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NATURAL GROUP ACTIONS ON TENSOR PRODUCTS OF THREE REAL VECTOR SPACES WITH FINITELY MANY ORBITS*

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Abstract. Let G be the direct product of the general linear groups of three real vector spaces U,V,W of dimensions l,m,n $(2 \le l \le m \le n < \infty)$. Consider the natural action of G on the tensor product of these spaces. The number of G-orbits in X is finite if and only if $l = 2$ and $m = 2$ or 3. In these cases the G-orbits and their connected components are classified, and the closure of each of the components is determined. The proofs make use of recent results of P.G. Parfenov, who solved the same problem for complex vector spaces.

Key words. Matrix pencils, Kronecker-Weierstrass theory, prehomogeneous vector spaces, relative invariants.

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1. Introduction. First we set up some notation. Let U, V, W be real vector spaces of dimension l, m, n respectively, where $2 \le l \le m \le n$ and denote by U^c, V^c, W^c their complexifications. Set $X = U \otimes_{\mathbb{R}} V \otimes_{\mathbb{R}} W$ and $X^c = U^c \otimes_{\mathbb{C}} V^c \otimes_{\mathbb{C}} W^c$. We consider the natural action of the group $G = \mathrm{GL}(U) \times \mathrm{GL}(V) \times \mathrm{GL}(W)$ on X, and that of $G^c = \mathrm{GL}(U^c) \times \mathrm{GL}(V^c) \times \mathrm{GL}(W^c)$ on X^c.

It is well known that the number of G^c-orbits in X^c is finite if and only if $l = 2$ and $m = 2$ or 3; see [9]. When $l = m = n = 2$ there are seven orbits, their representatives and isotropy subalgebras were described in [6, Proposition 5.19] and [2]. When $l = 2$ and $m = n = 3$ there are eighteen orbits, and their representatives were obtained in [7, Table 1, ℓ=2]. In a recent announcement [8], Parfenov gives a list of representatives of G^c-orbits for all the finite cases. Moreover, he has determined the closure of each of the G^c-orbits in X^c.

For any real vector space $\mathcal{V}$ denote by $\mathrm{GL}^+(\mathcal{V})$ the subgroup of all $a \in \mathrm{GL}(\mathcal{V})$ with positive determinant. For U,V,W as above, let $G^+ = \mathrm{GL}^+(U) \times \mathrm{GL}^+(V) \times \mathrm{GL}^+(W)$. In this paper we enumerate the G^+-orbits in X for the cases when $l = 2$ and $m = 2$ or 3 (see Theorem 5.1), and also find their closures (see Theorem 6.1 and the Appendix).

We found that for each G^c-orbit $\mathcal{O}^c \subset X^c$, $\mathcal{O}^c \cap X$ is either a single G-orbit, or a union of two G-orbits, so there are a few more G-orbits in X than G^c-orbits in X^c. Furthermore, in the cases where n is small, many of the G-orbits are not connected. Since G^+ is the identity component of G, the G^+-orbits are simply the connected components of the G-orbits, so we get a larger number of G^+-orbits. The additional orbits make the closure diagrams more complicated than in the complex case. With the restriction to G^+, orbits that are distinct for one space may be contained in a single orbit for a larger space. For this reason we have made several diagrams to

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capture all the cases, as opposed to the previous work of Parfenov [8] where one diagram could display all cases. Our main results are stated in Theorems 5.1 and 6.1. Theorem 5.1 gives the classification of the G and G^+-orbits in X for each m and n, and Theorem 6.1 discribes the closures of the G^+-orbits.

REMARK 1.1. Note that the G^+-orbits in X are also the orbits of the smaller group $\mathrm{SL}(U)\times\mathrm{SL}(V)\times\mathrm{SL}(W)\times\mathbb{R}_+^*$, where $\mathbb{R}_+^*$ is the multiplicative group of positive real numbers which acts on X by scalar multiplication. Since $l=2$, $-I\in\mathrm{SL}(U)$ and $(-I,I,I)\cdot\xi=-\xi$ for all $\xi\in X$. Consequently, there is a one-to-one correspondence between the nonzero orbits of G^+ in X and the orbits of $\mathrm{SL}(U)\times\mathrm{SL}(V)\times\mathrm{SL}(W)$ in the projective space $\mathbf{P}(X)$. Hence our results give also the classification of the orbits (and their closures) in $\mathbf{P}(X)$ under the action of $\mathrm{SL}(U)\times\mathrm{SL}(V)\times\mathrm{SL}(W)$.

2. Preliminaries. We fix a basis $\{u_1,u_2,...,u_l\}$ of U, a basis $\{v_1,v_2,...,v_m\}$ of V, and $\{w_1,w_2,...,w_n\}$ of W. Then the elements $e_{ijk}=u_i\otimes v_j\otimes w_k$ form a basis of X (and X^c). So, using summation convention, every tensor $\xi\in X^c$ can be written as a linear combination $\xi=\xi^{ijk}e_{ijk}$. An element $a\in\mathrm{GL}(U^c)$ is identified with its matrix representation (a_i^p) with respect to the above basis, so that $a(u_i)=a_i^pu_p$. Thus p is the row index and i the column index in the matrix (a_i^p). Similar conventions are used for $b\in\mathrm{GL}(V^c)$ and $c\in\mathrm{GL}(W^c)$.

We assume, from now on, that $l=2$ and $m=2$ or 3. The tensor ξ will be associated with the pair (A_ξ,B_ξ) of $m\times n$ matrices $A_\xi=(\xi^{1jk})$ and $B_\xi=(\xi^{2jk})$, where we agree that j is the row index and k the column index. We often replace the pair (A_ξ,B_ξ) by the matrix pencil $\lambda A_\xi+\mu B_\xi$ where λ and μ are indeterminates.

For $\eta=(a,b,c)\cdot\xi$ we have, using summation convention, $\eta^{pqr}=a_i^pb_j^qc_k^r\xi^{ijk}$. When we look at the matrix pencils associated with each tensor, they are related by $\lambda A_\eta+\mu B_\eta=b(\lambda(a_1^1A_\xi+a_2^1B_\xi)+\mu(a_1^2A_\xi+a_2^2B_\xi))c^T$, where c^T denotes the *transpose* of c. So our group acts by left and right multiplication by nonsingular matrices, along with a nonsingular linear change in variables λ and μ.

The equivalence classes of matrix pencils, i.e., the orbits under left and right multiplication by invertible constant matrices (matrices that do not depend on λ or μ) have been studied in an old theory by Kronecker and Weierstrass, which is valid over any field. These correspond to the orbits in our X under the smaller group $\mathrm{GL}(V)\times\mathrm{GL}(W)$. From this we derive the following result:

For any matrices A and B, denote by $A\oplus B$ the matrix $\begin{bmatrix} A & 0\\ 0 & B\end{bmatrix}$, with the obvious generalization for more than two matrices. We allow matrices having no rows or columns. Thus if, say, B is a 0 by k matrix, then $A\oplus B=[A\ 0]$ is the matrix obtained from A by appending k zero columns.

For any nonnegative integer ε let L_ε be the $\varepsilon\times(\varepsilon+1)$ matrix

$$\begin{bmatrix} \lambda & \mu & 0 & ... & 0\\ 0 & \lambda & \mu & ... & 0\\ \vdots & & & & \\ 0 & 0 & ... & \lambda & \mu \end{bmatrix}.$$

Then every matrix pencil is equivalent to one of the form

$$C \oplus L_{\varepsilon_1} \oplus L_{\varepsilon_2} \oplus ... \oplus L_{\varepsilon_p} \oplus L^T_{\eta_1} \oplus L^T_{\eta_2} \oplus ... \oplus L^T_{\eta_q}\,,$$

where C is a square matrix pencil with non-vanishing determinant, which will be called the *regular core* of the pencil. Furthermore the minimal indices ε_i and η_j are uniquely determined up to permutation, as are the elementary divisors of C. As well, it was shown by Ja'Ja [5] that the minimal indices of a pencil remain unchanged by the action of $\mathrm{GL}(U^c)$. The elementary divisors of C on the other hand are transformed by a nonsingular linear change in variables λ and μ. As pointed out by Atkinson [1], there is an error in Ja'ja's paper [5], which results in his Theorems 2, 3 and 4 being false. However the proof of his Theorem 1, quoted below, is correct. (We state it only for the case of the complex field.)

THEOREM 2.1. [5, Theorem 1] *Given an $m \times n$ singular pencil of matrices $\lambda A_\xi + \mu B_\xi$ over $\mathbb{C}$, the Kronecker minimal indices are invariant under the action of G^c.*

We also need a theorem of Atkinson [1]. To state Atkinson's theorem, we must first introduce some of his notation. His theorem deals with the case $m = n$ and $\det(\lambda A_\xi + \mu B_\xi) \neq 0$ (the so called *regular case*). Let $D_k(\lambda, \mu)$ be the greatest common divisor of all $k \times k$ minors of $\lambda A_\xi + \mu B_\xi$. Then the classical homogeneous invariant polynomials are defined by

$$i_k(\lambda,\mu) = \frac{D_{n-k+1}(\lambda,\mu)}{D_{n-k}(\lambda,\mu)}, \quad 1 \le k \le n.$$

It can be shown that these are in fact polynomials, and furthermore that $i_k(\lambda, \mu)$ divides $i_{k-1}(\lambda, \mu)$. Factor these invariant polynomials into powers of, say r, distinct irreducible polynomials:

$$\begin{aligned} i_1(\lambda,\mu) &= \phi_1(\lambda,\mu)^{\tau_{11}} \phi_2(\lambda,\mu)^{\tau_{12}} \dots \phi_r(\lambda,\mu)^{\tau_{1r}}, \\ i_2(\lambda,\mu) &= \phi_1(\lambda,\mu)^{\tau_{21}} \phi_2(\lambda,\mu)^{\tau_{22}} \dots \phi_r(\lambda,\mu)^{\tau_{2r}}, \\ \\ i_n(\lambda,\mu) &= \phi_1(\lambda,\mu)^{\tau_{n1}} \phi_2(\lambda,\mu)^{\tau_{n2}} \dots \phi_r(\lambda,\mu)^{\tau_{nr}}, \end{aligned}$$

where $\tau_{st} \le \tau_{s-1,t}$, and $\tau_{1s} > 0$. Then define the family of vectors

$$v_1 = (\tau_{11}, \dots, \tau_{n1}), \quad v_2 = (\tau_{12}, \dots, \tau_{n2}), \ \dots, \ v_r = (\tau_{1r}, \dots, \tau_{nr}).$$

(Again we specify the field to be $\mathbb{C}$.)

THEOREM 2.2. [1] *Two regular tensors ξ_1 and ξ_2 lie in the same G^c-orbit if and only if*

(a) *The familly of vectors defined above is the same for $\lambda A_{\xi_1} + \mu B_{\xi_1}$ as for $\lambda A_{\xi_2} + \mu B_{\xi_2}$, and*

(b) *The action of $\mathrm{GL}(U^c)$ can simultaneously take each irreducible invariant factor of $\lambda A_{\xi_1} + \mu B_{\xi_1}$ to an irreducible invariant factor of $\lambda A_{\xi_2} + \mu B_{\xi_2}$ whose associated vector (as above) is the same.*

Since we force $m \le 3$, and work in $\mathbb{C}$, in our case (b) is always satisfied (provided that (a) holds). We can then list representatives for each G^c-orbit of regular cores:

$C_1 : [.]$ (the empty core)

$C_2 : [\lambda]$

$$C_3 : \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}, \qquad C_4 : \begin{bmatrix} \lambda & \mu \\ 0 & \lambda \end{bmatrix}, \qquad C_5 : \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix},$$

$$C_6 : \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}, \ C_7 : \begin{bmatrix} \lambda & \mu & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}, \ C_8 : \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \mu \end{bmatrix},$$

$$C_9 : \begin{bmatrix} \lambda & \mu & 0 \\ 0 & \lambda & \mu \\ 0 & 0 & \lambda \end{bmatrix}, \ C_{10} : \begin{bmatrix} \lambda & \mu & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \mu \end{bmatrix}, \ C_{11} : \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \lambda+\mu \end{bmatrix}.$$

Once these cores are known, Ja'ja's theorem allows us to combine them with the singular pencils L_ε and L_η^T to get representatives for all the orbits in X^c under G^c. See Table 3.1 (these orbits are also listed by Parfenov; see [8]).

Remark 2.3. There is a small error in Parfenov's paper. His results are correct if we interchange orbit 8 with orbit 11 and orbit 9 with 13. However, his ordering is nice in that the orbits are in order of non-decreasing dimension for large n. For this reason we prefer to use his numbering and change the statement of his Theorems 2 and 3. The necessary changes are: The representatives for the case $2 \times 2 \times 3$ are 1-7, 11, 13, not 1-9; the representatives for the cases $2 \times 2 \times n, n \geq 4$, are 1-7, 11, 13, 19, not 1-9, 19; in the graph of abuttings, there should be no arrow from 19 to 9, but instead an arrow from 19 to 13.

3. G^c-orbits in X^c. In this section we list the dimensions of each G^c orbit, and relative invariants for the action of G^c on X^c where they exist. The calculation of these relative invariants gives rise to quadratic forms which will prove useful later on.

In order to calculate the dimensions of each orbit, we first calculated the dimension of the stabilizer of the chosen representative in G^c. These calculations are fairly straightforward, so we will include only one example.

In Table 3.1 we list our representatives for the G^c-orbits for each $m = 2$ or 3 and $n \geq m$. The representatives are written for the smallest possible values of m and n only. To get the orbits for a particular set of parameters, take all representatives with smaller or equal dimension and add zeros to make them fit.

Example 3.1. Calculation of the stabilizer of representative 6 (see Table 3.1). We must solve the equation

$$\begin{bmatrix} b_1^1 & b_2^1 \\ b_1^2 & b_2^2 \end{bmatrix} \begin{bmatrix} a_1^1\lambda + a_1^2\mu & a_2^1\lambda + a_2^2\mu \\ 0 & a_1^1\lambda + a_1^2\mu \end{bmatrix} \begin{bmatrix} c_1^1 & c_1^2 \\ c_2^1 & c_2^2 \end{bmatrix} = \begin{bmatrix} \lambda & \mu \\ 0 & \lambda \end{bmatrix}$$

under the condition that the matrices

$$a = \begin{bmatrix} a_1^1 & a_2^1 \\ a_1^2 & a_2^2 \end{bmatrix}, \ b = \begin{bmatrix} b_1^1 & b_2^1 \\ b_1^2 & b_2^2 \end{bmatrix}, \ \text{and} \ c = \begin{bmatrix} c_1^1 & c_2^1 \\ c_1^2 & c_2^2 \end{bmatrix}$$

are nonsingular. In order to simplify the calculation, we let $d = (c^T)^{-1}$ and look at the equation

Table 3.1
Orbits in X^c and their dimensions

	Orbit	Dimension		Orbit	Dimension
1	$[.]$	0	15	$\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \mu \end{bmatrix}$	$3m+3n-3$
2	$[\lambda]$	$m+n$	16	$\begin{bmatrix} \lambda & \mu & 0 \\ 0 & \lambda & \mu \\ 0 & 0 & \lambda \end{bmatrix}$	$3m+3n-2$
3	$\begin{bmatrix} \lambda \\ \mu \end{bmatrix}$	$2m+n-1$	17	$\begin{bmatrix} \lambda & \mu & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \mu \end{bmatrix}$	$3m+3n-1$
4	$\begin{bmatrix} \lambda & \mu \end{bmatrix}$	$m+2n-1$	18	$\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \lambda+\mu \end{bmatrix}$	$3m+3n$
5	$\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$	$2m+2n-3$	19	$\begin{bmatrix} \lambda & \mu & 0 & 0 \\ 0 & 0 & \lambda & \mu \end{bmatrix}$	$2m+4n-4$
6	$\begin{bmatrix} \lambda & \mu \\ 0 & \lambda \end{bmatrix}$	$2m+2n-1$	20	$\begin{bmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & \mu \end{bmatrix}$	$3m+4n-6$
7	$\begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}$	$2m+2n$	21	$\begin{bmatrix} \lambda & \mu & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & \mu \end{bmatrix}$	$3m+4n-4$
8	$\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \\ 0 & \mu \end{bmatrix}$	$3m+2n-2$	22	$\begin{bmatrix} \lambda & 0 & 0 & 0 \\ 0 & \mu & 0 & 0 \\ 0 & 0 & \lambda & \mu \end{bmatrix}$	$3m+4n-3$
9	$\begin{bmatrix} \lambda & 0 \\ \mu & \lambda \\ 0 & \mu \end{bmatrix}$	$3m+2n-1$	23	$\begin{bmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & \mu & 0 \\ 0 & 0 & \lambda & \mu \end{bmatrix}$	$3m+4n-2$
10	$\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$	$3m+3n-8$	24	$\begin{bmatrix} \lambda & \mu & 0 & 0 \\ 0 & \lambda & \mu & 0 \\ 0 & 0 & \lambda & \mu \end{bmatrix}$	$3m+4n-1$
11	$\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & \mu \end{bmatrix}$	$2m+3n-2$	25	$\begin{bmatrix} \lambda & 0 & 0 & 0 & 0 \\ 0 & \lambda & \mu & 0 & 0 \\ 0 & 0 & 0 & \lambda & \mu \end{bmatrix}$	$3m+5n-6$
12	$\begin{bmatrix} \lambda & \mu & 0 \\ 0 & 0 & \lambda \\ 0 & 0 & \mu \end{bmatrix}$	$3m+3n-4$	26	$\begin{bmatrix} \lambda & \mu & 0 & 0 & 0 \\ 0 & \lambda & \mu & 0 & 0 \\ 0 & 0 & 0 & \lambda & \mu \end{bmatrix}$	$3m+5n-4$
13	$\begin{bmatrix} \lambda & \mu & 0 \\ 0 & \lambda & \mu \end{bmatrix}$	$2m+3n-1$	27	$\begin{bmatrix} \lambda & \mu & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda & \mu \end{bmatrix}$	$3m+6n-9$
14	$\begin{bmatrix} \lambda & \mu & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$	$3m+3n-4$			

$$\begin{bmatrix} b_1^1 & b_2^1 \\ b_1^2 & b_2^2 \end{bmatrix} \begin{bmatrix} a_1^1\lambda + a_1^2\mu & a_2^1\lambda + a_2^2\mu \\ 0 & a_1^1\lambda + a_1^2\mu \end{bmatrix} = \begin{bmatrix} \lambda & \mu \\ 0 & \lambda \end{bmatrix} \begin{bmatrix} d_1^1 & d_2^1 \\ d_1^2 & d_2^2 \end{bmatrix}.$$

This leads to eight equations:

$$\begin{array}{c} a_1^1 b_1^1 = d_1^1,\ a_2^1 b_1^1 + a_1^1 b_2^1 = d_2^1, \\ a_1^2 b_1^1 = d_1^2,\ a_2^2 b_1^1 + a_1^2 b_2^1 = d_2^2, \\ a_1^1 b_1^2 = d_1^2,\ a_2^1 b_1^2 + a_1^1 b_2^2 = d_2^2, \\ a_1^2 b_1^2 = 0,\ a_2^2 b_1^2 + a_1^2 b_2^2 = 0. \end{array}$$

Solving this system is not difficult, and we end up with the solution

$$a = \begin{bmatrix} a_1^1 & a_2^1 \\ 0 & a_2^2 \end{bmatrix},\ b = \begin{bmatrix} b_1^1 & b_2^1 \\ 0 & \frac{a_2^2 b_1^1}{a_1^1} \end{bmatrix},\ d = \begin{bmatrix} a_1^1 b_1^1 & a_2^1 b_1^1 + a_1^1 b_2^1 \\ 0 & a_2^2 b_1^1 \end{bmatrix}.$$

From this we see that the dimension of the stabilizer is 5, so the dimension of the orbit is $4+4+4-5=7$.

To find the dimension for general m and n we can use the following lemma.

LEMMA 3.2. *Let $\xi \in X^c$ be such that both the rows and columns of $\lambda A_\xi + \mu B_\xi$ are linearly independent over $\mathbb{C}$, and let δ be the dimension of $G^c \cdot \xi$. Let $V^* \supset V^c$ and $W^* \supset W^c$ be complex vector spaces of dimension $m^* \geq m$ and $n^* \geq n$, respectively. Let $X^* = U^c \otimes_{\mathbb{C}} V^* \otimes_{\mathbb{C}} W^*$ and $G^* = \mathrm{GL}(U^c) \times \mathrm{GL}(V^*) \times \mathrm{GL}(W^*)$. Then the dimension δ^* of $G^* \cdot \xi$ is $m(m^*-m) + n(n^*-n) + \delta$.*

Proof. Let $(a, b^*, c^*) \in G^*$. Write $b^* = \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix}$, where B_1 is $m \times m$. Also, taking $d^* = (c^{*T})^{-1}$, put $d^* = \begin{bmatrix} D_1 & D_2 \\ D_3 & D_4 \end{bmatrix}$, where D_1 is $n \times n$. The equation for the stabilizer then becomes:

$$\begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix} \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \lambda A_\xi + \mu B_\xi & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} D_1 & D_2 \\ D_3 & D_4 \end{bmatrix},$$

where $E = \lambda(a_1^1 A_\xi + a_2^1 B_\xi) + \mu(a_1^2 A_\xi + a_2^2 B_\xi)$.

Clearly B_2, B_4, D_3, D_4 can be anything, since they are multiplied by zero. By the independence of rows and columns of $\lambda A_\xi + \mu B_\xi$ it is clear that $B_3 = 0$ and $D_2 = 0$. The remaining condition for the equation to hold is just that the triple (a, B_1, D_1) is in the stabilizer of ξ for G^c acting on X^c. So the dimension of the stabilizer of ξ in G^* is $m^*(m^*-m) + n^*(n^*-n)$ plus the dimension of the stabilizer of ξ in G^c. Hence the dimension δ^* of $G^* \cdot \xi$ is

$$\delta^* = \dim(G^*) - [m^*(m*-m) + n^*(n^*-n) + \dim(G) - \delta] = m(m^*-m) + n(n^*-n) + \delta$$

as required. $\square$

Note that each of the representatives we have chosen has both its rows and columns linearly independent over $\mathbb{C}$, all zero rows and columns are removed, so by calculating the dimension in the smallest possible space X we can then easily find the dimension of the orbit for any larger space.

For instance, in Example 3.1 of orbit 6, $n = m = 2$ and $\delta = 7$. So for a $2 \times m^* \times n^*$ dimensional space with $m^*, n^* \geq 2$, orbit 6 has dimension $2m^* + 2n^* - 1$.

Now we need some more notation. For any vector space $\mathcal{V}$ over $\mathbb{C}$, denote by $S(\mathcal{V}^*)$ the algebra of all polynomial functions from $\mathcal{V}$ to $\mathbb{C}$. $S(\mathcal{V}^*) = \mathbb{C} \oplus S^1(\mathcal{V}^*) \oplus S^2(\mathcal{V}^*) \oplus \ldots$ where, for each k, $S^k(\mathcal{V}^*)$ is the space of all homogeneous polynomial functions of degree k from $\mathcal{V}$ to $\mathbb{C}$. There is a natural action of $\mathrm{GL}(\mathcal{V})$ on $S(\mathcal{V}^*)$ by $a \cdot f(v) = f(a^{-1} \cdot v)$ where $a \in \mathrm{GL}(\mathcal{V})$ and $v \in \mathcal{V}$. Each $S^k(\mathcal{V}^*)$ is invariant under the action, so for each k we get a group action on $S^k(\mathcal{V}^*)$. We say that elements in the same orbit under this action are *equivalent* to each other.

A *relative invariant* is a homogeneous polynomial function $f : X^c \to \mathbb{C}$ such that $g \cdot f(\xi) = \chi(g) f(\xi)$ for some character χ of G^c and arbitrary $g \in G^c$ and $\xi \in X^c$. Now we move on to the calculation of our relative invariants. Note that the pair (G^c, X^c) is a *prehomogeneous vector space* (PV), i.e., there exists a G^c-orbit which is open and dense in X^c. Its complement, the union of all the other orbits, is called the *singular set* of the space. In all cases, the action of G^c on X^c is irreducible.

If the singular set is a hypersurface, then there exists a unique (up to a scalar multiple) irreducible relative invariant f on X^c such that the singular set is precisely the zero locus of f. In such cases the PV is said to be *regular*.

Our basic reference for the theory of PV spaces is [9] which the reader should consult for more details. By consulting the tables in this paper, we see that (G^c, X^c) is a regular PV only in the six cases $2 \times 2 \times n, n = 2, 3, 4$ and $2 \times 3 \times n, n = 3, 4, 6$. In the cases $2 \times 2 \times 2$, $2 \times 2 \times 4$, $2 \times 3 \times 3$, and $2 \times 3 \times 6$ the relative invariant f is described in [9, Section 7]. In the remaining two cases, $2 \times 2 \times 3$ and $2 \times 3 \times 4$, one can apply [9, Proposition 18] to compute the relative invariants. Except in the cases $2 \times 2 \times 4$ and $2 \times 2 \times 6$, these relative invariants are special cases of hyperdeterminants, which are studied extensively in [4], specifically see Proposition 1.4 of Chapter 14.

In many of the spaces considered, we have constructed a quadratic form p which will be useful later on in proving that certain orbits are not contained in the closures of other orbits of larger dimension. We use the notation $|M|$ to mean to determinant of the matrix, or matrix pencil, M. Here are the results:

Case $2 \times 2 \times 2$: For each tensor $\xi \in X^c$, our p is $|\lambda A_\xi + \mu B_\xi|$, and the relative invariant f is the discriminant of this quadratic form. Its character χ is $\chi(a, b, c) = |a|^2 |b|^2 |c|^2$. This means that if $\eta = (a, b, c) \cdot \xi$, then $f(\eta) = \chi(a, b, c) f(\xi)$ for all $(a, b, c) \in G^c$.

Case $2 \times 2 \times 3$: Let

$$P_\xi = \begin{bmatrix} \xi^{111} & \xi^{121} \\ \xi^{211} & \xi^{221} \end{bmatrix}, \; Q_\xi = \begin{bmatrix} \xi^{112} & \xi^{122} \\ \xi^{212} & \xi^{222} \end{bmatrix}, \; R_\xi = \begin{bmatrix} \xi^{113} & \xi^{123} \\ \xi^{213} & \xi^{223} \end{bmatrix}.$$

Our p is $|\lambda P_\xi + \mu Q_\xi + \nu R_\xi|$, and the relative invariant f is the discriminant of p. Its character is $\chi(a, b, c) = |a|^3 |b|^3 |c|^2$.

Case $2 \times 2 \times 4$: In this case, our relative invariant f is simply

$$f(\xi) = \begin{vmatrix} \xi^{111} & \xi^{121} & \xi^{211} & \xi^{221} \\ \xi^{112} & \xi^{122} & \xi^{212} & \xi^{222} \\ \xi^{113} & \xi^{123} & \xi^{213} & \xi^{223} \\ \xi^{114} & \xi^{124} & \xi^{214} & \xi^{224} \end{vmatrix},$$

which has character $\chi(a,b,c) = |a|^2|b|^2|c|$. This expression can also be obtained in a similar method as before, yielding p. To do this, define P_ξ, Q_ξ, and R_ξ as in the previous case, and let $S_\xi = \begin{bmatrix} \xi^{114} & \xi^{124} \\ \xi^{214} & \xi^{224} \end{bmatrix}$. We let $p = |\lambda P_\xi + \mu Q_\xi + \nu R_\xi + \sigma S_\xi|$.

The discriminant of p is then a relative invariant. However, in this case the discriminant turns out to be f^2, so f is really the minimal invariant.

Case $2 \times 3 \times 3$: If we put the tensor ξ in pencil form, $\lambda A_\xi + \mu B_\xi$, then the relative invariant f is the discriminant of the binary cubic form $|\lambda A_\xi + \mu B_\xi|$. Its character is $\chi(a,b,c) = |a|^6\,|b|^4\,|c|^4$.

Case $2 \times 3 \times 4$: We let

$$q = \begin{vmatrix} \xi^{121} & \xi^{122} & \xi^{123} & \xi^{124} \\ \xi^{131} & \xi^{132} & \xi^{133} & \xi^{134} \\ \xi^{221} & \xi^{222} & \xi^{223} & \xi^{224} \\ \xi^{231} & \xi^{232} & \xi^{233} & \xi^{234} \end{vmatrix}, \quad r = \begin{vmatrix} \xi^{111} & \xi^{112} & \xi^{113} & \xi^{114} \\ \xi^{131} & \xi^{132} & \xi^{133} & \xi^{134} \\ \xi^{211} & \xi^{212} & \xi^{213} & \xi^{214} \\ \xi^{231} & \xi^{232} & \xi^{233} & \xi^{234} \end{vmatrix},$$

$$s = \begin{vmatrix} \xi^{111} & \xi^{112} & \xi^{113} & \xi^{114} \\ \xi^{121} & \xi^{122} & \xi^{123} & \xi^{124} \\ \xi^{211} & \xi^{212} & \xi^{213} & \xi^{214} \\ \xi^{221} & \xi^{222} & \xi^{223} & \xi^{224} \end{vmatrix},$$

$$\alpha = -\begin{vmatrix} \xi^{111} & \xi^{112} & \xi^{113} & \xi^{114} \\ \xi^{131} & \xi^{132} & \xi^{133} & \xi^{134} \\ \xi^{211} & \xi^{212} & \xi^{213} & \xi^{214} \\ \xi^{221} & \xi^{222} & \xi^{223} & \xi^{224} \end{vmatrix} - \begin{vmatrix} \xi^{111} & \xi^{112} & \xi^{113} & \xi^{114} \\ \xi^{121} & \xi^{122} & \xi^{123} & \xi^{124} \\ \xi^{211} & \xi^{212} & \xi^{213} & \xi^{214} \\ \xi^{231} & \xi^{232} & \xi^{233} & \xi^{234} \end{vmatrix},$$

$$\beta = \begin{vmatrix} \xi^{121} & \xi^{122} & \xi^{123} & \xi^{124} \\ \xi^{131} & \xi^{132} & \xi^{133} & \xi^{134} \\ \xi^{211} & \xi^{212} & \xi^{213} & \xi^{214} \\ \xi^{221} & \xi^{222} & \xi^{223} & \xi^{224} \end{vmatrix} + \begin{vmatrix} \xi^{111} & \xi^{112} & \xi^{113} & \xi^{114} \\ \xi^{121} & \xi^{122} & \xi^{123} & \xi^{124} \\ \xi^{221} & \xi^{222} & \xi^{223} & \xi^{224} \\ \xi^{231} & \xi^{232} & \xi^{233} & \xi^{234} \end{vmatrix},$$

$$\gamma = -\begin{vmatrix} \xi^{121} & \xi^{122} & \xi^{123} & \xi^{124} \\ \xi^{131} & \xi^{132} & \xi^{133} & \xi^{134} \\ \xi^{211} & \xi^{212} & \xi^{213} & \xi^{214} \\ \xi^{231} & \xi^{232} & \xi^{233} & \xi^{234} \end{vmatrix} - \begin{vmatrix} \xi^{111} & \xi^{112} & \xi^{113} & \xi^{114} \\ \xi^{131} & \xi^{132} & \xi^{133} & \xi^{134} \\ \xi^{221} & \xi^{222} & \xi^{223} & \xi^{224} \\ \xi^{231} & \xi^{232} & \xi^{233} & \xi^{234} \end{vmatrix}.$$

Then our quadratic form is $p = q\lambda^2 + r\mu^2 + s\nu^2 + \alpha\mu\nu + \beta\nu\lambda + \gamma\lambda\mu$. Our relative invariant is f, the discriminant of p. It has degree 12 and character $\chi(a,b,c) = |a|^6\,|b|^4\,|c|^3$.

Case $2 \times 3 \times 6$: The relative invariant f is given by

$$f(\xi) = \begin{vmatrix} \xi^{111} & \xi^{121} & \xi^{131} & \xi^{211} & \xi^{221} & \xi^{231} \\ \xi^{112} & \xi^{122} & \xi^{132} & \xi^{212} & \xi^{222} & \xi^{232} \\ \xi^{113} & \xi^{123} & \xi^{133} & \xi^{213} & \xi^{223} & \xi^{233} \\ \xi^{114} & \xi^{124} & \xi^{134} & \xi^{214} & \xi^{224} & \xi^{234} \\ \xi^{115} & \xi^{125} & \xi^{135} & \xi^{215} & \xi^{225} & \xi^{235} \\ \xi^{116} & \xi^{126} & \xi^{136} & \xi^{216} & \xi^{226} & \xi^{236} \end{vmatrix}.$$

It has degree 6 and character $\chi(a,b,c) = |a|^3\,|b|^2\,|c|$.

REMARK 3.3. For the cases $2 \times 2 \times n$ with $n > 4$ and $2 \times 3 \times n$ with $n = 5$ or $n > 6$ there is no relative invariant as mentioned above. This also follows from the fact that there is no orbit of co-dimension one, or in the cases where $lm < n$ from the following argument (due to a referee).

When $lm < n$ any $\xi \in X^c$ can be written as $\xi = \sum u_i \otimes v_j \otimes w_{ij}$ where $w_{ij} \in W^c$. Since $lm < n$ we can choose a direct decomposition $W^c = H \oplus L$ where H is a hyperplane containing all the w_{ij}'s. Let $c_t \in \mathrm{SL}(W^c)$ act on H as multiplication by t and on L as multiplication be t^{1-n}, so $|c_t| = 1$. Then $\lim_{t\to 0}(I,I,c_t)\cdot\xi = 0$. So the zero vector of X^c is in the closure of $\mathrm{SL}(W^c)\cdot\xi$ and hence every relative invariant must be a constant.

4. G-orbits in X. We now determine the equivalence classes of regular cores in the real case. So, assume that $\xi \in X$ is such that the associated pencil $\lambda A_\xi + \mu B_\xi$ is regular, i.e., $m = n \le 3$ and $f(\lambda,\mu) = |\lambda A_\xi + \mu B_\xi| \ne 0$. If $f(\lambda,\mu)$ splits (into a product of linear factors) then, just as in the complex case, the orbit $G\cdot\xi$ contains a representative whose associated pencil is one of the pencils C_1-C_{11} listed in Section 3. Otherwise, either $m = n = 2$ and $f(\lambda,\mu)$ is irreducible, or $m = n = 3$ and $f = f_1 f_2$ where f_1 is a linear form, and f_2 an irreducible quadratic form. Then we claim that $G\cdot\xi$ has a representative whose associated pencil is one of the following:

$$C_5' : \begin{bmatrix} \lambda & \mu \\ -\mu & \lambda \end{bmatrix},\; C_{11}' : \begin{bmatrix} \lambda & \mu & 0 \\ -\mu & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}.$$

These cores are labeled C_5' and C_{11}' since if we embed X in X^c then C_5 and C_5' represent tensors in the same G^c-orbit, as do C_{11} and C_{11}'.

We give a sketch of the argument justifying the above claim. First let $m = n = 2$. Clearly we may assume that $A_\xi = I$. By replacing B_ξ by a similar matrix, we may further assume that $B_\xi = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}$, $\beta > 0$. If we let $a = \begin{bmatrix} 1 & 0 \\ -\alpha/\beta & 1/\beta \end{bmatrix}$, we find that $(a,I,I)\cdot\xi$ has C_5' as its associated pencil.

Now, let $m = n = 3$. Again, we may assume that $A_\xi = I$. Further, by a change of variables, we may take $f_1(\lambda,\mu) = \lambda$. By similarity, we can transform B_ξ to the form $B_\xi = \begin{bmatrix} \alpha & \beta & 0 \\ -\beta & \alpha & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $\beta > 0$.

TABLE 4.1
Extra G-orbits in X

	Orbit	Dimension
7′	$\begin{bmatrix} \lambda & \mu \\ -\mu & \lambda \end{bmatrix}$	$2m+2n$
18′	$\begin{bmatrix} \lambda & \mu & 0 \\ -\mu & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$	$3m+3n$
22′	$\begin{bmatrix} \lambda & \mu & 0 & 0 \\ -\mu & \lambda & 0 & 0 \\ 0 & 0 & \lambda & \mu \end{bmatrix}$	$3m+4n-3$

If we then take $a = \begin{bmatrix} \alpha^2+\beta^2 & -\alpha \\ 0 & \beta \end{bmatrix}$, $b = ((\alpha^2+\beta^2)I - \alpha B_\xi)^{-1}$, then it is easy to verify that C'_{11} is the associated pencil of $(a, b, I) \cdot \xi$.

The representatives given in Table 3.1 are in fact all real, so, for the G-orbits in X coming from cores C_1 through C_{11}, we use the same representatives. From the two new cores, we get three G-orbits in X that do not have representatives in Table 3.1. They are given in Table 4.1.

5. G^+-orbits in X. Since G^+ is the identity component of G, the G^+-orbits in X are just the connected components of the G-orbits. So, for $\xi \in X$, the number of G^+-orbits contained in $G \cdot \xi$ is $[G : G^+G_\xi]$ where G_ξ represents the stabilizer of ξ in G. By looking at G_ξ, this quickly allows us to determine all the orbits in X under the action by G^+. Again, the calculations are straightforward, so only a few examples are included. Figuring out the G^+-orbits from the stabilizers is simplified by the following notation:

For an element (a, b, c) of G, we define the *sign signature* of (a, b, c) to be the triple (s_1, s_2, s_3) where s_1 is the sign of the determinant of a, either $+$ or $-$, s_2 the sign of the determinant of b, and s_3 the sign of the determinant of c.

To illustrate this, consider orbit 6. In the case $2 \times 2 \times 2$ the only possible sign signatures of elements in the stabilizer of our representative are $(+, +, +)$ and $(-, -, -)$ (see Example 3.1). So we see that this one G-orbit splits into four G^+-orbits. To find representatives, we can choose elements of G with sign signatures, for instance, $(+, +, +)$, $(+, +, -)$, $(+, -, +)$ and $(+, -, -)$. We then act on our representative for the G-orbit with each of this four group elements to get the representatives of the four distinct G^+-orbits. For simplicity we always use the identity for our element with signature $(+, +, +)$.

However, in the case $2 \times 2 \times n$ with $n > 2$ the stabilizer contains elements with four different sign signatures, $(+, +, +)$, $(+, +, -)$, $(-, -, +)$ and $(-, -, -)$, so we only get two orbits under G^+. In the case $2 \times 3 \times n$ for $n \geq 3$, the stabilizer contains elements with all possible sign signatures, so the G-orbit is also a single G^+-orbit. This is in

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TABLE 5.1
Orbits split by G^+ for the $2 \times 2 \times 2$ case

3_{++} $\begin{bmatrix} \lambda & 0 \\ \mu & 0 \end{bmatrix}$		3_{-+} $\begin{bmatrix} \lambda & 0 \\ -\mu & 0 \end{bmatrix}$	
4_{++} $\begin{bmatrix} \lambda & \mu \\ 0 & 0 \end{bmatrix}$	4_{+-} $\begin{bmatrix} \lambda & -\mu \\ 0 & 0 \end{bmatrix}$		
5_{++} $\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$	5_{+-} $\begin{bmatrix} \lambda & 0 \\ 0 & -\lambda \end{bmatrix}$		
6_{++} $\begin{bmatrix} \lambda & \mu \\ 0 & \lambda \end{bmatrix}$	6_{+-} $\begin{bmatrix} -\lambda & \mu \\ 0 & \lambda \end{bmatrix}$	6_{-+} $\begin{bmatrix} \lambda & \mu \\ 0 & -\lambda \end{bmatrix}$	6_{--} $\begin{bmatrix} \lambda & -\mu \\ 0 & \lambda \end{bmatrix}$
$7'_{++}$ $\begin{bmatrix} \lambda & \mu \\ -\mu & \lambda \end{bmatrix}$	$7'_{+-}$ $\begin{bmatrix} -\lambda & \mu \\ \mu & \lambda \end{bmatrix}$	$7'_{-+}$ $\begin{bmatrix} \lambda & \mu \\ \mu & -\lambda \end{bmatrix}$	$7'_{--}$ $\begin{bmatrix} \lambda & -\mu \\ \mu & \lambda \end{bmatrix}$

fact the pattern for all the orbits, and it turns out that for the cases $2 \times 3 \times n$ with $n \geq 7$, the G-orbits are all connected.

It turns out that for every space X considered, and every $\xi \in X$, for each G^+-orbit $O^* \subset G \cdot \xi$ it is always possible to choose some $g \in G$ with sign signature $(+, s_2, s_3)$ for some s_2 and s_3 such that $g \cdot \xi \in O^*$. So, a G^+-orbit is uniquely defined by a representative of the G-orbit which contains it, along with the values of s_2 and s_3. We denote a G^+-orbit by the number of the G-orbit which contains it, with s_2 and s_3 as subscripts. So in the $2 \times 2 \times 2$ case orbit 6_{-+} is the orbit with representative $\begin{bmatrix} \lambda & \mu \\ 0 & -\lambda \end{bmatrix}$, since this representative can be obtained from $\begin{bmatrix} \lambda & \mu \\ 0 & \lambda \end{bmatrix}$ by acting with an element of G with sign signature $(+, -, +)$. In the cases where the G^+-orbits are the same as the G-orbits, we do not include the signs. Furthermore, in the $2 \times 3 \times n$ cases it turns out that the G^+-orbits are uniquely determined by s_3, so we only include one sign in the subscript. In choosing a representative we can take $s_1 = s_2 = +$.

In total, there are twelve G-orbits which, for some values of m and n, are not connected, and hence get split when the group is restricted to G^+. Since they can split to different extents depending on the parameters m and n, we have included four Tables (5.1-5.4) to capture the different cases.

The cases $2 \times 2 \times n$ for $n \geq 3$ are fairly similar, so we combine them in one table. For each of the orbits except $19_{+\pm}$ the G^+-orbits remain the same for all $n \geq 3$. Orbit 19 only occurs for $n \geq 4$, and remains a single G^+-orbit for $n \geq 5$ while it splits into two distinct G^+-orbits only in the case $n = 4$.

The quadratic forms $p : X^c \to \mathbb{C}$ introduced in Section 3 can be restricted to X to obtain real quadratic forms $X \to \mathbb{R}$. We denote this restriction also by p. These real quadratic forms are not G^+-invariant, but their equivalence class is. This is easy to verify in all cases except for $2 \times 3 \times 4$, for which we refer to Lemma 7.1. In the

TABLE 5.2
Orbits split by G^+ for the $2 \times 2 \times n$ cases for $n \geq 3$

3_{++} $\begin{bmatrix} \lambda \\ \mu \end{bmatrix}$ 6_{++} $\begin{bmatrix} \lambda & \mu \\ 0 & \lambda \end{bmatrix}$ $7'_{++}$ $\begin{bmatrix} \lambda & \mu \\ -\mu & \lambda \end{bmatrix}$ 13_{++} $\begin{bmatrix} \lambda & \mu & 0 \\ 0 & \lambda & \mu \end{bmatrix}$		3_{-+} $\begin{bmatrix} \lambda \\ -\mu \end{bmatrix}$ 6_{-+} $\begin{bmatrix} \lambda & \mu \\ 0 & -\lambda \end{bmatrix}$ $7'_{-+}$ $\begin{bmatrix} \lambda & \mu \\ \mu & -\lambda \end{bmatrix}$ 13_{-+} $\begin{bmatrix} \lambda & -\mu & 0 \\ 0 & \lambda & \mu \end{bmatrix}$
19_{++} $\begin{bmatrix} \lambda & \mu & 0 & 0 \\ 0 & 0 & \lambda & \mu \end{bmatrix}$	19_{+-} $\begin{bmatrix} \lambda & \mu & 0 & 0 \\ 0 & 0 & \lambda & -\mu \end{bmatrix}$	Only for $n = 4$

TABLE 5.3
Orbits split by G^+ for the $2 \times 3 \times 4$ case

19_{+} $\begin{bmatrix} \lambda & \mu & 0 & 0 \\ 0 & 0 & \lambda & \mu \end{bmatrix}$	19_{-} $\begin{bmatrix} \lambda & \mu & 0 & 0 \\ 0 & 0 & \lambda & -\mu \end{bmatrix}$
21_{+} $\begin{bmatrix} \lambda & \mu & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & \mu \end{bmatrix}$	21_{-} $\begin{bmatrix} \lambda & \mu & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & -\mu \end{bmatrix}$
$22'_{+}$ $\begin{bmatrix} \lambda & \mu & 0 & 0 \\ -\mu & \lambda & 0 & 0 \\ 0 & 0 & \lambda & \mu \end{bmatrix}$	$22'_{-}$ $\begin{bmatrix} \lambda & \mu & 0 & 0 \\ -\mu & \lambda & 0 & 0 \\ 0 & 0 & \lambda & -\mu \end{bmatrix}$
24_{+} $\begin{bmatrix} \lambda & \mu & 0 & 0 \\ 0 & \lambda & \mu & 0 \\ 0 & 0 & \lambda & \mu \end{bmatrix}$	24_{-} $\begin{bmatrix} \lambda & \mu & 0 & 0 \\ 0 & \lambda & \mu & 0 \\ 0 & 0 & \lambda & -\mu \end{bmatrix}$

TABLE 5.4
Orbits split by G^+ for the $2 \times 3 \times 6$ case

27_{+} $\begin{bmatrix} \lambda & \mu & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda & \mu \end{bmatrix}$	27_{-} $\begin{bmatrix} \lambda & \mu & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda & -\mu \end{bmatrix}$

Table 5.5
G^+-orbits in X

Dimension	G	G^+	Representatives
$2\times2\times2$	8	17	1,2, $3_{\pm+}$, $4_{+\pm}$, $5_{+\pm}$, $6_{\pm\pm}$, $7'_{\pm\pm}$, 7
$2\times2\times3$	10	14	1, 2, 4, 5, 7, 11, $3_{\pm+}$, $6_{\pm+}$, $7'_{\pm+}$, $13_{\pm+}$
$2\times2\times4$	11	16	1, 2, 4, 5 ,7, 11, $3_{\pm+}$, $6_{\pm+}$, $7'_{\pm+}$, $13_{\pm+}$, $19_{+\pm}$
$2\times2\times n, n\geq5$	11	15	1, 2, 4, 5, 7, 11, $3_{\pm+}$, $6_{\pm+}$, $7'_{\pm+}$, $13_{\pm+}$, 19
$2\times3\times3$	20	20	1-18, $7'$, $18'$
$2\times3\times4$	27	31	1-18, $7'$, $18'$, $19_\pm$, 20, $21_\pm$, 22, $22'_\pm$, 23, $24_\pm$
$2\times3\times5$	29	29	1-26, $7'$, $18'$, $22'$
$2\times3\times6$	30	31	1-26, $7'$, $18'$, $22'$, $27_\pm$
$2\times3\times n, n\geq7$	30	30	1-27, $7'$, $18'$, $22'$

spaces X for which we have defined p, for each $\xi\in X$ we denote by p_ξ the quadratic form associated to ξ.

For a real quadratic form we define the *signature* to be (q,r,s) where q, r, and s are the number of $+1$'s, -1's, and 0's in the diagonalization of p. It is well known that such triples (q,r,s) parameterize the equivalence classes of real quadratic forms. It can also be shown that taking a limit of a sequence $g_t\cdot\xi$ a 1 or -1 can be changed to zero, but its sign cannot change, and a zero must remain a zero. In other words, q and r can decrease, but never increase. So, if the signature of p_η has q (or r) larger than the value for p_ξ then we know that η is not contained in the closure of $G^+\cdot\xi$.

We are now ready to state our first theorem:

Theorem 5.1. *For each case $2\times2\times n$ and $2\times3\times n$, the number of G-orbits, the number of connected components for these, and representatives for each component G^+-orbit are given in Table 5.5. The first column gives the case, the second gives the number of G-orbits, the third the number of G^+-orbits, and in the last column we list the G^+-orbits.*

Proof. In the preceding discussion, we have successfully found all G-orbits in X. The method we outlined above for finding the component G^+-orbits relied on the ability to calculate the stabilizer in G of our representative for each G-orbit. This works, but for the three representatives in Table 4.1 the calculations are somewhat difficult. We shall therefore present a proof that the component G^+-orbits for these three cases are as stated.

We first consider the G-orbit $7'$.

Case $2\times2\times2$: We claim that the G-orbit $7'$ consists of four G^+-orbits, $7'_{\pm\pm}$.

We calculated that the stabilizer of our representative ξ for orbit $7'$ has two connected components. Its identity component is the direct product of two copies of the multiplicative group $\mathbb{C}^*$. The first copy consists of the elements (a,b,c) with

$$a=\frac{1}{x^2+y^2}\begin{bmatrix} x & -y\\ y & x\end{bmatrix},\ b=\begin{bmatrix} x & y\\ -y & x\end{bmatrix},\ c=I,$$

and the second with

$$a = \frac{1}{x^2+y^2}\begin{bmatrix} x & y \\ -y & x \end{bmatrix},\ b = I,\ c = \begin{bmatrix} x & y \\ -y & x \end{bmatrix},$$

where $(x, y) \neq (0, 0)$. As a representative for the non-identity component, we can take the element

$$a = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},\ b = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},\ c = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Hence the possible sign signatures of elements in the stabilizer are $(+, +, +)$ and $(-, -, -)$ and our claim is proven.

Case $2 \times 2 \times n, n \geq 3$: As in our calculations for the dimension of each orbit, we see that the stabilizer of ξ consists of the elements $(a, b, c) \in G$ with $c = \begin{bmatrix} C_1 & C_2 \\ 0 & C_4 \end{bmatrix}$, where (a, b, C_1) belongs to the stabilizer of ξ in the case $2 \times 2 \times 2$. Thus the sign signatures (s_1, s_2, s_3) of elements in the stabilizer are arbitrary except that $s_1 = s_2$. Hence $7'$ is the union of two G^+-orbits, $7'_{++}$ and $7'_{-+}$.

Case $2 \times 3 \times n$: It is not hard to find elements of the stabilizer with all possible sign signatures, so the G-orbit $7'$ is connected.

For the G-orbit $18'$, we can find elements of the stabilizer with all possible sign signatures. So $18'$ is always connected.

Finally we consider the G-orbit $22'$.

In the $2 \times 3 \times 5$ case we can find elements of the stabilizer with all possible sign signature, so for $n \geq 5$ the G-orbit $22'$ is connected. For $n = 4$ we can find elements of the stabilizer with any sign signature such that $s_3 = +$. So there are at most two components, $22'_+$ and $22'_-$. We see that the signature of our quadratic form p is different on $22'_+$ and $22'_-$. So, once we prove that p is invariant on G^+-orbits up to equivalence, we will have proven that for the case $2 \times 3 \times 4$, the G-orbit $22'$ has exactly two component G^+-orbits. This is done in Lemma 7.1. □

6. Closure diagrams. We note that if $\eta \in X$ is in the closure of an orbit $G^+ \cdot \xi$, then for all $g \in G^+$, $g \cdot \eta$ must also be in the closure of $G^+ \cdot \xi$. So the closure of any orbit in X is a union of itself with a number of smaller dimensional orbits. Hence to prove that one G^+-orbit O_2 is contained in the closure of another G^+-orbit O_1 we need only display a family $g_\varepsilon \in G^+$, $\varepsilon > 0$, such that $g_\varepsilon \cdot \xi \to \eta$ as $\varepsilon \to 0^+$ for some $\xi \in O_1$ and $\eta \in O_2$. Furthermore, to show that O_2 is not contained in the closure of O_1 it is sufficient to choose one $\eta \in O_2$ and show that it is not in the closure of O_1.

Theorem 6.1. *The closures of the G^+-orbits in X are as shown in the diagrams in the Appendix.*

The following is a proof that all the containments claimed in the diagrams are correct. The proof that there are no further containments is more difficult, and is given in the next section. We use the notation $O_1 \to O_2$ to mean that orbit O_2 is contained in the closure of orbit O_1, and $O_1 \not\to O_2$ to mean that orbit O_2 is not contained in the closure of orbit O_1.

Case $2\times2\times2$: To simplify things, note that the symmetric group S_3 acts naturally on X by permuting the tensor indices i, j, k, and that this action preserves G^+-orbits. So we can let S_3 act on the set of G^+-orbits in X. We can use a similar trick for the natural action of G/G^+ on the G^+-orbits. Let H be the group of permutations of our orbits generated by these two actions. Then H acts naturally on our diagrams, and it must take any pair of orbits (O_1, O_2) with $O_1 \to O_2$ to another pair (O_1', O_2') with $O_1' \to O_2'$. We then only need to show that one line in each H-orbit of lines exists (or fails to exist), and the others follow.

It can be seen that all of the lines $(7'_{s_1s_2}, 6_{s_1s_2})$ are in the same H-orbit, as are all the lines from 7, all the lines from orbits of dimension 7 to orbits of dimension 5, and all the lines from orbits of dimension 5 to the orbit of dimension 4. So all that needs to be shown to prove all the lines exist is that $7'_{++} \to 6_{++}$, $7 \to 6_{++}$, $6_{++} \to 5_{++}$, $5_{++} \to 2$, and $2 \to 1$.

We shall give the family $g_\varepsilon = (a, b, c)_\varepsilon \in G^+$ in each case by specifying its components a, b, c as functions of ε. To find these families, we proceeded by performing elementary row and column operations, along with changes in variables in λ and μ until we get something close to the representative of the smaller orbit. So, for $7'_{++} \to 6_{++}$ we may use the following sequence:

$$\begin{bmatrix} \lambda & \mu \\ -\mu & \lambda \end{bmatrix} \longrightarrow \begin{bmatrix} \varepsilon\lambda & \mu \\ -\varepsilon\mu & \lambda \end{bmatrix} \longrightarrow \begin{bmatrix} \varepsilon\lambda & \mu \\ -\varepsilon^2\mu & \varepsilon\lambda \end{bmatrix} \longrightarrow \begin{bmatrix} \lambda & \mu \\ -\varepsilon^2\mu & \lambda \end{bmatrix}.$$

Hence $a = \begin{bmatrix} \varepsilon^{-1} & 0 \\ 0 & 1 \end{bmatrix}$, $b = \begin{bmatrix} 1 & 0 \\ 0 & \varepsilon \end{bmatrix}$, and $c = \begin{bmatrix} \varepsilon & 0 \\ 0 & 1 \end{bmatrix}$ gives the required family.

For the rest of the cases we will not include the sequence of steps, just the final expressions for a, b, c in terms of ε.

$7 \to 6_{++}$: $a = \begin{bmatrix} 1 & \varepsilon^{-1} \\ 0 & 1 \end{bmatrix}$, $b = \begin{bmatrix} 1 & 1 \\ 0 & \varepsilon \end{bmatrix}$, $c = \begin{bmatrix} 1 & 0 \\ -\varepsilon^{-1} & 1 \end{bmatrix}$.

$6_{++} \to 5_{++}$: $a = \begin{bmatrix} 1 & 0 \\ 0 & \varepsilon \end{bmatrix}$, $b = c = I$.

$5_{++} \to 2$: $a = I$, $b = \begin{bmatrix} 1 & 0 \\ 0 & \varepsilon \end{bmatrix}$, $c = I$.

$2 \to 1$: $a = \varepsilon I$, $b = c = I$.

$2 \times 2 \times 3$: All the lines starting with orbits of dimension at most 10 follow from the previous case. The action of G/G^+ interchanges orbit 13_{++} with 13_{-+} and orbit $7'_{++}$ with $7'_{-+}$ simultaneously, while fixing 11 and 7, so the only new lines that need justification are $13_{++} \to 7'_{++}$, $13_{++} \to 11$, and $11 \to 7$.

$13_{++} \to 7'_{++}$: $a = b = I$, $c = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & \varepsilon \end{bmatrix}$.

$13_{++} \to 11$: $a = \begin{bmatrix} \varepsilon^{-1} & 0 \\ 0 & 1 \end{bmatrix}$, $b = \begin{bmatrix} \varepsilon & 0 \\ 0 & 1 \end{bmatrix}$, $c = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \varepsilon & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

$11 \to 7$: $a = b = I$, $c = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -\varepsilon & 0 \end{bmatrix}$.

$2\times 2\times 4$: Since G/G^+ permutes transitively the four pairs $(19_{+\pm}, 13_{\pm+})$ we need only show that $19_{++} \to 13_{++}$.

$19_{++} \to 13_{++}$: $a = b = I$, $c = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\varepsilon & 0 \end{bmatrix}$.

$2\times 3\times 3$: We have a lot of new lines, so only a few of the more difficult ones are explicitly included. The others are similar.

$18 \to 17$: $a = \begin{bmatrix} -\varepsilon^{-1} & \varepsilon^{-1} \\ 0 & -1 \end{bmatrix}$, $b = \begin{bmatrix} \varepsilon & \varepsilon & 0 \\ 0 & -\varepsilon^2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$, $c = \begin{bmatrix} -1 & 0 & 0 \\ -\varepsilon^{-1} & -\varepsilon^{-1} & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

$17 \to 16$: $a = \begin{bmatrix} -1 & \varepsilon^{-1} \\ 0 & -1 \end{bmatrix}$, $b = \begin{bmatrix} 1 & 0 & \varepsilon^{-1} \\ 0 & 1 & -1 \\ 0 & 0 & \varepsilon \end{bmatrix}$, $c = \begin{bmatrix} -1 & 0 & 0 \\ -\varepsilon^{-1} & -1 & 0 \\ 0 & -\varepsilon^{-1} & 1 \end{bmatrix}$.

$16 \to 9$: $a = -I$, $b = \begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$, $c = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & \varepsilon \end{bmatrix}$.

$2\times 3\times 4$: Again we only include the more difficult lines.

$22 \to 21_+$: $a = \begin{bmatrix} 1 & \varepsilon^{-1} \\ 0 & 1 \end{bmatrix}$, $b = \begin{bmatrix} 1 & 1 & 0 \\ 0 & \varepsilon & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $c = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\varepsilon^{-1} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -\varepsilon^{-1} & 1 \end{bmatrix}$.

$24_+ \to 22'_+$: $a = I$, $b = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \varepsilon^{-1} & 0 & \varepsilon^{-1} \end{bmatrix}$, $c = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & \varepsilon & 0 \\ 0 & 0 & 0 & \varepsilon \end{bmatrix}$.

The cases with $n \geq 5$ do not pose any significant problems with regards to proofs of containment, so they are all left to the reader.

7. Proofs of non-containment. Proving that a smaller orbit is not contained in the closure of a larger one is more interesting, and we have used several different types of arguments. It is clear that if in X a G^+-orbit O_1 is contained in the closure of another G^+-orbit O_2, then in X^c the G^c-orbit which includes O_1 is contained in the closure of the G^c-orbit which includes O_2. Using this observation, the following results follow from Parfenov's work, [8]:

$4 \not\to 3$, $10 \not\to 3$, $10 \not\to 4$, $19 \not\to 8$, $12 \not\to 9$, $20 \not\to 9$, $20 \not\to 12$,
$20 \not\to 13$, $16 \not\to 15$, $21 \not\to 18$, $21 \not\to 18'$, $25 \not\to 24$.

As well, numbering the orbits as we have, no orbit with a larger label is ever contained in the closure of an orbit with a smaller label. Also note that if a G-orbit O_2 is not contained in the closure of O_1, then no G^+-orbit contained in O_2 can be contained in the closure of a G^+-orbit contained in O_1. So these results carry over even if the orbits split.

We need the following two lemmas.

Lemma 7.1. *The quadratic form p, as defined for the case $2\times 3\times 4$, is invariant up to equivalence on G^+-orbits.*

Proof. For each $\xi \in X$, we use the notation p_ξ to mean the quadratic form p calculated for ξ. Fix some $\xi \in X$. We prove p_ξ is invariant under $\mathrm{GL}^+(U), \mathrm{GL}^+(V)$, and $\mathrm{GL}^+(W)$ separately.

First, let $a \in GL^+(U)$. It is not hard to see that each coefficient of p is multiplied by a factor of $|a|^2$ when a acts on ξ. With a little calculation, we see that for $c \in \mathrm{GL}^+(W)$, each coefficient of p is multiplied by a factor of $|c|$ when c acts on ξ. Since c has positive determinant, this clearly doesn't change the equivalence class of p. So it remains to show that the equivalence class is invariant under the action of $\mathrm{GL}^+(V)$.

Here, we note that $\mathrm{GL}^+(V)$ is generated by the diagonal matrices of positive determinant, along with matrices with ones on the diagonal and one non-zero entry either just above or just below the diagonal. So we need only prove invariance for matrices of these types.

First, the case where b is a diagonal matrix, $b = \begin{bmatrix} t_1 & 0 & 0 \\ 0 & t_2 & 0 \\ 0 & 0 & t_3 \end{bmatrix}$, $t_1t_2t_3 > 0$.

We calculate how each of the coefficients of p transforms. Let $q', r', s', \alpha', \beta', \gamma'$ be the coefficients of $p_{b\cdot\xi}$ corresponding to $q, r, s, \alpha, \beta, \gamma$. We see that:

$$q' = t_2^2t_3^2q,\ r' = t_1^2t_3^2r,\ s' = t_1^2t_2^2s,\ \alpha' = t_1^2t_2t_3\alpha,\ \beta' = t_1t_2^2t_3\beta,\ \gamma' = t_1t_2t_3^2\gamma.$$

So, $p_{b\cdot\xi}(\lambda, \mu, \nu) = p_\xi(t_2t_3\lambda, t_1t_3\mu, t_1t_2\nu)$. This is a linear change of variables in λ, μ, and ν, so the equivalence class of p is not changed.

Now we look at the case $b = \begin{bmatrix} 1 & t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Then the new coefficients are given by:

$$q' = q,\ r' = r - t\alpha + t^2q,\ s' = s,\ \alpha' = \alpha - t\beta,\ \beta' = \beta,\ \gamma' = \gamma - 2tq.$$

It can then be verified that $p_{b\cdot\xi}(\lambda, \mu, \nu) = p_\xi(\lambda - t\mu, \mu, \nu)$. Again, this is a change of variables in λ, μ, ν so the equivalence class of p is unchanged. The other three types of matrices of this form with t in the position $(2, 3), (2, 1)$, or $(3, 2)$ follow in the same way, so we see that the equivalence class of p is invariant under the action of $\mathrm{GL}^+(V)$, and hence under the action of the whole group G^+. □

LEMMA 7.2. *Let W' be a subspace of W and $\xi, \eta \in U \otimes V \otimes W'$. If η is contained in the closure of $G^+ \cdot \xi$, then η is contained in the closure of* $\mathrm{GL}^+(U) \times \mathrm{GL}^+(V) \times \mathrm{GL}(W') \cdot \xi$.

REMARK 7.3. This is a variation of Parfenov's Theorem 1 in [8].

Proof. We may assume that $\{w_1, w_2, \ldots, w_{n'}\}$, the first n' basis vectors for W, form a basis for W'. Choose a sequence $g_s = (a_s, b_s, c_s) \in G^+$ such that $g_s \cdot \xi \to \eta$. For each s write $c_s = \begin{bmatrix} c'_s & X_s \\ Y_s & Z_s \end{bmatrix}$, where c'_s is an $n' \times n'$ matrix and the other dimensions are as needed.

Notice that, since $\xi^{ijk}=0$ for $k>n'$, $g_s\cdot\xi=(a_s,b_s,\begin{bmatrix} c_s' & 0\\ Y_s & I\end{bmatrix})\cdot\xi$. Furthermore, Y_s only affects components of $g_s\cdot\xi$ with $k>n'$, so since $\eta^{ijk}=0$ for $k>n'$, we see that $(a_s,b_s,c_s')\cdot\xi$ also approaches η in the space $U\otimes V\otimes W'$. □

For the following cases we use the quadratic forms p introduced in Section 2 when we were calculating relative invariants.

Case $2\times2\times2$: We show that $7'_{++}\not\to 5_{+-}$. By acting on the orbits by both S_3 and G/G^+ as described at the beginning of Section 6 we get proofs of all non-containments between orbits of dimension 8 and orbits of dimension 5.

To see that $7'_{++}\not\to 5_{+-}$ simply note that for the representatives we have chosen, the signature of p for the orbit $7'_{++}$ is $(2,0,0)$ while the signature for 5_{+-} is $(0,1,0)$, which completes the proof by the argument stated just before Theorem 5.1.

Case $2\times2\times3$: Using p as defined in Section 2 we see that its signature for the orbit 13_{++} is $(1,2,0)$, for $7'_{-+}$ is $(2,0,1)$, and for 3_{-+} is $(0,1,0)$. So $13_{++}\not\to 7'_{-+}$ and $7'_{-+}\not\to 3_{++}$. Using the natural action of G/G^+ we also get $13_{-+}\not\to 7'_{++}$ and $7'_{++}\not\to 6_{-+}$, which completes the proof that there are no more lines in the diagram.

Cases $2\times2\times n, n\geq4$: All non-containments in these cases follow from the cases $2\times2\times2$ and $2\times2\times3$ using Lemma 7.2.

Case $2\times3\times3$: Everything here follows immediately from Parfenov's work except $12\not\to7'$ and $14\not\to7'$.

$12\not\to7'$: Note that $\xi\in U\otimes v_1\otimes W+U\otimes V\otimes w_3$ where ξ is our representative for orbit 12. So if ξ' is in the closure of orbit 12, there must be some $v\in V$ and $w\in W$ such that $\xi'\in U\otimes v\otimes W+U\otimes V\otimes w$. We now show that our representative η for orbit $7'$ is not of this type, which will complete the proof.

Assume that $\eta\in U\otimes v\otimes W+U\otimes V\otimes w$ for some $v\in V$ and $w\in W$. Using summation convention, we have $v=d^jv_j$, $w=r^kw_k$, and

$$\eta=x^{ik}u_i\otimes v\otimes w_k+y^{ij}u_i\otimes v_j\otimes w=(x^{ik}d^j+y^{ij}r^k)e_{ijk}.$$

By equating the coefficients for $i,j,k=1,2$, we obtain the system

$$\begin{aligned} x^{11}d^1+y^{11}r^1&=1, & x^{21}d^1+y^{21}r^1&=0,\\ x^{12}d^1+y^{11}r^2&=0, & x^{22}d^1+y^{21}r^2&=1,\\ x^{11}d^2+y^{12}r^1&=0, & x^{21}d^2+y^{22}r^1&=-1,\\ x^{12}d^2+y^{12}r^2&=1, & x^{22}d^2+y^{22}r^2&=0.\end{aligned}$$

By eliminating the x^{ik}, we obtain

$$\begin{aligned} (y^{11}d^2-y^{12}d^1)r^1&=d^2, & (y^{21}d^2-y^{22}d^1)r^1&=d^1,\\ (y^{11}d^2-y^{12}d^1)r^2&=-d^1, & (y^{21}d^2-y^{22}d^1)r^2&=d^2.\end{aligned}$$

Hence $r^1d^1+r^2d^2=r^2d^1-r^1d^2=0$. But

$$(r^1d^1+r^2d^2)^2+(r^2d^2-r^1d^2)^2=((r^1)^2+(r^2)^2)((d^1)^2+(d^2)^2),$$

so we see that $r^1=r^2=0$ or $d^1=d^2=0$. In both cases our system of equations is inconsistent. So we have that $12\not\to7'$.

$14\not\to7'$: Let ξ and η be our representatives for the G^+-orbits 14 and $7'$ respectively. By specializing $\lambda=0$, $\mu=1$, the pencil $\lambda A_\xi+\mu B_\xi$ gives the matrix B_ξ of rank 1. On the other hand every non-trivial specialization of $\lambda A_\eta + \mu B_\eta$ produces a matrix of rank 2.

Case $2 \times 3 \times 4$: Using the previous cases, along with Lemma 7.2 and Parfenov's work, we get everything except: $23 \not\to 22'_\pm$, $24_+ \not\to 22'_-$, $24_- \not\to 22'_+$, $22'_+ \not\to 19_-$, $22'_- \not\to 19_+$, $22 \not\to 18'$, and $22'_\pm \not\to 18$.

By directly evaluating the coefficients for our representatives, we find the signatures of p for the following orbits: $24_+ : (1,2,0)$; $23 : (1,1,0)$; $22'_- : (2,0,0)$; $19_+ : (0,1,2)$.

It follows from this that $24_+ \not\to 22'_-$, $22'_- \not\to 19_+$, and $23 \not\to 22'_\pm$. From the action of G/G^+ we also see that $24_- \not\to 22'_+$ and $22'_+ \not\to 19_-$.

In order to handle the remaining cases, we recall a property of real binary cubic forms $c_0\lambda^3 + c_1\lambda^2\mu + c_2\lambda\mu^2 + c_3\mu^3$. The discriminant D of this form is given by

$$D = c_1^2 c_2^2 + 18 c_0 c_1 c_2 c_3 - 4(c_0 c_2^3 + c_1^3 c_3) - 27 c_0^2 c_3^2.$$

The cubic form splits (into linear factors) if and only if $D \geq 0$.

$22 \not\to 18'$: Let ξ and η be our representatives of the G^+-orbits 22 and $18'$ respectively. The greatest common divisor of the 3×3 minors of $\lambda A_\xi + \mu B_\xi$ is $\lambda\mu$. Consequently, every 3×3 minor of this pencil splits, and so its discriminant is ≥ 0. Clearly this is also true for all ξ' in the orbit $G^+ \cdot \xi$, and consequently for all ξ' in the closure of this orbit. On the other hand the first 3×3 minor of η is $\lambda(\lambda^2 + \mu^2)$ which has discriminant -4. Hence η is not in the closure of $G^+ \cdot \xi$.

$22'_\pm \not\to 18$: Let ξ and η be our representatives for the G^+-orbits $22'_\pm$ and 18 respectively. The greatest common divisors of the 3×3 minors of $\lambda A_\xi + \mu B_\xi$ is $\lambda^2 + \mu^2$. Consequently, every 3×3 minor of this pencil has discriminant ≤ 0. As above this is also true for all ξ' in the closure of $G^+ \cdot \xi$. On the other hand the first 3×3 minor of $\lambda A_\eta + \mu B_\eta$ is $\lambda\mu(\lambda + \mu)$ which has discriminant 1. Hence η is not in the closure of $G^+ \cdot \xi$. This completes the $2 \times 3 \times 4$ case.

For the cases $2 \times 3 \times n$ with $n \geq 5$ all non-containments follow from smaller cases, or from Parfenov's paper [8], so the proof is now complete. □

8. Appendix. In figures 1-4 we exhibit the closure diagrams for G^+-orbits in X in the cases $2 \times 2 \times 2$, $2 \times 2 \times 4$, $2 \times 3 \times 4$, and $2 \times 3 \times 6$. In these diagrams, the closure of an orbit is the union of itself with all orbits that can be reached from it by following lines in a downward direction in the diagram. The numbers on the left hand side show the dimensions of the orbits at each level. Some modifications need to be made to obtain the diagrams for cases not explicitly shown. These are as follows:

$2 \times 2 \times 3$: Drop orbits $19_\pm$ from the diagram for $2 \times 2 \times 4$ (Figure 2).

$2 \times 2 \times n, n \geq 5$: Combine orbits 19_+ and 19_- to form the single orbit 19 in the diagram $2 \times 2 \times 4$ (Figure 2). Both orbits 13_+ and 13_- are in the closure of orbit 19.

$2 \times 3 \times 3$: Take the portion of the diagram for the cases $2 \times 3 \times 4$ (Figure 3) consisting of orbits present in the smaller space (see Table 5.5).

$2 \times 3 \times 5$: Take the part of the diagram for $2 \times 3 \times 6$ (Figure 4) consisting of orbits present in the smaller space (see Table 5.5).

$2 \times 3 \times n, n \geq 7$: Combine orbits 27_+ and 27_- in the diagram for $2 \times 3 \times 6$ (Figure 4) to get a single orbit 27. Orbit 26 is in the closure of orbit 27.

In the cases where the diagrams need to be modified, the dimensions of the orbits change. They are given in Tables 3.1 and 4.1.

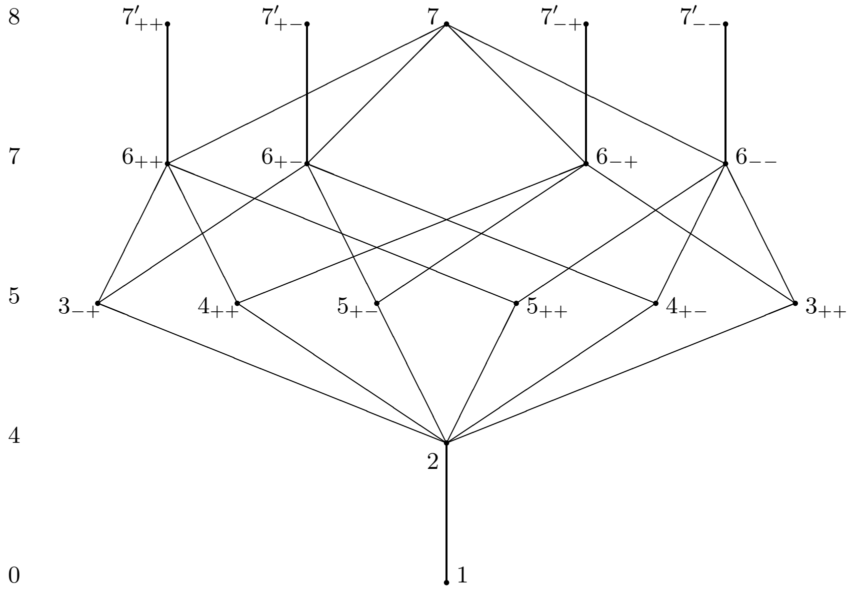

Figure 1: $\ell = m = n = 2$

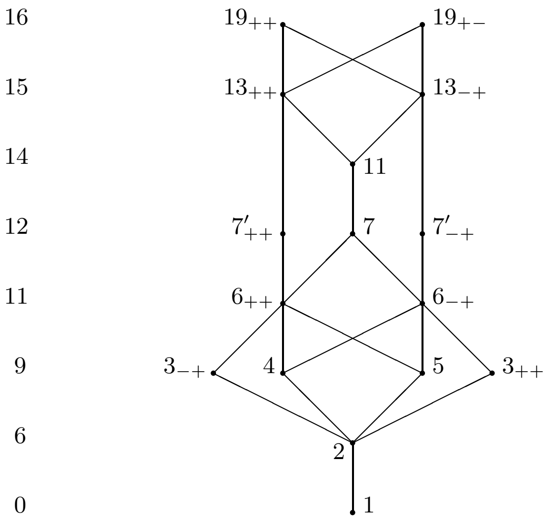

Figure 2: $\ell = m = 2,\ n = 4$

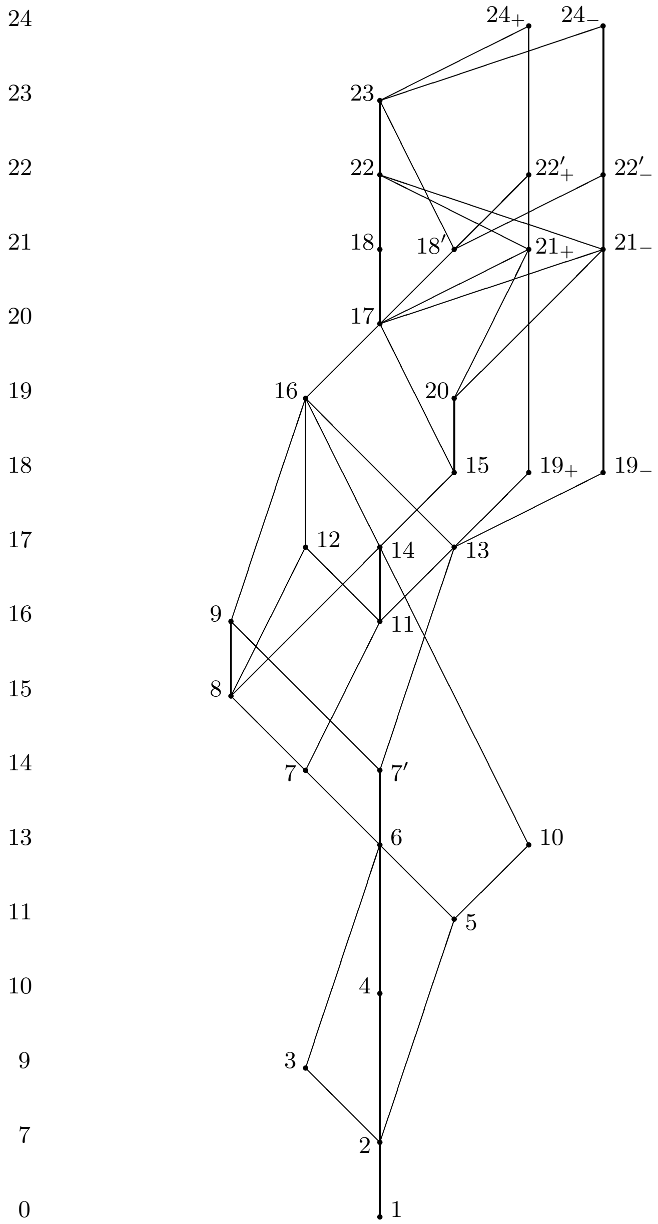

Figure 3: $\ell = 2,\ m = 3,\ n = 4$

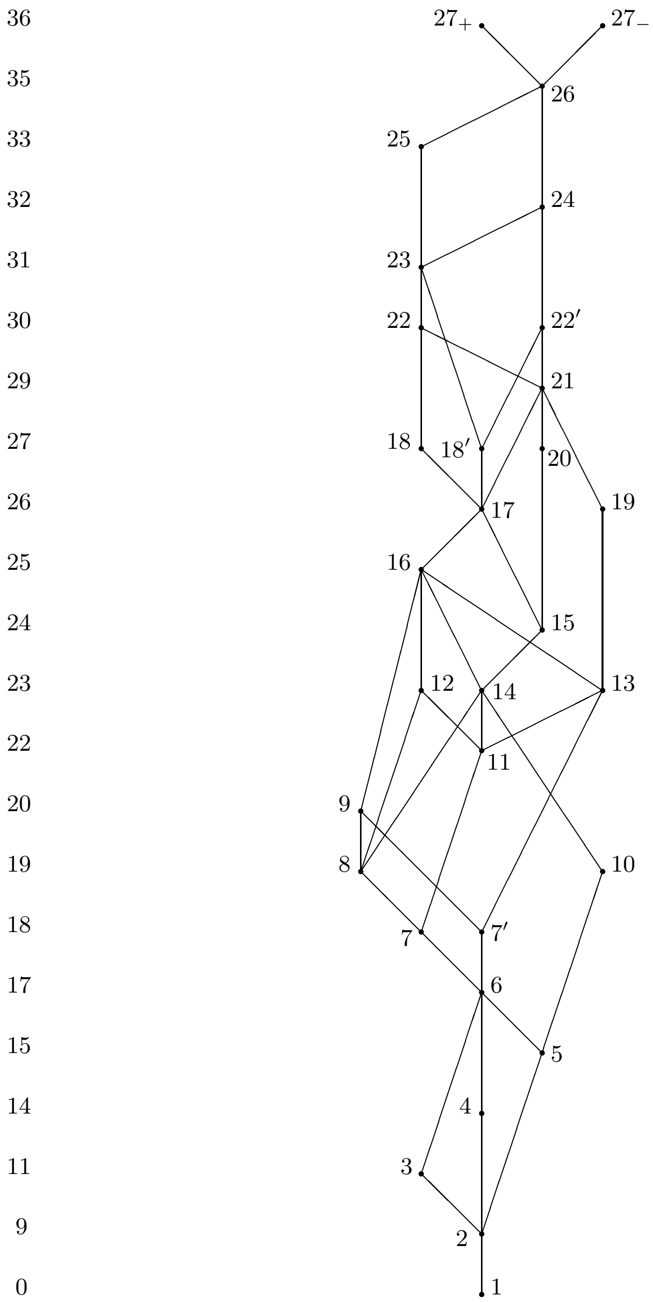

Figure 4: $\ell = 2,\ m = 3,\ n = 6$

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