(\omega, z)$ in (5) and consider the set

$$M = \{z | z \in C, \quad p_n(z) = 0 \quad \text{or} \quad D(z) = 0\}.$$

By (4) and (5), it is easy to see that M is a finite set. Suppose that $z^* \in \mathbb{C} \setminus M$. Then the polynomial $P(\omega, z^*)$ with respect to ω must have n distinct zeros. Hence, the set of special zero points of $P(\omega,z)$ is a subset of M. Therefore, $P(\omega,z)$ has at most finitely many special zero points. □

For any $z\in\mathbb{C}\setminus M$, by Lemma 2.5, the polynomial $P(\omega,z)$ with respect to ω must have n distinct roots r_i $(i=1,2,\dots,n)$. By the hypothesis, r_i $(i=1,2,\dots,n)$ are also the roots of $Q(\omega,z)$. Hence, the degree for polynomial $Q(\omega,z)$ with respect to ω is greater than or equal to n.

Assume that

$$Q(\omega,z)=\sum_{k=0}^{m} q_k(z)\omega^k,$$

where $q_k(z)$ $(k=0,1,\dots,m)$ are polynomials in z, $q_m(z)\not\equiv 0$, and $m\geq n$.

By the division theory for polynomials in one variable, we have

$$Q(\omega,z)=h(\omega,z)\cdot P(\omega,z), \tag{6}$$

where

$$\begin{cases} h(\omega,z)=\sum_{k=0}^{m-n} h_k(z)\omega^k,\\ h_{m-n}(z)=q_m(z)/p_n(z),\\ h_{m-n-1}(z)=\frac{1}{p_n(z)}[q_{m-1}(z)-\frac{q_m(z)p_{n-1}(z)}{p_n(z)}]=\frac{q_{m-1}*(z)}{p_n^2(z)},\\ \dots\\ h_{m-n-i}(z)=\frac{q_{m-i}*(z)}{p_n^{i+1}(z)},\\ h_0(z)=\frac{q_n*(z)}{p_n^{m-n-1}(z)}. \end{cases} \tag{7}$$

Notice that the polynomials $q^*_{m-i}(z)$ could be obtained from $q_k(z)$ and $p_k(z)$ by applying the operations of addition, subtraction, multiplication, and division. The denominators and numerators of (7) may have common factors.

Suppose that $u(z)$ is a polynomial with the least degree, such that $u(z)\cdot h(\omega,z)$ is a polynomial in $\mathbb{C}[x,z]$. That is,

$$u(z)h(\omega,z)=G_1(\omega,z), \tag{8}$$

where $u(z)$ and $G_1(\omega,z)$ are polynomials in $\mathbb{C}[\omega,z]$. Note that there is no nontrivial common factor between $u(z)$ and $G_1(\omega,z)$. By (6) and (8), we get

$$u(z)\cdot Q(\omega,z)=G_1(\omega,z)\cdot P(\omega,z). \tag{9}$$

If $u(z)$ is a nonzero constant, then we obtain the desired result. If $u(z)$ is a nonconstant polynomial, since $P(\omega,z)$ is irreducible, by Lemma 2.2, $u(z)$ must divide $G_1(\omega,z)$. This yields a contradiction with the above assumption that $u(z)$ and $G_1(\omega,z)$ have no nontrivial common factor in $\mathbb{C}[x,z]$. Therefore, $u(z)$ must be a nonzero constant. Letting $G(\omega,z)=[\frac{1}{u(z)}]\cdot G_1(\omega,z)$, from (9) we obtain (3). So the proof of Theorem 2.1 is complete. □

3. Main Results. Consider the autonomous system (1). Assume that (1) has the algebraic curve solution

$$P(x,y)=\sum_0^m a_i(x)y^i=0, \tag{10}$$

where $a_i(x)$ $(i=0,1,\ldots,m)$ are polynomials and $a_m(x)\not\equiv 0$. $P(x,y)$ is irreducible in $\mathbb{C}$; $a_i(x)$ $(i=0,1,\ldots,m)$ are all relatively prime in $\mathbb{C}$.

By the division theorem, there exist the polynomials $\alpha(x)$ and $\beta(x)$ such that

$$\begin{aligned}\left.\frac{dP}{dt}\right|_{(1)} &= \left.\left(\frac{\partial P}{\partial x}\frac{dx}{dt}+\frac{\partial P}{\partial y}\frac{dy}{dt}\right)\right|_{(1)}\\ &= \left[\sum_0^m a_i'(x)y^i\right]\cdot y+\left[\sum_0^m [ia_i(x)y^{i-1}][g(x)+f(x)y]\right]\\ &= [\alpha(x)+\beta(x)y]P(x,y),\end{aligned}$$

that is,

$$\begin{aligned}&\sum_0^m a_i'(x)y^{i+1}+\sum_0^m ia_i(x)f(x)y^i+\sum_0^m ia_i(x)g(x)y^{i-1}\\ &=[\alpha(x)+\beta(x)y]\left(\sum_0^m a_i(x)y^i\right).\end{aligned} \tag{11}$$

Equating the coefficients of y^{m+1} on both sides of (11), we obtain that $a_m'(x)=a_m(x)\beta(x)$, i.e., $\beta(x)=0$, $a_m(x)=$ nonzero constant. To simplify the complicated computation, we assume that $a_m(x)=1$. Equating the coefficients of y^i $(i=m,m-1,\ldots,1,0)$ on both sides of (11), we have

$$\begin{cases} a_{m-1}'(x)=\alpha(x)-mf(x),\\ \cdots\\ a_j'(x)=a_{j+1}(x)[\alpha(x)-(j+1)f(x)]-(j+2)a_{j+2}(x)g(x)\\ \qquad (j=m-2,m-3,\ldots,1,0),\\ \\ 0=a_0(x)\alpha(x)-a_1(x)g(x).\end{cases} \tag{12}$$

Under the conditions $l>\xi$ ($l=\deg f(x)$, $\xi=\deg g(x)$) or $l=\xi$ and $f(x)\not\equiv cg(x)$ $(c\neq 0, c\in\mathbb{C})$, denote $\deg a_t(x)=\max_{0\leq i\leq m}[\deg a_i(x)]$, $t\in\mathbb{Z}$. The leading term in x on the right-hand side of (11) is $a_t(x)\alpha(x)y^t$. The combination of the terms including y^t on the left-hand side of (11) is $[a_{t-1}'(x)+ta_t(x)f(x)+(t+1)a_{t+1}(x)g(x)]y^t$. Hence, we obtain that $\deg\alpha(x)=\deg f(x)$. Otherwise, assume that $\deg\alpha(x)=l_0<l$. Then, by the first equation of (12), second equation of (12), …, mth equation of (12), respectively, we deduce that $a_{m-i}(x)=i(l+1)$ $(i=m-1,\ldots,0)$. By the last equation of (12), i.e.,

$$a_0(x)\alpha(x)=a_1(x)g(x), \tag{13}$$

we have $\deg a_0(x)+l_0 = \deg a_1+\xi$, i.e., $\xi = l+l_0+1$. This yields a contradiction with our previous assumption $l \geq \xi$. If we assume that $\deg \alpha(x) = l_0 > l$, the argument is identical.

CLAIM 3.1. *Suppose that $l > \xi$ or $l = \xi$ and $f(x) \not\equiv cg(x)$ $(c \neq 0, c \in \mathbb{C})$, and that* (1) *has the algebraic curve solution as $P(x,y) = \sum_0^m a_i(x)y^i = 0$. Then $\beta(x) \equiv 0$ and there is some k $(0 < k \leq m,\ k \in \mathbb{Z})$ such that $\alpha(x) = kf(x)$.*

Proof. We have already proved in this section that $\beta(x) \equiv 0$. Next, by way of contradiction, we prove that $\alpha(x) = kf(x)$ $(0 < k \leq m, k \in \mathbb{Z})$.

Suppose that $\alpha(x) \neq kf(x)$ $(0 < k \leq m,\ k \in \mathbb{Z})$. Let $\deg[\alpha(x) - \delta f(x)] = \min_{0\leq j\leq m} \deg[\alpha(x) - jf(x)] = n_0$ $(\delta, n_0 \in \mathbb{Z},\ 0 < \delta \leq m, n_0 \geq 0)$. By (13), we have

$$\deg a_1(x) = \deg a_0(x) + (l - \xi).$$

Again, by (12),

$$a'_j(x) = a_{j+1}(x)[\alpha(x) - (j+1)f(x)] - (j+2)a_{j+2}(x)g(x). \tag{14}$$

Letting $j = 0, 1, \ldots, \delta - 2$, respectively, we have

$$\deg a_i(x) = \deg a_0(x) + i(l - \xi) \qquad (i = 1, 2, \ldots, \delta). \tag{15}$$

Using the first equation of (12), we deduce

$$\deg a_{m-1}(x) = l + 1.$$

Letting $j = m-2, \ldots, \delta, \delta - 1$, respectively, we have

$$\begin{cases} \deg a_i(x) = (m-i)(l+1) \qquad (\delta \leq i \leq m-2), \\ \deg a_{\delta-1}(x) = (m-\delta)(l+1) + n_0 + 1. \end{cases} \tag{16}$$

By (15) and (16), we have

$$\begin{cases} \deg a_\delta(x) = \delta(l-\xi) + \deg a_0(x) = (m-\delta)(l+1), \\ \deg a_{\delta-1}(x) = (\delta-1)(l-\xi) + \deg a_0(x) = (m-\delta)(l+1) + n_0 + 1. \end{cases} \tag{17}$$

Thus, from (17) we deduce that $\xi - l = n_0 + 1 > 0$. This contradicts the previous assumption $l \geq \xi$. Therefore, there exists some k $(0 < k \leq m,\ k \in \mathbb{Z})$ such that $\alpha(x) = kf(x)$. ◻

Since $\alpha(x) = kf(x)$, taking $j = k-1$ in (14), we have

$$a'_{k-1}(x) = -(k+1)a_{k+1}(x)g(x), \tag{18}$$

and from (12), by using the first equation, second equation, ..., $(m-k)$th equation, respectively, we have

$$\deg a_i(x) = (m-i)(l+1) \qquad (k \leq i \leq m-1). \tag{19}$$

Combining (18) and (19), we get

$$\begin{aligned}\deg a_{k-1}(x) &= \deg a_{k+1}(x) + \deg g(x) + 1 \\ &= (m-k)(l+1) + \xi - l.\end{aligned} \tag{20}$$

On the other hand, taking $j = 0, 1, \ldots, k-2$ in (14), we have

$$\deg a_i(x) = \deg a_0(x) + i(l - \xi) \quad (i = 1, 2, \ldots, k). \tag{21}$$

Thus, we obtain the following result.

THEOREM 3.2. *If $l > \xi$, then the second-order polynomial autonomous system (1) has no algebraic curve solution in $\mathbb{C}$.*

Proof. Suppose that the second-order polynomial autonomous system (1) has the algebraic curve solution of the form (10), and that u_i $(i = 0, 1, \ldots, m)$ are the leading coefficients of $a_i(x)$.

Next, we discuss the value k by the following two steps:

(I) Assume that $k = m$. Then by the first equation of (12), we deduce that $a_{m-1}(x) = \text{constant}$.

(1) In the case $a_{m-1}(x) = 0$, using the second equation, third equation, ..., and mth equation of (12), respectively, we have

$$\begin{cases} \deg a_{m-2}(x) = \xi + 1, \\ \deg a_{m-3}(x) = (\xi + 1) + (l + 1), \\ \deg a_{m-4}(x) = (\xi + 1) + 2(l + 1), \\ \ldots \\ \deg a_0(x) = (\xi + 1) + (m - 2)(l + 1). \end{cases}$$

Then we have

$$\begin{cases} \deg[a_0(x)\alpha(x)] = (\xi + 1) + (m - 2)(l + 1) + l, \\ \deg[a_1(x)g(x)] = (\xi + 1) + (m - 3)(l + 1) + \xi. \end{cases} \tag{22}$$

It is easy to see that (22) contradicts (13).

(2) In the case $a_{m-1}(x) = \text{constant} \neq 0$, similarly, using the second equation, third equation, ..., and mth equation of (12), respectively, we have

$$\begin{cases} \deg a_{m-2}(x) = l + 1, \\ \deg a_{m-3}(x) = 2(l + 1), \\ \deg a_{m-4}(x) = 3(l + 1), \\ \ldots \\ \deg a_0(x) = (m - 1)(l + 1). \end{cases}$$

Then we have

$$\begin{cases} \deg[a_0(x)\alpha(x)] = (m-1)(l+1)+l, \\ \deg[a_1(x)g(x)] = (m-2)(l+1)+\xi. \end{cases} \tag{23}$$

It is easy to see that (23) also contradicts (13).

(II) Assume that $1 \le k < m$. Then by the first equation of (12), we have

$$a_{m-1}(x) = \frac{k-m}{l+1}c_l x^{l+1} + \cdots,$$

i.e.,

$$u_{m-1} = \frac{k-m}{l+1}c_l,$$

and using the second equation, third equation, ..., and $(m-k)$th equation of (12), respectively, we have

$$\begin{cases} a_{m-2}(x) = \frac{(k-m)(k-m+1)}{2!\cdot(l+1)^2}c_l^2 x^{2(l+1)} + a_{m-2}^{(2l+1)}(x), \\ a_{m-3}(x) = \frac{(k-m)(k-m+1)(k-m+2)}{3!\cdot(l+1)^3}c_l^3 x^{3(l+1)} + a_{m-3}^{(3l+2)}(x), \\ \cdots \\ a_{k+1}(x) = \frac{(k-m)(k-m+1)(k-m+2)\cdots(-2)}{(m-k-1)!\cdot(l+1)^{m-k-1}}c_l^{m-k-1}x^{(m-k-1)(l+1)} \\ \qquad +a_{k+1}^{((m-k-1)(l+1)-1)}(x), \\ a_k(x) = \frac{(k-m)(k-m+1)(k-m+2)\cdots(-2)\cdot(-1)}{(m-k)!\cdot(l+1)^{m-k}}c_l^{m-k}x^{(m-k)(l+1)} \\ \qquad +a_k^{((m-k)(l+1)-1)}(x), \end{cases} \tag{24}$$

where $a_i^{(j)}(x)$ are the polynomials in x with degree at most j.

By (18) and (24), we have

$$u_{k-1} = -(k+1)\cdot u_{k+1}\cdot g_\xi \cdot \frac{1}{(m-k)(l+1)+\xi-l}$$

$$= -(k+1)\cdot \frac{(k-m)(k-m+1)(k-m+2)\cdots(-2)}{(m-k-1)!\cdot(l+1)^{m-k-1}}c_l^{m-k-1}\cdot g_\xi \cdot \frac{1}{(m-k)(l+1)+\xi-l}.$$

When we take $j = k-2$ in (14), we obtain

$$a'_{k-2}(x) = a_{k-1}(x)f(x) - ka_k(x)g(x). \tag{25}$$

By (20), (21), and (25), we have that $\deg a_{k-2}(x) < (m-k)(l+1)+\xi+1$. Otherwise, we have the following.

(i) Assume that $\deg a_{k-2}(x) = (m-k)(l+1)+\xi+1$. By the $(k-2)$th equation from the bottom of (12), the $(k-3)$th equation from the bottom of (12), ..., the second equation from the bottom of (12), we deduce

$$\begin{cases} \deg a_{k-3}(x) = (m-k+1)(l+1)+\xi+1, \\ \deg a_j(x) = (m-j-2)(l+1)+\xi+1 \\ \qquad\qquad (j = k-2, k-3, \ldots, 1), \\ \deg a_0(x) = (m-2)(l+1)+\xi+1. \end{cases}$$

Since $\deg a_0(x)\alpha(x) = (m-2)(l+1)+\xi+l+1$, $a_1(x)g(x) = (m-3)(l+1)+\xi+\xi+1$, by (13) we deduce that $2l+1 = \xi$. This contradicts our previous assumption $l > \xi$.

(ii) Similarly, $\deg a_{k-2}(x) > (m-k)(l+1)+\xi+1$ is impossible. Hence, the coefficient of $x^{(m-k)(l+1)+\xi}$ on the right-hand side of (25) must be zero, that is,

$$u_{k-1} \cdot c_l \cdot [k-(k-1)] = k \cdot u_k \cdot g_\xi,$$

that is,

$$-(k+1) \cdot \frac{(k-m)(k-m+1)(k-m+2)\cdots(-2)}{(m-k-1)! \cdot (l+1)^{m-k-1}} c_l^{m-k-1} \cdot g_\xi \cdot \frac{1}{(m-k)(l+1)+\xi-l} \cdot c_l$$
$$= k \cdot \frac{(k-m)(k-m+1)(k-m+2)\cdots(-2)\cdot(-1)}{(m-k)! \cdot (l+1)^{m-k}} c_l^{m-k} \cdot g_\xi.$$

Simplifying, we have

$$\frac{k+1}{(m-k)(l+1)+\xi-l} = \frac{k}{(m-k)(l+1)},$$

that is,

$$(\xi+1)k = m(l+1). \tag{26}$$

Since $l > \xi$, from (26) we deduce that $k > m$. This yields the contradiction with Claim 3.1. Therefore, the second-order polynomial autonomous system (1) has no algebraic curve solution in $\mathbb{C}$ under the condition $l > \xi$. The proof is complete. ∎

THEOREM 3.3. *If $l = \xi$ and $f(x) \not\equiv cg(x)$ $(c \neq 0)$, then the second-order polynomial autonomous system* (1) *has no algebraic curve solution in* $\mathbb{C}$.

Proof. Suppose that the second-order polynomial autonomous system (1) has the algebraic curve solution of the form (10). The proof is similar to that of Theorem 3.2.

First, we prove that $k \neq m$. By way of contradiction, we assume that $k = m$. Then by the first equation of (12), we deduce that $a_{m-1}(x) = \text{constant}$.

(a) In the case $a_{m-1}(x)=0$, using the same argument as in case (1) of Theorem 3.2, we have

$$\begin{cases} \deg a_{m-2}(x) = l+1, \\ \deg a_{m-3}(x) = 2(l+1), \\ \deg a_{m-4}(x) = 3(l+1), \\ \dots \\ \deg a_0(x) = (m-3)(l+1). \end{cases}$$

Then we have

$$\begin{cases} \deg[a_0(x)\alpha(x)] = (m-1)(l+1)+l, \\ \deg[a_1(x)g(x)] = (m-2)(l+1)+\xi. \end{cases} \tag{27}$$

It is easy to see that (27) contradicts (13).

(b) In the case $a_{m-1}(x)=c_0\neq 0$, since $f(x)\not\equiv cg(x)$ $(c\neq 0)$, we have that $a'_{m-2}(x)=c_0f(x)-mg(x)\neq 0$. Denote that $\deg[c_0f(x)-mg(x)]=t_0\geq 0$. By (12), we deduce

$$\begin{cases} \deg a_{m-2}(x) = t_0+1, \\ \deg a_{m-3}(x) = t_0+1+(l+1), \\ \deg a_{m-4}(x) = t_0+1+2(l+1), \\ \dots \\ \deg a_0(x) = t_0+1+(m-2)(l+1). \end{cases}$$

Then we have

$$\begin{cases} \deg[a_0(x)\alpha(x)] = (m-2)(l+1)+t_0+1+l, \\ \deg[a_1(x)g(x)] = (m-3)(l+1)+l. \end{cases} \tag{28}$$

It is easy to see that (28) also contradicts $t_0\geq 0$. By (a) and (b), we conclude that $k\neq m$. By virtue of Claim 3.1, there exists some k $(1\leq k<m)$ such that $\alpha(x)=kf(x)$.

Second, we prove that under the conditions $l=\xi$, $f(x)\not\equiv cg(x)$ $(c\neq 0)$, and $(1\leq k<m)$, (1) has no algebraic curve solution in the complex domain $\mathbb{C}$. Recall that $f(x)=\sum_{i=0}^{l} c_ix^i$ and $g(x)=\sum_{j=0}^{l} g_jx^j$. Using the second equation, third

equation, ..., and $(m-k+2)$th equation of (12), we deduce

$$\left\{\begin{array}{l} a_{m-1}(x) = \frac{k-m}{l+1} c_l x^{l+1} + a_{m-1}^{(l)}(x), \\ a_{m-2}(x) = \frac{(k-m)(k-m+1)}{2!\cdot(l+1)^2} c_l^2 x^{2(l+1)} + a_{m-2}^{(2l+1)}(x), \\ \cdots \\ a_{k+1}(x) = \frac{(k-m)(k-m+1)\cdot(-2)}{(m-k-1)!\cdot(l+1)^{m-k-1}} c_l^{m-k-1} x^{(m-k-1)(l+1)} \\ \qquad + a_{k+1}^{((m-k-1)(l+1)-1)}(x), \\ a_k(x) = \frac{(k-m)(k-m+1)\cdot(-2)\cdot(-1)}{(m-k)!\cdot(l+1)^{m-k}} c_l^{m-k} x^{(m-k)(l+1)} \\ \qquad + a_k^{((m-k)(l+1)-1)}(x), \\ a_{k-1}(x) = \frac{-(k+1)(k-m)(k-m+1)\cdot(-2)\cdot(-1)}{(m-k)!\cdot(l+1)^{m-k}} c_l^{m-k-1} \cdot g_l x^{(m-k)(l+1)} \\ \qquad + a_{k-1}^{((m-k)(l+1)-1)}(x), \end{array}\right. \tag{29}$$

where $a_i^{(j)}(x)$ are polynomials in x with the degree at most j.

On the other hand, by (21) we have

$$\deg a_0(x) = \deg a_1(x) = \cdots = \deg a_k(x). \tag{30}$$

By (20) and (30), we deduce that $\deg a_{k-2}(x) < (m-k)(l+1)+l$. Hence, the coefficient of $x^{(m-k)(l+1)+l}$ on the right-hand side of (25) must be zero. Thus, by (25) and (29), we have

$$\begin{aligned} &\frac{-(k+1)(k-m)(k-m+1)\cdot(-2)\cdot(-1)}{(m-k)!\cdot(l+1)^{m-k}} c_l^{m-k-1} \cdot g_l \cdot c_l \\ &= k \cdot \frac{(k-m)(k-m+1)\cdot(-2)\cdot(-1)}{(m-k)!\cdot(l+1)^{m-k}} \cdot c_l^{m-k} \cdot g_l. \end{aligned} \tag{31}$$

Since $k \neq m$, (31) implies that $k = k+1$. Obviously, this is impossible. Therefore, (1) has no algebraic curve solution in $\mathbb{C}$. □

4. Conclusion. The whole paper [3] discussed the nonexistence of the algebraic curve solution for system (1) in the case when $f(x) = \varepsilon(1-x^2)$ and $g(x) = -x$ (which is equivalent to (2)) by the method of undetermined coefficients. It is easily seen that this case is only one of the special cases of Theorem 3.2. Also by our theorems, when $\deg f(x) \geq \deg g(x)$ ($f(x) \not\equiv cg(x)$ if $\deg f(x) = \deg g(x)$), we know that the Lienard equation

$$\frac{d^2x}{dt^2} + f(x)\frac{dx}{dt} + g(x) = 0$$

has no algebraic curve solution in $\mathbb{C}$. Therefore, it is not integrable.

It is easily seen that the method of the analysis of algebraic curves and the division theorem for two variables in $\mathbb{C}$ is really an effective approach for investigating the existence of the algebraic curve solution of the polynomial autonomous systems, and the integrability of some nonlinear differential equations. Note that [13–19] were concerned with the rational solutions of some second-order ordinary differential equations. When the coefficients of these ordinary differential equations are polynomials, using the method in this paper, we can obtain the same results.

We also can apply this method to investigate the following polynomial autonomous systems:

$$\begin{cases} \dot{x} = E(x,y), \\ \dot{y} = F(x,y), \end{cases}$$

where $E(x,y)$ and $F(x,y)$ are polynomials in x and y in $\mathbb{C}$. When $E(x,y)$ and $F(x,y)$ are special polynomials in x and y (for example, when $E(x,y)$ and $F(x,y)$ are quadratic polynomials with y), we conjecture that some interesting and useful results may be obtained.

5. Acknowledgements. The author thanks Prof. Goong Chen, Prof. Keying Guan, and the referees for their help and useful suggestions.

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