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ON PROPERTIES OF SOME MATRIX SPLITTINGS*

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Abstract. Properties of some splittings of a monotone matrix A are discussed. These properties are fundamental hypotheses in the proofs of comparison theorems. This fact is illustrated by several results given in the literature.

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This paper is closely related to [10]. The following definitions of splittings are used.

DEFINITION 1.1. Let $M, N \in \mathbb{R}^{n\times n}$. Then the decomposition $A = M - N$ is called
(a) a *regular splitting of* A if $M^{-1} \geq 0$ and $N \geq 0$,
(b) a *nonnegative splitting of* A if $M^{-1} \geq 0$, $M^{-1}N \geq 0$ and $NM^{-1} \geq 0$,
(c) a *weak nonnegative splitting of* A if $M^{-1} \geq 0$ and either $M^{-1}N \geq 0$ (the *first type*) or $NM^{-1} \geq 0$ (the *second type*),
(d) a *weak splitting of* A if M is nonsingular, $M^{-1}N \geq 0$ and $NM^{-1} \geq 0$,
(e) a *weaker splitting of* A if M is nonsingular and either $M^{-1}N \geq 0$ (the *first type*) or $NM^{-1} \geq 0$ (the *second type*),
(f) a *convergent splitting of* A if $\varrho(M^{-1}N) = \varrho(NM^{-1}) < 1$.

The splittings defined in the successive items extend successively a class of splittings of $A = M - N$ for which the matrices N and M^{-1} may lose the properties of nonnegativity. Distinguishing both types of weak nonnegative and weaker splittings leads to further extensions allowing us to analyze cases when $M^{-1}N$ may have negative entries even if NM^{-1} is a nonnegative matrix.

It is necessary to mention that the definition assumed in item (b) is equivalent to the definition of weak regular splitting of A introduced originally by Ortega and Rheinboldt [6]. However, some authors, for instance Berman and Plemmons [1], using the same name "weak regular splitting", restrict this definition to its weaker version based on the conditions $M^{-1} \geq 0$ and $M^{-1}N \geq 0$ without the condition $NM^{-1} \geq 0$, which corresponds to a weak nonnegative splitting of the first type defined in item (c).

In the literature there are many comparison theorems proven under different hypotheses for the splittings of the above definition.

Conditions ensuring that a splitting of $A = M - N$ is convergent are unknown in a general case. However, as is well known [8, 9], the splittings defined in first three

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items of Definition 1.1 are convergent if and only if $A^{-1} \geq 0$.

The properties of weak nonnegative splittings are summarized in the following theorem.

THEOREM 1.2. [8] *Let $A = M - N$ be a weak nonnegative splitting of A. If $A^{-1} \geq 0$, then:*
1. $A^{-1} \geq M^{-1}$.
2. $\varrho(M^{-1}N) = \varrho(NM^{-1}) < 1$.
3. *If $M^{-1}N \geq 0$, then $A^{-1}N \geq M^{-1}N$ and if $NM^{-1} \geq 0$, then $NA^{-1} \geq NM^{-1}$.*
4. $\varrho(M^{-1}N) = \dfrac{\varrho(A^{-1}N)}{1+\varrho(A^{-1}N)} = \dfrac{\varrho(NA^{-1})}{1+\varrho(NA^{-1})} < 1$.

Conversely, if $\varrho(M^{-1}N) < 1$, then $A^{-1} \geq 0$.

The relation in item 4 was obtained by Varga [7] for regular splittings of A. From the above theorem we can conclude the following corollary.

COROLLARY 1.3. *Each weak nonnegative (as well as nonnegative and regular) splitting of $A = M - N$ is convergent if and only if $A^{-1} \geq 0$. In other words, if A is not a monotone matrix, it is impossible to construct a convergent weak nonnegative splitting.*

It is obvious that when two weak nonnegative splittings of a monotone matrix $A = M_1 - N_1 = M_2 - N_2$ (i.e., $A^{-1} \geq 0$) are of the same type, the inequality

$$N_2 \geq N_1 \tag{1.1}$$

implies either

$$A^{-1}N_2 \geq A^{-1}N_1 \geq 0$$

or

$$N_2A^{-1} \geq N_1A^{-1} \geq 0.$$

Hence, we have $\varrho(A^{-1}N_2) \geq \varrho(A^{-1}N_1)$ and by Theorem 1.2 we can conclude that $\varrho(M_1^{-1}N_1) \leq \varrho(M_2^{-1}N_2)$.

Usually, for weak nonnegative splittings the matrices $M_1^{-1}N_1 \geq 0$ or $N_2M_2^{-1} \geq 0$ are accompanied by the conditions $N_1A^{-1} \ngeq 0$ or $A^{-1}N_2 \ngeq 0$, respectively. When both weak nonnegative splittings are of different types, one of the matrices $A^{-1}N_2$ and $A^{-1}N_1$ or N_2A^{-1} and N_1A^{-1} may have negative entries, which does not allow us to conclude that the inequality $\varrho(M_1^{-1}N_1) \leq \varrho(M_2^{-1}N_2)$ is satisfied. Let us assume that $M_1^{-1}N_1 \geq 0$ and $N_2M_2^{-1} \geq 0$ which implies that $A^{-1}N_1 \geq 0$ and $N_2A^{-1} \geq 0$, then from (1.1) we have

$$A^{-1}N_2 \geq A^{-1}N_1 \geq 0 \quad \text{and} \quad N_2A^{-1} \geq N_1A^{-1} \ngeq 0$$

which leads to the conclusion that the second splitting should be a nonnegative splitting. In the case when $N_1M_1^{-1} \geq 0$ and $M_2^{-1}N_2 \geq 0$, similar considerations allow us to conclude again that the second splitting should be a nonnegative splitting.

Since for each weak nonnegative splitting we have $A^{-1}N \geq M^{-1}N$ or $NA^{-1} \geq NM^{-1}$, it may occur that, for instance, $NA^{-1} \geq 0$ with $NM^{-1} \ngeq 0$. Just such an

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example of splitting can be obtained using the examples of Climent and Perea [2] as follows.

$$A = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & -2 \\ -1 & 0 & 2 \end{bmatrix} = M_1 - N_1 = M_2 - N_2, \text{where} A^{-1} = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3/2 & 1 \\ 1 & 1 & 1 \end{bmatrix} > 0,$$

and

$$M_1 = \begin{bmatrix} 1 & -3/2 & 3/4 \\ 0 & 2 & -201/100 \\ -1 & -1/2 & 3 \end{bmatrix}, \; N_1 = \begin{bmatrix} 0 & 1/2 & -1/4 \\ 0 & 0 & -1/100 \\ 0 & -1/2 & 1 \end{bmatrix},$$

$$M_1^{-1} = \frac{1}{696}\begin{bmatrix} 999 & 825 & 303 \\ 402 & 750 & 402 \\ 400 & 400 & 400 \end{bmatrix},$$

$$M_2 = \begin{bmatrix} 1 & -3/2 & 3/4 \\ 0 & 2 & -2 \\ 0 & -1/2 & 3 \end{bmatrix}, \; N_2 = \begin{bmatrix} 0 & 1/2 & -1/4 \\ 0 & 0 & 0 \\ 1 & -1/2 & 1 \end{bmatrix},$$

$$M_2^{-1} = \frac{1}{40}\begin{bmatrix} 40 & 33 & 12 \\ 0 & 24 & 16 \\ 0 & 4 & 16 \end{bmatrix},$$

$$M_1^{-1}N_1 = \frac{1}{696}\begin{bmatrix} 0 & 348 & 45 \\ 0 & 0 & 309 \\ 0 & 0 & 296 \end{bmatrix}, \; N_1M_1^{-1} = \frac{1}{696}\begin{bmatrix} 101 & 275 & 101 \\ -4 & -4 & -4 \\ 199 & 25 & 199 \end{bmatrix},$$

$$M_2^{-1}N_2 = \frac{1}{20}\begin{bmatrix} 6 & 17 & 1 \\ 8 & -4 & 8 \\ 8 & -4 & 8 \end{bmatrix}, \; N_2M_2^{-1} = \frac{1}{40}\begin{bmatrix} 0 & 11 & 4 \\ 0 & 0 & 0 \\ 40 & 17 & 20 \end{bmatrix},$$

$$A^{-1}N_2 = \frac{1}{4}\begin{bmatrix} 4 & 2 & 2 \\ 4 & 0 & 3 \\ 4 & 0 & 3 \end{bmatrix} \geq A^{-1}N_1 = \frac{1}{200}\begin{bmatrix} 0 & 100 & 92 \\ 0 & 0 & 147 \\ 0 & 0 & 148 \end{bmatrix} \geq 0,$$

$$N_2A^{-1} = \begin{bmatrix} 1/4 & 1/2 & 1/4 \\ 0 & 0 & 0 \\ 5/2 & 9/4 & 3/2 \end{bmatrix} \geq N_1A^{-1} = \begin{bmatrix} 1/4 & 1/2 & 1/4 \\ -1/100 & -1/100 & -1/100 \\ 1/2 & 1/4 & 1/2 \end{bmatrix} \ngeq 0.$$

In this example the first splitting is weak nonnegative of the first type with $N_1A^{-1} \ngeq 0$ but the second splitting is weak nonnegative of the second type with $A^{-1}N_2 \geq 0$ and $N_2A^{-1} \geq 0$. Since in this case assumption (1.1) is satisfied and moreover $A^{-1}N_2 \geq A^{-1}N_1 \geq 0$, we can conclude that

$$0.4253 = \varrho(M_1^{-1}N_1) < \varrho(M_2^{-1}N_2) = 0.6531. \tag{1.2}$$

The above result indicates existing a subclass of weak nonnegative splittings with stronger conditions $A^{-1}N \geq 0$ and $NA^{-1} \geq 0$ which leads to the following reformulation of weak nonnegative splittings.

Definition 1.4. Let $M, N \in \mathbb{R}^{n\times n}$. Then the decomposition $A = M - N$ is called:
(c') a *strict weak nonnegative splitting of* A if $M^{-1} \geq 0$ and either $M^{-1}N \geq 0$ (the *first type*) or $NM^{-1} \geq 0$ (the *second type*), where $A^{-1}N \geq 0$ and $NA^{-1} \geq 0$,
(c") a *nonstrict weak nonnegative splitting of* A if $M^{-1} \geq 0$ and either $M^{-1}N \geq 0$ (the *first type*) or $NM^{-1} \geq 0$ (the *second type*), where either $A^{-1}N \geq 0$ and $NA^{-1} \ngeq 0$, or $A^{-1}N \ngeq 0$ and $NA^{-1} \geq 0$, respectively.

A similar reformulation may be done for the case of weaker splittings.

In the case of the weaker condition $M_1^{-1} \geq M_2^{-1}$ the contrary behavior is observed. As is demonstrated on examples in [8], when both weak nonnegative splittings of a monotone matrix A are the same type, with $M_1^{-1} \geq M_2^{-1}$ (or even $M_1^{-1} > M_2^{-1}$) it may occur that $\varrho(M_1^{-1}N_1) > \varrho(M_2^{-1}N_2)$. The conclusions for the case of weak nonnegative splittings are summarized in the following corollary.

Corollary 1.5. *Let $A = M_1 - N_1 = M_2 - N_2$ be two weak nonnegative splittings, where $A^{-1} \geq 0$, then:*
(a) *the assumption $N_2 \geq N_1$ allows us to prove that $\varrho(M_1^{-1}N_1) \leq \varrho(M_2^{-1}N_2)$ if one of splittings is at least strict, this assumption may be not valid when both splittings are nonstrict of different types,*
(b) *if both splittings are of the same type, then the assumption $M_1^{-1} \geq M_2^{-1} \geq 0$ is not a sufficient condition for proving that $\varrho(M_1^{-1}N_1) \leq \varrho(M_2^{-1}N_2)$.*

For nonnegative splittings we have the following result.

Theorem 1.6. [8] *Let $A = M_1 - N_1 = M_2 - N_2$ be two nonnegative splittings of A, where $A^{-1} \geq 0$. If $M_1^{-1} \geq M_2^{-1}$, then*

$$\varrho(M_1^{-1}N_1) \leq \varrho(M_2^{-1}N_2).$$

If $A^{-1} > 0$ and $M_1^{-1} > M_2^{-1}$, then

$$\varrho(M_1^{-1}N_1) < \varrho(M_2^{-1}N_2).$$

But for different types of weak nonnegative splittings there is the similar result.

Theorem 1.7. [8] *Let $A = M_1 - N_1 = M_2 - N_2$ be two weak nonnegative splittings of different types, that is, either $M_1^{-1}N_1 \geq 0$ and $N_2M_2^{-1} \geq 0$ or $N_1M_1^{-1} \geq 0$ and $M_2^{-1}N_2 \geq 0$, where $A^{-1} \geq 0$. If $M_1^{-1} \geq M_2^{-1}$, then*

$$\varrho(M_1^{-1}N_1) \leq \varrho(M_2^{-1}N_2).$$

If $A^{-1} > 0$ and $M_1^{-1} > M_2^{-1}$, then

$$\varrho(M_1^{-1}N_1) < \varrho(M_2^{-1}N_2).$$

Remark 1.8. Obviously, the case of two mixed splittings of $A = M_1 - N_1 = M_2 - N_2$ (i.e., when one of them is nonnegative and the second is weak nonnegative) is fulfilled by the assumptions of Theorem 1.7.

Theorem 1.7 is valid independently if the splittings are strict or nonstrict. As is well known for weak nonnegative splittings the condition $N_2 \geq N_1$ implies the condition $M_1^{-1} \geq M_2^{-1} \geq 0$ but the contrary statement may be not valid [8]. In

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the example considered above we have $M_1^{-1} > M_2^{-1} \geq 0$, which implies the strict inequality in (1.2).

In the case of the Berman and Plemmons' weak regular splitting [1], corresponding to the weak nonnegative splitting of the first type, Elsner [3] showed that the assumption $M_1^{-1} \geq M_2^{-1} \geq 0$ may be not a sufficient hypothesis for ensuring the inequality $\varrho(M_1^{-1}N_1) \leq \varrho(M_2^{-1}N_2)$ and he stated result of Theorem 1.7 for the case when one of the splittings is regular. This means that Elsner restored the need of the condition $NM^{-1} \geq 0$ sticking originally in the Ortega and Rheinboldt's definition of weak regular splitting. It is evident that Elsner's result is a particular case of Theorem 1.7.

The Ortega and Rheinboldt's definition of weak regular splitting is used by Lanzkron, Rose, and Szyld [4] and they have proved the following theorem.

Theorem 1.9. ([4, Theorem 3.1]) *Let $A = M_1 - N_1 = M_2 - N_2$ be convergent weak regular splittings (that is, nonnegative splittings) such that*

$$M_1^{-1} \geq M_2^{-1},$$

and let x and z be the nonnegative Frobenius eigenvectors of $M_1^{-1}N_1$ and $M_2^{-1}N_2$, respectively. If $N_2z \geq 0$ or if $N_1x \geq 0$ with $x > 0$, then

$$\varrho(M_1^{-1}N_1) \leq \varrho(M_2^{-1}N_2).$$

As can be deduced from Corollary 1.3, the term "'convergent" is equivalent to the assumption that $A^{-1} \geq 0$. Since $M^{-1} \geq 0$, $M^{-1}N \geq 0$ and $\varrho(M^{-1}N) < 1$ by the assumption, then

$$A^{-1} = [I - M^{-1}N]^{-1}M^{-1} = [I + M^{-1}N + (M^{-1}N)^2 + (M^{-1}N)^3 + \ \dots\ ]M^{-1} \geq 0$$

and conversely, if $A^{-1} \geq 0$, then $\varrho(M^{-1}N) < 1$.

As follows from Theorem 1.6, the hypothesis $M_1^{-1} \geq M_2^{-1}$ is a sufficient condition in this theorem and the assumptions $N_2z \geq 0$ or $N_1x \geq 0$ with $x > 0$ are superfluous because each one follows from the properties of nonnegative splittings. For each nonnegative splitting of $A = M - N$, where $A^{-1} \geq 0$ and $1 > \lambda = \varrho(M^{-1}N) \geq 0$, one can write

$$M^{-1}Nx = \lambda x, \ \text{ where } \ x \geq 0,$$

or equivalently

$$\begin{aligned} Nx &= \lambda Mx, \\ NM^{-1}Mx &= \lambda Mx, \\ NM^{-1}y &= \lambda y. \end{aligned}$$

Since NM^{-1} is a nonnegative matrix, then its eigenvector $y = Mx \geq 0$, hence

$$Nx = \lambda y \geq 0.$$

Thus, this theorem supplied with additional but completely superfluous conditions, is equivalent to Theorem 1.6.

REMARK 1.10. We point out that Theorem 1.9 was restated in [5, Theorem 3.5] but for weak nonnegative splittings of the first type. In that reformulation, the conditions $N_2 z \geq 0$ or $N_1 x \geq 0$ are not superfluous.

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