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# AN UPPER BOUND ON ALGEBRAIC CONNECTIVITY OF GRAPHS WITH MANY CUTPOINTS*

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**Abstract.** Let $G$ be a graph on $n$ vertices which has $k$ cutpoints. A tight upper bound on the algebraic connectivity of $G$ in terms of $n$ and $k$ for the case that $k > n/2$ is provided; the graphs which yield equality in the bound are also characterized. This completes an investigation initiated by the author in a previous paper, which dealt with the corresponding problem for the case that $k \leq n/2$.

**Key words.** Laplacian matrix, algebraic connectivity, cutpoint.

**AMS subject classifications.** 05C50, 15A48

**1. Introduction and Preliminaries.** Let $G$ be a graph on $n$ vertices. Its *Laplacian matrix* $L$ can be written as $L = D - A$, where $A$ is the $(0, 1)$ adjacency matrix of $G$, and $D$ is the diagonal matrix of vertex degrees. There is a wealth of literature on Laplacian matrices in general (see, e.g., the survey by Merris [9]), and on their eigenvalues in particular. It is straightforward to see that $L$ is a positive semidefinite singular M-matrix, with the all-ones vector $\mathit{1}$ as a null vector. Further, Fiedler [5] has shown that if $G$ is connected, then the remaining eigenvalues of $L$ are positive. Motivated by this observation, the second smallest eigenvalue of $L$ is known as the *algebraic connectivity of* $G$; throughout this paper, we denote the algebraic connectivity of $G$ by $\alpha(G)$. The eigenvectors of $L$ corresponding to $\alpha(G)$ have come to be known as *Fiedler vectors* for $G$.

We list here a few of the well-known properties of algebraic connectivity; these can be found in [5]. Since $\alpha(G)$ is the second smallest eigenvalue of $L$, it follows that $\alpha(G) = min\{y^T Ly | y^T \mathit{1} = 0, y^T y = 1\}$. Further, if we add an edge into $G$ to form $\tilde{G}$, then $\alpha(G) \leq \alpha(\tilde{G})$. Finally, if $G$ has vertex connectivity $c \leq n-2$, then $\alpha(G) \leq c$. In particular, if $G$ has a *cutpoint* - that is, a vertex whose deletion (along with all edges incident with it) yields a disconnected graph - then we see that $\alpha(G) \leq 1$.

Motivated by this last observation, Kirkland [7] posed the following problem: if $G$ is a graph on $n$ vertices which has $k$ cutpoints, find an attainable upper bound on $\alpha(G)$. In [7], such a bound is constructed for the case that $1 \leq k \leq n/2$, and the graphs attaining the bound are characterized. The present paper is a continuation of the work in [7]; here we give an attainable upper bound on $\alpha(G)$ when $n/2 < k \leq n-2$, and explicitly describe the equality case.

The technique used in this paper relies on the analysis of the various connected components which arise from the deletion of a cutpoint. We now briefly outline that technique. Suppose that $G$ is a connected graph and that $v$ is a cutpoint of $G$. The *components at* $v$ are just the connected components of $G-v$, the (disconnected) graph

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which is produced when we delete $v$ and all edges incident with it. For a connected component $C$ at $v$, the *bottleneck matrix for* $C$ is the inverse of the principal submatrix of $L$ induced by the vertices of $C$. It is straightforward to see that the bottleneck matrix $B$ for $C$ is entrywise positive, and so it has a Perron value, $\rho(B)$, and we occasionally refer to $\rho(B)$ as the *Perron value of* $C$. If the components at $v$ are $C_1, \cdots, C_m$, then we say that $C_j$ is a *Perron component at* $v$ if its Perron value is maximum amongst those of the connected components at $v$. We note that there may be several Perron components at a vertex.

The following result, which pulls together several facts established in [4] and [1], shows how the viewpoint of Perron components can be used to describe both $\alpha(G)$ and the corresponding Fiedler vectors. Throughout this paper, $J$ denotes the all-ones matrix, $O$ denotes the zero matrix (possibly a vector), and the orders of both $J$ and $O$ will be apparent from the context. We use $\rho(M)$ to denote the Perron value of any square entrywise nonnegative matrix $M$, while $\lambda_1(S)$ denotes the largest eigenvalue of any symmetric matrix $S$. We refer the reader to [3] for the basics on nonnegative matrices, and to [6] for background on symmetric matrices.

PROPOSITION 1.1. *Let $G$ be a connected graph having vertex $v$ as a cutpoint. Suppose that the components at $v$ are $C_1, \cdots, C_m$, with bottleneck matrices $B_1, \cdots, B_m$, respectively. If $C_m$ is a Perron component at $v$, then there exists a unique $\gamma \geq 0$ such that*

$$\rho\left(\left[\begin{array}{cccc|c} B_1 & O & \cdots & O & O \\ O & B_2 & \cdots & O & O \\ \vdots & & \ddots & \vdots & \vdots \\ O & \cdots & O & B_{m-1} & O \\ \hline O & O & \cdots & O & 0 \end{array}\right] + \gamma J\right) = \lambda_1(B_m - \gamma J) = \frac{1}{\alpha(G)}. \tag{1.1}$$

*Further, we have $\gamma = 0$ if and only if there are two or more Perron components at $v$. Finally, $y$ is a Fiedler vector for $G$ if and only if it can be written as $\left[\begin{array}{c} y_1 \\ \hline y_2 \end{array}\right]$ where $y_1$ is an eigenvector of* $\left[\begin{array}{cccc|c} B_1 & O & \cdots & O & O \\ O & B_2 & \cdots & O & O \\ \vdots & & \ddots & \vdots & \vdots \\ O & \cdots & O & B_{m-1} & O \\ \hline O & O & \cdots & O & 0 \end{array}\right] + \gamma J$ *corresponding to $\rho$, $y_2$ is an eigenvector of $B_m - \gamma J$ corresponding to $\lambda_1$, and where $1^T y_1 + 1^T y_2 = 0$.*

We emphasize that in both of the partitioned matrices appearing in Proposition 1.1, the last diagonal block is $1 \times 1$.

REMARK 1.2. Note that if $\gamma > 0$ in Proposition 1.1, then necessarily the entries of $y_1$ either all have the same sign, or they are all 0, while the signs of the entries in $y_2$ depend on the specifics of $B_m$ and $\gamma$. If $\gamma = 0$, then the nonzero entries of $y_1$ correspond to entries in Perron vectors of bottleneck matrices of Perron components at $v$ amongst $C_1, \cdots, C_{m-1}$, while $y_2$ is either a Perron vector for the bottleneck matrix of the (Perron) component $C_m$, or is the zero vector.

REMARK 1.3. We observe here that Proposition 1.1 holds even if $v$ is not a cutpoint. In that case $m = 1$, so that the matrix whose Perron value we compute on the left side of (1.1) is the $1 \times 1$ matrix $[\gamma]$, while $B_m$ is interpreted as the inverse of the principal submatrix of the Laplacian induced by the vertices of $G - v$.

The next result follows readily from Proposition 1.1; the proof is a variation on that of Theorems 2.4 and 2.5 of [4].

COROLLARY 1.4. *Let $G$ be a connected graph with a cutpoint $v$, and suppose that there are just two components at $v$. Let $B$ be the bottleneck matrix of a component $C$ at $v$ which is not the unique Perron component at $v$. Form a new graph $\tilde{G}$ by replacing the component $C$ at $v$ by another component $\tilde{C}$ such that the corresponding bottleneck matrix satisfies $\rho\left(\left[\begin{array}{c|c} B & O \\ \hline O & 0 \end{array}\right] + \gamma J\right) > \rho\left(\left[\begin{array}{c|c} \tilde{B} & O \\ \hline O & 0 \end{array}\right] + \gamma J\right)$ for all $\gamma \geq 0$. Then $\alpha(G) < \alpha(\tilde{G})$.*

The following result can also be deduced from Proposition 1.1.

COROLLARY 1.5. *Let $G$ be a connected graph with a cutpoint $v$, and suppose that $C$ is a connected component at $v$. Let $G - C$ be the graph obtained from $G$ by deleting both $C$ and each edge between $v$ and any vertex of $C$. Then $\alpha(G) \leq \alpha(G - C)$.*

The following result will be useful in the sequel, and is a recasting of Lemma 6 of [2].

PROPOSITION 1.6. *Let $G$ be a connected graph with a cutpoint $v$. Suppose that we have two components $C_1, C_2$ at $v$ with corresponding Perron values $\rho_1$ and $\rho_2$, respectively. If $\rho_1 \leq \rho_2$, then $\alpha(G) \leq 1/\rho_1$. Further, if $\alpha(G) = 1/\rho_1$, then $\rho_1 = \rho_2$ and both $C_1$ and $C_2$ are Perron components at $v$.*

We close the section with a result from [4] which helps describe the structure of a bottleneck matrix when the component under consideration contains some cutpoints.

LEMMA 1.7. *Suppose that we have a component $C$ at a vertex $v$; suppose further that $C$ has $p$ vertices, and let $M = [M_{i,j}]_{1 \leq i,j \leq p}$ be the bottleneck matrix for $C$. Construct a new component at $v$ as follows: fix some integer $1 \leq k \leq p$ and select vertices $i = 1, \cdots, k$ of $C$; for each $1 \leq i \leq k$, add a component with bottleneck matrix $B_i$ at vertex $i$. Then the resulting component at $v$ has bottleneck matrix given by*

$$\left[\begin{array}{cccc|c} B_1 + M_{1,1}J & M_{1,2}J & \cdots & M_{1,k}J & \mathbf{1}e_1^T M \\ M_{2,1}J & B_2 + M_{2,2}J & \cdots & M_{2,k}J & \mathbf{1}e_2^T M \\ \vdots & & \ddots & \vdots & \vdots \\ M_{k,1}J & \cdots & M_{k,k-1}J & B_k + M_{k,k}J & \mathbf{1}e_k^T M \\ \hline Me_1\mathbf{1}^T & Me_2\mathbf{1}^T & \cdots & Me_k\mathbf{1}^T & M \end{array}\right].$$

**2. Main Results.** In order to construct our bound on algebraic connectivity, we first investigate some special classes of graphs; it will transpire that in fact these graphs are the extremizing ones for the problem at hand. In describing these graphs we will say that a graph $G_2$ is formed from a graph $G_1$ by *attaching a path on $q$*

*vertices at vertex* $v$ if $G_2$ differs from $G_1$ only in the existence of a new connected component at $v$: a path on $q$ vertices, where $v$ is adjacent to just one vertex in that component, namely to an end point of that path. We will refer to such a component as a *path attached at* $v$. We remark that the bottleneck matrix for a path on $q$ vertices attached at a vertex $v$ has the form

$$P_q \equiv \begin{bmatrix} q & q-1 & q-2 & \cdots & 2 & 1 \\ q-1 & q-1 & q-2 & \cdots & 2 & 1 \\ \vdots & \vdots & \ddots & & \vdots & \vdots \\ 2 & 2 & \cdots & & 2 & 1 \\ 1 & 1 & \cdots & & 1 & 1 \end{bmatrix}.$$

Given $q, m \in \mathbb{N}$ with $m \geq 2$, we form the following classes of graphs:
i) $E_0(q,m)$ is the graph formed by attaching a path on $q$ vertices to each vertex of the complete graph on $m$ vertices. (By an abuse of terminology, we will sometimes refer to $E_0(q,m)$ as a class of graphs.)
ii) $E_1(q,m)$ denotes the class of graphs formed as follows: start with a graph $H$ on $m+1$ vertices having a special cutpoint labeled $v_0$ which is adjacent to all other vertices of $H$, then attach a path on $q$ vertices at each vertex of $H - v_0$.
iii) For each $m \geq l \geq 2, E_l(q,m)$ denotes the class of graphs formed as follows: start with a graph $H$ on $m$ vertices which has at least $r$ vertices of degree $m-1$ for some $m \geq r \geq l$; select $r$ such vertices of degree $m-1$, and at each, attach a path on $q+1$ vertices; at each remaining vertex $i$ (where $1 \leq i \leq m-r$) of $H$, attach a path on $j_i \leq q$ vertices (possibly $j_i = 0$), subject to the condition that $r + \sum_{i=1}^{m-r} (j_i - q) = l$.

REMARK 2.1. Comparing constructions i) and iii), we see that in fact $E_m(q,m) = E_0(q+1,m)$; occasionally this fact will be notationally convenient in the sequel.

For each $l$ with $0 \leq l \leq m$, consider a graph $G \in E_l(q,m)$, and denote the size of its vertex set by $n$. We find from constructions i), ii) and iii) above that necessarily the number of cutpoints in $G$ is $k = (qn+l)/(q+1)$.

Next, given $q, m \in \mathbb{N}$ with $m \geq 2$, we define the following quantities, which will turn out to furnish our extremal values for algebraic connectivity:

$$\alpha_{0,q,m} = 1/\rho\left(\left[\begin{array}{c|c} P_q & O \\ \hline O & 0 \end{array}\right] + \frac{1}{m}J\right);$$

$$\alpha_{1,q,m} = 1/\rho\left(P_{q+1}\right);$$

and for each $2 \leq l \leq m$,

$$\alpha_{l,q,m} = 1/\rho\left(\left[\begin{array}{c|c} P_{q+1} & O \\ \hline O & 0 \end{array}\right] + \frac{1}{m}J\right).$$

REMARK 2.2. Observe that $\alpha_{0,q,m} > \alpha_{1,q,m} > \alpha_{2,q,m} = \alpha_{l,q,m}$ for $l \geq 3$, that $\alpha_{l,q,m}$ is strictly decreasing in $q$, and that $\alpha_{l,q,m}$ is strictly increasing in $m$ for $l \neq 1$.

The following result computes the algebraic connectivity for the graphs in $E_l(q,m)$ for each $l \geq 0$.

PROPOSITION 2.3. i) $\alpha(E_0(q,m)) = \alpha_{0,q,m}$. *Further, if any edge is deleted from* $E_0(q,m)$*, then the resulting graph has algebraic connectivity strictly less than* $\alpha_{0,q,m}$.

ii) *For any graph* $G \in E_1(q,m)$*, we have* $\alpha(G) = \alpha_{1,q,m}$. *Further if any edge incident with the special cutpoint* $v_0$ *is deleted from* $G$*, then the resulting graph has algebraic connectivity strictly less than* $\alpha_{1,q,m}$.

iii) *If* $l \geq 2$*, then for any graph* $G \in E_l(q,m)$*, we have* $\alpha(G) = \alpha_{l,q,m}$. *Further if any edge incident with a vertex of degree* $m$ *is deleted from* $G$*, then the resulting graph has algebraic connectivity strictly less than* $\alpha_{l,q,m}$.

*Proof.* i) Let $u$ be a vertex of $E_0(q,m)$ which has degree $m$. Then the non-Perron component at $u$ is the path on $q$ vertices, which has bottleneck matrix $P_q$. Further, it follows from Lemma 1.7 that the bottleneck matrix for the Perron component at $u$ is given by

$$B = \left[\begin{array}{cccc|c} P_q + \frac{2}{m}J & \frac{1}{m}J & \cdots & \frac{1}{m}J & \frac{1}{m}\mathbf{1}e_1^T(I+J) \\ \frac{1}{m}J & P_q + \frac{2}{m}J & \cdots & \frac{1}{m}J & \frac{1}{m}\mathbf{1}e_2^T(I+J) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{1}{m}J & \frac{1}{m}J & \cdots & P_q + \frac{2}{m}J & \frac{1}{m}\mathbf{1}e_{m-1}^T(I+J) \\ \hline \frac{1}{m}(I+J)e_1\mathbf{1}^T & \frac{1}{m}(I+J)e_2\mathbf{1}^T & \cdots & \frac{1}{m}(I+J)e_{m-1}\mathbf{1}^T & \frac{1}{m}(I+J) \end{array}\right].$$

We find that $B - \dfrac{1}{m}J$ is permutationally similar to a direct sum of $m-1$ copies of $\left[\begin{array}{c|c} P_q & O \\ \hline O & 0 \end{array}\right] + \dfrac{1}{m}J$. It now follows from Proposition 1.1 that $\alpha(E_0(q,m)) = \alpha_{0,q,m}$.

Let $w$ be another vertex of $E_0(q,m)$ of degree $m$. From Proposition 1.1 we see that the following construction yields a Fiedler vector $y$ of $E_0(q,m)$. Let $z$ be a positive Perron vector of $\left[\begin{array}{c|c} P_q & O \\ \hline O & 0 \end{array}\right] + \dfrac{1}{m}J$. Now let the subvector of $y$ corresponding to the vertices in the Perron component at $u$, along with $u$ itself, be given by $z$, let the subvector of $y$ corresponding to the direct summand of $B - \frac{1}{m}J$ which includes vertex $w$ be given by $-z$, and let the remaining entries of $y$ be 0. Note in particular that $y_u > 0 > y_w$. Thus if $L$ is the Laplacian matrix of the graph formed from $E_0(q,m)$ by deleting the edge between $u$ and $w$, we find that $y^T L y = \alpha_{0,q,m} y^T y - (y_u - y_w)^2 < \alpha_{0,q,m} y^T y$, so that the algebraic connectivity of that graph is less than $\alpha_{0,q,m}$.

ii) Consider the graph $D_1$ formed by attaching $m$ paths on $q+1$ vertices to the single vertex $v_0$. Evidently $D_1 \in E_1(q,m)$, and it is readily seen from Proposition 1.1 that $\alpha(D_1) = \alpha_{1,q,m}$. Further, since any $G \in E_1(q,m)$ can be formed by adding edges to $D_1$, we see that $\alpha(G) \geq \alpha_{1,q,m}$. Next, let $C$ be a connected component at $v_0$ in $G$. We claim that the Perron value of $C$ is at least $\rho(P_{q+1})$; once the claim is established, an application of Proposition 1.6 will then yield that $\alpha(G) = \alpha_{1,q,m}$. Since adding edges into $C$ can only decrease its Perron value (see, e.g., [8]), we need only establish the claim for the case that the vertices in $C$ adjacent to $v_0$ induce a complete subgraph, say on $a-1$ vertices. In that case, we find from Lemma 1.7 that the bottleneck matrix for $C$ has the form

$$B = \left[\begin{array}{cccc|c} P_q + \frac{2}{a}J & \frac{1}{a}J & \cdots & \frac{1}{a}J & \frac{1}{a}\mathit{1}e_1^T(I+J) \\ \frac{1}{a}J & P_q + \frac{2}{a}J & \cdots & \frac{1}{a}J & \frac{1}{a}\mathit{1}e_2^T(I+J) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{1}{a}J & \frac{1}{a}J & \cdots & P_q + \frac{2}{a}J & \frac{1}{a}\mathit{1}e_a^T(I+J) \\ \hline \frac{1}{a}(I+J)e_1\mathit{1}^T & \frac{1}{a}(I+J)e_2\mathit{1}^T & \cdots & \frac{1}{a}(I+J)e_a\mathit{1}^T & \frac{1}{a}(I+J) \end{array}\right].$$

Next we observe that $B$ is permutationally similar to

$$\left[\begin{array}{ccccc} qI + \frac{1}{a}(I+J) & (q-1)I + \frac{1}{a}(I+J) & \cdots & I + \frac{1}{a}(I+J) & \frac{1}{a}(I+J) \\ (q-1)I + \frac{1}{a}(I+J) & (q-1)I + \frac{1}{a}(I+J) & \cdots & I + \frac{1}{a}(I+J) & \frac{1}{a}(I+J) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{1}{a}(I+J) & \frac{1}{a}(I+J) & \cdots & \frac{1}{a}(I+J) & \frac{1}{a}(I+J) \end{array}\right],$$

where each block is $(a-1)\times(a-1)$. Since the rows in each block of this last matrix sum to the corresponding entry of $P_{q+1}$, it follows readily that the Perron value of $C$ is $\rho(P_{q+1})$. We thus conclude that $\alpha(G) = \alpha_{1,q,m}$.

Let $w$ be a vertex of $G$ which is adjacent to $v_0$. From Proposition 1.1 we see that the following construction yields a Fiedler vector $y$ for $G$. Let $z_1$ be a positive Perron vector for the bottleneck matrix of the (Perron) component at $v_0$ containing $w$, and let $z_2$ be a negative Perron vector for the bottleneck matrix of some other (Perron) component at $v_0$, normalized so that $\mathit{1}^T z_1 + \mathit{1}^T z_2 = 0$. Now let the subvectors of $y$ corresponding to those components at $v_0$ be $z_1$ and $z_2$, respectively, and let the remaining entries of $y$ be 0. Note in particular that $y_w > 0 = y_{v_0}$. Thus if $L$ is the Laplacian matrix of the graph formed from $G$ by deleting the edge between $v_0$ and $w$, we find that $y^T L y < \alpha_{1,q,m} y^T y$, so that the algebraic connectivity of that graph is less than $\alpha_{1,q,m}$.

iii) Suppose that $l \geq 2$, and that $G \in E_{l,q,m}$; then $G$ can be constructed by starting with a graph $H$ on $m$ vertices in which vertices $1, \cdots, r$ have degree $m-1$ (where $m \geq r \geq l$), attaching paths of length $q+1$ to vertices $1, \cdots, r$, and attaching paths of length $0 \leq j_i \leq q$ to vertex $i$, for each $i = r+1, \cdots, m$. Let $H_1$ be the complete graph on $m$ vertices and construct $G_1 \in E_{l,q,m}$ from $H_1$ via a procedure parallel to the construction of $G$. Let $H_2$ be the graph on $m$ vertices in which vertices $1, \cdots, r$ have degree $m-1$ and vertices $r+1, \cdots, m$ have degree $r$; now construct $G_2 \in E_{l,q,m}$ from $H_2$ via a procedure parallel to the construction of $G$. Observe that $G$ can be formed by adding edges to $G_2$, or by deleting edges from $G_1$; we thus find that $\alpha(G_1) \geq \alpha(G) \geq \alpha(G_2)$.

Let $u$ be a vertex of $G_1$ of degree $m$. Then the non-Perron component at $u$ is the path on $q+1$ vertices, which has bottleneck matrix $P_{q+1}$. Further, it follows from

Lemma 1.7 that the bottleneck matrix $B_1$ for the Perron component at $u$ has the form

$$\left[\begin{array}{cc|c} A_1 & \frac{1}{m}J & U_1 \\ \frac{1}{m}J & A_2 & U_2 \\ \hline \multicolumn{2}{c|}{U_3} & \frac{1}{m}(I+J) \end{array}\right],$$

where

$$A_1 = \begin{bmatrix} P_{q+1}+\frac{2}{m}J & \frac{1}{m}J & \cdots & \frac{1}{m}J \\ \frac{1}{m}J & \ddots & & \vdots \\ \vdots & & & \frac{1}{m}J \\ \frac{1}{m}J & \cdots & \frac{1}{m}J & P_{q+1}+\frac{2}{m}J \end{bmatrix}, \quad U_1 = \begin{bmatrix} \frac{1}{m}\mathbf{1}e_1^T(I+J) \\ \vdots \\ \vdots \\ \frac{1}{m}\mathbf{1}e_{r-1}^T(I+J) \end{bmatrix},$$

$$A_2 = \begin{bmatrix} P_{j_1}+\frac{2}{m}J & \frac{1}{m}J & \cdots & \frac{1}{m}J \\ \frac{1}{m}J & \ddots & & \vdots \\ \vdots & & & \frac{1}{m}J \\ \frac{1}{m}J & \cdots & \frac{1}{m}J & P_{j_{m-r}}+\frac{2}{m}J \end{bmatrix}, \quad U_2 = \begin{bmatrix} \frac{1}{m}\mathbf{1}e_r^T(I+J) \\ \vdots \\ \vdots \\ \frac{1}{m}\mathbf{1}e_{m-1}^T(I+J) \end{bmatrix}$$

and

$$U_3 = \left[\begin{array}{ccc} \frac{1}{m}(I+J)e_1\mathbf{1}^T & \cdots & \frac{1}{m}(I+J)e_{m-1}\mathbf{1}^T \end{array}\right].$$

Note that $B_1 - \dfrac{1}{m}J$ is permutationally similar to a direct sum of $r-1$ copies of $\left[\begin{array}{c|c} P_{q+1} & O \\ \hline O & 0 \end{array}\right] + \dfrac{1}{m}J$, along with the matrices $\left[\begin{array}{c|c} P_{j_i} & O \\ \hline O & 0 \end{array}\right] + \dfrac{1}{m}J$, $1 \le i \le m-r$.

It now follows from Proposition 1.1 that $\alpha(G_1) = \alpha_{l,q,m}$. From Proposition 1.1 we also see that the following construction yields a Fiedler vector $y$ for $G_1$. Let $z_1$ be a positive Perron vector for $\left[\begin{array}{c|c} P_{q+1} & O \\ \hline O & 0 \end{array}\right] + \dfrac{1}{m}J$, and let $z_2$ be a $\lambda_1$-eigenvector of $B_1 - \dfrac{1}{m}J$ with all nonpositive entries, normalized so that $\mathbf{1}^T z_1 + \mathbf{1}^T z_2 = 0$. (Observe that such a $z_2$ exists, since $B_1 - \dfrac{1}{m}J$ is a direct sum of positive matrices.) Now let the subvector of $y$ corresponding to the vertices in the Perron component at $u$, along with $u$ itself, be $z_2$, and let the remaining subvector of $y$ be $z_1$. In particular, for each vertex $w$ in the Perron component at $u$ in $G_1$, $y_u > 0 \geq y_w$; it now follows as above that if we delete an edge from $G$ which is incident with $u$, the resulting graph has algebraic connectivity strictly less than $\alpha_{l,q,m}$.

Next we consider the graph $G_2$, and again let $u$ be a vertex of $G_2$ of degree $m$. As above, the non-Perron component at $u$ is a path on $q+1$ vertices. Let $M = \left[\begin{array}{c|c} \frac{1}{m}(I_{r-1}+J) & \frac{1}{m}J \\ \hline \frac{1}{m}J & \frac{1}{r}I_{m-r} + \frac{r-1}{mr}J \end{array}\right]$. We find from Lemma 1.7 that the bottleneck matrix $B_2$ for the Perron component at $u$ can be written as

$$B_2 = \left[\begin{array}{cc|c} N_1 & N_3 & V_1 \\ N_3^T & N_2 & V_2 \\ \hline \multicolumn{2}{c|}{V_3} & M \end{array}\right],$$

where

$$N_1 = \begin{bmatrix} P_{q+1} + M_{1,1}J & M_{1,2}J & \dots & M_{1,r-1}J \\ M_{2,1}J & \ddots & & \vdots \\ \vdots & & & M_{r-2,r-1}J \\ M_{r-1,1}J & \dots & M_{r-1,r-2}J & P_{q+1} + M_{r-1,r-1}J \end{bmatrix},$$

$$N_2 = \begin{bmatrix} P_{j_1} + M_{r,r}J & M_{r,r+1}J & \dots & M_{r,m-1}J \\ M_{r+1,r}J & \ddots & & \vdots \\ \vdots & & & M_{m-2,m-1}J \\ M_{m-1,r} & \dots & M_{m-1,m-2}J & P_{j_{m-r}} + M_{m-1,m-1}J \end{bmatrix},$$

$$N_3 = \begin{bmatrix} M_{1,r}J & \dots & M_{1,m-1}J \\ \vdots & & \vdots \\ M_{r-1,r}J & \dots & M_{r-1,m-1}J \end{bmatrix}$$

and

$$V_1 = \begin{bmatrix} \mathit{1}e_1^T M \\ \vdots \\ \vdots \\ \mathit{1}e_{r-1}^T M \end{bmatrix}, \quad V_2 = \begin{bmatrix} \mathit{1}e_r^T M \\ \vdots \\ \vdots \\ \mathit{1}e_{m-1}^T M \end{bmatrix}, \quad V_3 = \begin{bmatrix} Me_1\mathit{1}^T & \cdots & Me_{m-1}\mathit{1}^T \end{bmatrix},$$

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with $M_{i,j}$ denoting the entry of $M$ in row $i$ and column $j$. Consequently, $B_2 - \frac{1}{m}J$ is permutationally similar to a direct sum of $r-1$ copies of $\left[\begin{array}{c|c} P_{q+1} & O \\ \hline O & 0 \end{array}\right] + \frac{1}{m}J$, along with the matrix

$$R = \left[\begin{array}{cccc|c} P_{j_1} + \frac{1}{r}I & O & \cdots & O & \frac{1}{r}\mathbf{1}e_1^T \\ \vdots & \ddots & & \vdots & \vdots \\ \vdots & & & O & \vdots \\ O & \cdots & O & P_{j_{m-r}} + \frac{1}{r}I & \frac{1}{r}\mathbf{1}e_{m-r}^T \\ \hline \frac{1}{r}e_1\mathbf{1}^T & \cdots & \cdots & \frac{1}{r}e_{m-r}\mathbf{1}^T & \frac{1}{r}I \end{array}\right] - \frac{1}{mr}J.$$

Now $R + \frac{1}{mr}J$ is permutationally similar to a direct sum of the matrices $\left[\begin{array}{c|c} P_{j_i} & O \\ \hline O & 0 \end{array}\right] + \frac{1}{r}J$ for $1 \le i \le m-r$, so we see that

$$\lambda_1(R) \le \lambda_1\left(R + \frac{1}{mr}J\right) < \rho\left(\left[\begin{array}{c|c} P_{q+1} & O \\ \hline O & 0 \end{array}\right] + \frac{1}{m}J\right).$$

In particular, we have

$$\lambda_1\left(B_2 - \frac{1}{m}J\right) = \rho\left(\left[\begin{array}{c|c} P_{q+1} & O \\ \hline O & 0 \end{array}\right] + \frac{1}{m}J\right)$$

and so considering the bottleneck matrices for the components at $u$, an application of Proposition 1.1 (with $\gamma = 1/m$) shows that $\alpha(G_2) = \alpha_{l,q,m}$. The result now follows from the fact that $\alpha(G_1) \ge \alpha(G) \ge \alpha(G_2)$. ☐

Remark 2.4. Observe that from the proof of Proposition 2.3, we find that in case ii), each graph in $E_1(q,m)$ has the property that at the special cutpoint $v_0$, every component is a Perron component, with Perron value equal to $\rho(P_{q+1})$.

The following lemma deals with a special case which arises in the proof of our main result.

Lemma 2.5. *Let $G$ be a connected graph on $n$ vertices having $k > n/2$ cutpoints, such that $k = (qn+l)/(q+1)$ for some $q \ge 1$ and $l \ge 0$. Suppose that at each cutpoint $u$ of $G$ there are exactly two components, that one of those components, say $C$, is not the unique Perron component at $u$, and that $C$ is a path attached at $u$. Then $\alpha(G) \le \alpha_{l,q,n-k}$, and equality holds if and only if $G \in E_l(q, n-k)$.*

*Proof.* It is straightforward to show by induction on $n$ that since at each cutpoint there are two components, one of which is an attached path, the graph $G$ can be constructed as follows: begin with a graph $H$ on $n-k$ vertices which has no cutpoints, and for some $1 \le m \le n-k$, select $m$ vertices of $H$, say vertices $1, \cdots, m$; for each $1 \le i \le m$, attach a path of length $j_i$ at vertex $i$. In order to facilitate notation in the sequel, we will let $j_i = 0$ for $i = m+1, \cdots, n-k$ in the case that $m < n-k$. The graph thus constructed has $k = \sum_{i=1}^{m} j_i = \sum_{i=1}^{n-k} j_i$ cutpoints and $n-k+\sum_{i=1}^{n-k} j_i = \sum_{i=1}^{n-k}(j_i+1) = n$ vertices. From the hypothesis we may also assume without loss of generality that for each $1 \le i \le n-k$, the path on $j_i$ vertices attached at vertex $i$ is not the unique Perron component at vertex $i$.

If $m = 1$ then $j_1 = k$ and since $n-2 \ge k = (qn+l)/(q+1)$, we find that $n \ge 2q+l+2$. Since $n \ge 2q+l+2$, we find that $(qn+l)/(q+1) \ge 2q+l$; further it is clear that if $q \ge 2$ then $2q+l \ge q+2$, while if $q = 1$ then necessarily $l \ge 1$, since our hypothesis asserts that $n/2 < k = (qn+l)/(q+1)$, and again we see that $2q+l \ge q+2$. Thus we have $k = (qn+l)/(q+1) \ge 2q+l \ge q+2$. In particular, since the path on $k$ vertices attached at vertex 1 is not the unique Perron component, we have $\alpha(G) \le 1/\rho(P_k) \le 1/\rho(P_{q+2}) < \alpha_{l,q,n-k}$.

Henceforth we assume that $m \ge 2$. Note that as above, if some $j_i \ge q+2$, then $\alpha(G) < \alpha_{l,q,n-k}$. So henceforth we also suppose that $j_i \le q+1, i = 1, \cdots, m$. If each $j_i$ is at most $q$, then note that $mq \ge \sum_{i=1}^{m} j_i = k$, while $n = \sum_{i=1}^{m} j_i + n - k$. Since $(q+1)k = qn+l$, it follows that $mq \ge \sum_{i=1}^{m} j_i = q(n-k)+l \ge mq+l$. We deduce that $l = 0$, that $m = n-k$ and that each $j_i = q$. Observe now that by adding edges (if necessary) into $G$, we can construct $E_0(q, n-k)$. The conclusion now follows from Proposition 2.3.

Next we assume that at least one $j_i$ is equal to $q+1$. If there are $r \ge 2$ such $j_i$'s, $j_1, \cdots, j_r$ say, then note that $l = (q+1)k - qn = (q+1)\sum_{i=1}^{n-k} j_i - q\sum_{i=1}^{n-k}(j_i+1) = r + \sum_{i=r+1}^{n-k}(j_i - q)$. Thus, by adding edges into $G$ (if necessary) we can construct a graph in $E_l(q, n-k)$. The conclusion then follows from Proposition 2.3.

Finally, suppose that just one $j_i$ is equal to $q+1$, say $j_1 = q+1$. If some $j_i$ is at most $q-1$, then we see that $(q+1)+(q-1)+(m-2)q \ge \sum_{i=1}^{m} j_i = q(n-k)+l \ge qm+l$. Thus $l = 0$, but then we have $\alpha(G) \le 1/\rho(P_{q+1}) < \alpha_{0,q,n-k}$. On the other hand, if each $j_i = q$ for each $2 \le i \le m$, then we have $mq+1 = q(n-k)+l$. Note that if $n-k > m$, then $q+l \le 1$, contradicting the fact that $k > n/2$. Thus it must be the case that $n-k = m$, so that $l = 1$. Observing that by adding edges to $G$ if necessary, we can construct a graph in $E_1(q, n-k)$, the conclusion then follows from Proposition 2.3. ☐

We are now ready to present the main result of this paper.

THEOREM 2.6. *Let $G$ be a connected graph on $n$ vertices which has $k$ cutpoints. Suppose that $k > n/2$, say with $k = (qn+l)/(q+1)$ for some positive integer $q$ and nonnegative integer $l$. Then $\alpha(G) \le \alpha_{l,q,n-k}$. Furthermore, equality holds if and only $G \in E_l(q, n-k)$.*

*Proof.* We proceed by induction on $n$, and since the proof is somewhat lengthy, we first give a brief outline of our approach. After establishing the base case for the induction, we then assume the induction hypothesis, and deal with the case that at some cutpoint of $G$, there is a component on at least two vertices containing no cutpoints of $G$. Next, we cover the case that $l \geq 3$. We follow that by a discussion of the case that $0 \leq l \leq 2$ and that at some cutpoint of $G$ there are at least three components. We then suppose that $0 \leq l \leq 2$, and that at each cutpoint $v$ of $G$ there are exactly two components (note that one of those components is not the unique Perron component at $v$). We deal with the case that for some cutpoint $v$ of $G$ there is a component which is not the unique Perron component at $v$, and which is not an attached path. The last remaining case is then covered by Lemma 2.5.

As noted above, we will use induction on $n$. Note that since $(n+1)/2 \leq k \leq n-2$ we see that the smallest admissible case is $n = 5$. This yields $k = 3$, so we have $q = 1$ and $l = 1$. In that instance, $G$ is the path on 5 vertices, so that $\alpha(G) = 1/\rho(P_2) = \alpha_{1,1,2} = \alpha_{l,q,n-k}$; note also that $G \in E_1(1,2) = E_l(q, n-k)$ in this case.

Now we suppose that $n \geq 6$ and that the result holds for all graphs on at most $n-1$ vertices. Let $v$ be a cutpoint of $G$ at which there is a component $C$ which contains no cutpoints of $G$ and suppose that $C$ has $n_1 \geq 2$ vertices. We claim that in this case, $\alpha(G) < \alpha_{l,q,n-k}$. To see the claim, note that the graph $G-C$ has at least $k-1$ cutpoints and exactly $n-n_1$ vertices; since $k-1 = (q(n-n_1)+l-1+q(n_1-1))/(q+1)$, we find from Corollary 1.5 and the induction hypothesis that $\alpha(G) \leq \alpha(G-C) \leq \alpha_{l-1+q(n_1-1),q,n-n_1-k+1}$. Since $q(n_1-1) \geq 1$ and $n_1 \geq 2$, we find from Remark 2.2 that $\alpha_{l-1+q(n_1-1),q,n-n_1-k+1} \leq \alpha_{l,q,n-k}$, with strict inequality if either $q(n_1-1) > 1$ or $l \neq 1$. Thus it remains only to establish the claim when $q(n_1-1) = 1$ and $l = 1$ - i.e. when $n_1 = 2$, $l = 1$ and $q = 1$. From the induction hypothesis, either $\alpha(G-C) < \alpha_{1,1,n-k-1}$, in which case we are done, or $G - C \in E_1(1, n-k-1)$. In that case, note that at the special cutpoint $v_0$ of $G-C$, there are at least two Perron components, each of Perron value $\rho(P_2)$. Note also that in $G$, $v$ cannot be the same as $v_0$, otherwise $G$ has fewer than $(qn+l)/(q+1) = (n+1)/2$ cutpoints. Thus we see that in $G$, there is at least one component at $v_0$ with Perron value $\rho(P_2)$, and another with Perron value larger than $\rho(P_2)$. The claim now follows from Proposition 1.6.

Henceforth we will assume that any component at a cutpoint $v$ which does not contain a cutpoint of $G$ must necessarily consist of a single vertex. Suppose now that $l \geq 3$; select a cutpoint $v$ of $G$ at which one of the components is a single (pendant) vertex, and form $\tilde{G}$ by deleting that pendant vertex. Since $\tilde{G}$ has at least $k-1$ cutpoints and $n-1$ vertices, we find as above that $\alpha(G) \leq \alpha(\tilde{G}) \leq \alpha_{l-1,q,n-k} = \alpha_{l,q,n-k}$ (the last since $l \geq 3$), yielding the desired inequality on $\alpha(G)$. Further, if $\alpha(G) = \alpha_{l,q,n-k}$ then necessarily $\tilde{G}$ has exactly $k-1$ cutpoints (otherwise $\alpha(\tilde{G}) \leq \alpha_{l-1,q,n-k-1} < \alpha_{l,q,n-k}$, the last inequality from Remark 2.2), and $\alpha(\tilde{G}) = \alpha_{l-1,q,n-k}$. Thus by the induction hypothesis, $\tilde{G} \in E_{l-1}(q, n-k)$. Further, $G$ is formed from $\tilde{G}$ by adding a pendant vertex $p$ at one of the pendant vertices of $\tilde{G}$. Consider the construction of $\tilde{G}$ described in iii): if $p$ is added at the end of a path on $j_i \leq q$ vertices, then $G \in E_l(q, n-k)$, and we are done; if $p$ is added at the end of a path on $q+1$ vertices, then in $G$ there is a vertex $u$ (the root of that path) at which there are two components: one with Perron value $\rho(P_{q+2})$ and the other with Perron value at least $\rho\left(\left[\begin{array}{c|c} P_{q+1} & O \\ \hline O & 0 \end{array}\right] + \frac{1}{n-k}J\right)$. It now follows from Proposition 1.6 that $\alpha(G) < \alpha_{l,q,n-k}$, contrary to our assumption. We have thus established the result for $l \geq 3$.

Henceforth we assume that $0 \leq l \leq 2$. Suppose that at a cutpoint $v$ of $G$ there are $m \geq 3$ components, say $C_1, \cdots, C_m$, where $C_i$ contains $n_i$ vertices and $k_i$ cutpoints of $G$, $1 \leq i \leq m$. For each such $i$, we see that $G - C_i$ has $n - n_i$ vertices and $k - k_i$ cutpoints. Suppose that for each $1 \leq i \leq m$ we have $k-k_i \leq (q(n-n_i)+l-1)/(q+1)$. Summing these inequalities, we find that $mk-k+1 \leq (q(mn-n+1)+m(l-1))/(q+1)$, so that $(m-1)k \leq (q(m-1)n+ml-m-1)/(q+1) \leq (m-1)(qn+l-1)/(q+1)$, the last inequality following from the fact that $l \leq 2$. Thus $k < (qn+l)/(q+1)$, contrary to our hypothesis. We conclude that for some $i$ we must have $k-k_i \geq (q(n-n_i)+l)/(q+1)$. But then we have $\alpha(G) \leq \alpha(G - C_i) \leq \alpha_{l,q,n-n_i-k+k_i} \leq \alpha_{l,q,n-k}$, with the last inequality being strict in the case that $l = 0$ or 2 (by Remark 2.2). We thus find that $\alpha(G) \leq \alpha_{l,q,n-k}$. Suppose now that $\alpha(G) = \alpha_{l,q,n-k}$. Then as remarked above, we must have $l = 1$; further, we necessarily have $k - k_i = (q(n - n_i) + l)/(q + 1)$ and $G - C_i \in E_1(q, n-k-n_i+k_i)$ by the induction hypothesis. Let $v_0$ denote the special cutpoint of $G - C_i$, at which each component is a Perron component, having Perron value $\rho(P_{q+1})$. If $v \neq v_0$, then we find that in $G$, the cutpoint $v_0$ has one component with Perron value greater than $\rho(P_{q+1})$ and at least one component with Perron value equal to $\rho(P_{q+1})$; from Proposition 1.6, we conclude that $\alpha(G) < \alpha_{1,q,n-k}$, contrary to our assumption. Thus necessarily $v = v_0$ and so the graph $G - C_i$ is constructed as described in ii). In particular, for each $j \neq i$, $C_j$ satisfies $k_j = qn_j/(q + 1)$, and so the analysis above also applies to the graph $G - C_j$. Consequently, $G - C_j \in E_l(q, n - k - n_j + k_j)$, from which it follows that $G \in E_1(q, n - k)$, as desired.

Henceforth we assume that at each cutpoint of $G$, there are just two components. Let $u$ be a cutpoint of $G$, and suppose that there is a component $C$ at $u$ which is not the unique Perron component at $u$, and which is not a path attached at $u$. Consider the subgraph induced by the vertices of $C \cup u$ and let $w$ be a cutpoint of $G$ in that subgraph which is farthest from $u$ (possibly $w = u$) such that at $w$, there is a component $\hat{C}$ which is not the unique Perron component at $w$ in $G$, and which is not a path attached at $w$. Observe that $\hat{C}$ contains at least one cutpoint of $G$ (since we are dealing with the case that a component without any cutpoints is a path on one vertex). Further, at each cutpoint in $\hat{C}$, the component not containing $w$ is an attached path, otherwise there is a cutpoint $t$ farther from $u$ than $w$, such that at $t$, there is a component $\hat{C}$ which is not the unique Perron component at $t$, and which is not a path attached at $t$, contrary to the fact that $w$ is a cutpoint farthest from $u$ with that property.

We claim that if this is the case, then either $\alpha(G) < \alpha_{l,q,n-k}$ or $l = 1$ and $G \in E_1(q, n-k)$. Since adding edges into $G$ cannot decrease its algebraic connectivity, it is enough to prove the claim in the case that $\hat{C}$ is constructed by taking a complete graph on vertices $1, \cdots, m+x$, attaching a path of length $j_i \geq 1$ at vertex $i$, $1 \leq i \leq m$ (we admit the possibility that $x$ may be 0), and ensuring that $w$ is adjacent to each of vertices $1, \cdots, m + x$. Observe that necessarily, $m + x \geq 2$, otherwise $\hat{C}$ would be a path attached at $w$. If some $j_i \geq q+1$, it follows readily that

$$\alpha(G) \leq 1/\rho\left(\left[\begin{array}{c|c} P_{q+1} & O \\ \hline O & 0 \end{array}\right] + \frac{1}{m+x}J\right) < \alpha_{l,q,n-k},$$

where the last inequality holds since $m+x < n-k$. So we suppose that $j_i \leq q$ for $1 \leq i \leq m$. Next, form $G'$ from $G$ by replacing the component $\hat{C}$ at $w$ by a path on $j_1$ vertices attached at $w$. Since the bottleneck matrix $\hat{B}$ for $\hat{C}$ satisfies $\rho\left(\left[\begin{array}{c|c} \hat{B} & O \\ \hline O & 0 \end{array}\right] + \gamma J\right) > \rho\left(\left[\begin{array}{c|c} P_{j_1} & O \\ \hline O & 0 \end{array}\right] + \gamma J\right)$ for any nonnegative $\gamma$ (the strict inequality following from the fact that the order of $\hat{B}$ is strictly greater than $j_1$) , we find from Corollary 1.4 that $\alpha(G) < \alpha(G')$. Note that $G'$ has $k-1-\sum_{i=2}^{m} j_i$ cutpoints and $n-m-x-\sum_{i=2}^{m} j_i$ vertices. Further,

$$k-1-\sum_{i=2}^{m} j_i = \left(q(n-m-x-\sum_{i=2}^{m} j_i) + \sum_{i=2}^{m}(q-j_i) + qx - 1 + l\right)/(q+1).$$

In particular, if $x \geq 1$, then by the induction hypothesis and Remark 2.2, $\alpha(G) < \alpha(G') \leq \alpha_{l,q,n-k-m-x+1}$, yielding the desired inequality. If $x = 0$, then necessarily $m \geq 2$ (otherwise $\hat{C}$ is a path) and so if $j_i < q$ for some $2 \leq i \leq m$, we again find that $\alpha(G) < \alpha_{l,q,n-k}$. An analogous argument applies if $x = 0$ and $j_1 < q$, so it remains only to consider the case that $x = 0$ and $j_i = q$ for $1 \leq i \leq m$.

In that case, the bottleneck matrix $\hat{B}$ for $\hat{C}$ can be written as

$$\left[\begin{array}{ccccc} qI + \frac{1}{m+1}(I+J) & (q-1)I + \frac{1}{m+1}(I+J) & \cdots & I + \frac{1}{m+1}(I+J) & \frac{1}{m+1}(I+J) \\ (q-1)I + \frac{1}{m+1}(I+J) & (q-1)I + \frac{1}{m+1}(I+J) & \cdots & I + \frac{1}{m+1}(I+J) & \frac{1}{m+1}(I+J) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{1}{m+1}(I+J) & \frac{1}{m+1}(I+J) & \cdots & \frac{1}{m+1}(I+J) & \frac{1}{m+1}(I+J) \end{array}\right],$$

where each block is $m \times m$. Further, each block of $\hat{B}$ has constant row sums which are equal to the corresponding entry in $P_{q+1}$, and it then follows that $\rho(\hat{B}) = \rho(P_{q+1})$, while for each positive $\gamma$,

$$\rho\left(\left[\begin{array}{c|c} \hat{B} & O \\ \hline O & 0 \end{array}\right] + \gamma J\right) = \rho\left(\left[\begin{array}{c|c} P_{q+1} & O \\ \hline O & 0 \end{array}\right] + m\gamma J\right) > \rho\left(\left[\begin{array}{c|c} P_{q+1} & O \\ \hline O & 0 \end{array}\right] + \gamma J\right).$$

If there are two Perron components at $w$ in $G$, then an analogous argument on the other Perron component at $w$ (i.e., the component not equal to $\hat{C}$) reveals that either $\alpha(G) < \alpha_{l,q,n-k}$ or that $l = 1$ and $G \in E_1(q, n-k)$. On the other hand, if there is a unique Perron component at $w$ in $G$, form $G''$ from $G$ by replacing $\hat{C}$ by a path on $q+1$ vertices; it follows from Proposition 1.1 that $\alpha(G) < \alpha(G'')$. Observe that $G''$

has $k-(m-1)q$ cutpoints and $n-(m-1)(q+1)$ vertices. Since

$$k-(m-1)q=\frac{q(n-(m-1)(q+1))+l}{q+1},$$

we find from the induction hypothesis that $\alpha(G'')\le\alpha_{l,q,n-k-m+1}$, thus completing the proof of the claim.

From the forgoing, we now need only consider the case that at each cutpoint of $G$ there are just two components, and that for any cutpoint $u$, there is a component which is not the unique Perron component at $u$, and which is a path attached at $u$. The conclusion now follows from Lemma 2.5. □

REMARK 2.7. The hypothesis of Theorem 2.6 is stated for any integers $q$ and $l$ such that $q\ge 1$, $l\ge 0$ and $k=(qn+l)/(q+1)$, but it is straightforward to see that the resulting bound on $\alpha(G)$ is tightest when $q$ is as large as possible and that equality is attainable only in that case. Observe that if $l\ge n-k$, say $l=n-k+i$, then we find that $k=((q+1)n+i)/(q+2)$, so the case that $q$ is as large as possible is equivalent to the case that $l<n-k$. That case is easily seen to correspond to $q=\lfloor k/(n-k)\rfloor$ and $l=k-(n-k)\lfloor k/(n-k)\rfloor$. Thus we see that if $G$ has $n$ vertices and $k>n/2$ cutpoints, then $\alpha(G)\le\alpha_{k-(n-k)\lfloor k/(n-k)\rfloor,\lfloor k/(n-k)\rfloor,n-k}$, with equality if and only if $G\in E_{k-(n-k)\lfloor k/(n-k)\rfloor}(\lfloor k/(n-k)\rfloor,n-k)$.

While Theorem 2.6 gives us the upper bound $\alpha_{l,q,n-k}$ in terms of Perron values, the following result makes the value of $\alpha_{l,q,n-k}$ a little more explicit.

PROPOSITION 2.8. *Suppose that $q\in\mathbb{N}$, and that $m\ge 1$. Then there exists a unique $\theta_0\in\left[\frac{\pi}{2q+3},\frac{\pi}{2q+1}\right]$ such that $(m-1)\cos((2q+1)\theta_0/2)+\cos((2q+3)\theta_0/2)=0$. Furthermore,*

$$1/\rho\left(\left[\begin{array}{c|c}P_q & O\\ \hline O & 0\end{array}\right]+\frac{1}{m}J\right)=2(1-\cos(\theta_0)).$$

*Proof.* It is straightforward to see that the function $(m-1)\cos((2q+1)\theta_0/2)+\cos((2q+3)\theta_0/2)$ is decreasing from $(m-1)\cos((2q+1)\pi/(2(2q+3)))\ge 0$ to $\cos((2q+3)\pi)/(2(2q+1)))<0$ for $\theta\in\left[\frac{\pi}{2q+3},\frac{\pi}{2q+1}\right]$, so the existence and uniqueness of $\theta_0$ follows readily.

Further, we have

$$\left(\left[\begin{array}{c|c}P_q & O\\ \hline O & 0\end{array}\right]+\frac{1}{m}J\right)^{-1}=M\equiv\left[\begin{array}{cccccc}1 & -1 & 0 & \cdots & 0 & 0\\ -1 & 2 & -1 & 0 & \cdots & 0\\ \vdots & & \ddots & & & \vdots\\ 0 & 0 & \cdots & -1 & 2 & -1\\ 0 & 0 & \cdots & 0 & -1 & m+1\end{array}\right],$$

so that $1/\rho\left(\left[\begin{array}{c|c}P_q & O\\ \hline O & 0\end{array}\right]+\frac{1}{m}J\right)$ is the smallest eigenvalue of $M$. Observe that $M$ is an M-matrix. Further, since

$$\cos((i-1)\theta_0+\theta_0/2)+\cos((i+1)\theta_0+\theta_0/2)=2\cos(i\theta_0+\theta_0/2)\cos(\theta_0)$$

for each $i = 0, \cdots, q$, we find that the vector $v = \begin{bmatrix} \cos(\theta_0/2) \\ \cos(3\theta_0/2) \\ \vdots \\ \cos((2q+1)\theta_0/2) \end{bmatrix}$ is an eigenvector of $M$ corresponding to the eigenvalue $2(1-\cos(\theta_0))$. Since $v$ is an eigenvector with all positive entries, it corresponds to the smallest eigenvalue of $M$, and the result now follows. □

COROLLARY 2.9. *For each* $q \in \mathbb{N}$, $\alpha_{1,q,n-k} = 2(1-\cos(\frac{\pi}{2q+3}))$.

*Proof.* Since $1/\rho(P_{q+1})$ corresponds to the case $m = 1$ in Proposition 2.8, the conclusion follows. □

REMARK 2.10. The principal results of [7] assert that for a graph $G$ on $n$ vertices with $k$ cutpoints, we have: i) if $k = 1$, then $\alpha(G) \le 1$, with equality if and only if the single cutpoint $v_0$ is adjacent to all other vertices of $G$; ii) if $2 \le k \le n/2$, then $\alpha(G) \le 2(n-k)/(n-k+2+\sqrt{(n-k)^2+4})$, with equality if and only if $G$ is constructed by taking a graph on $n-k$ vertices which has $k$ vertices of degree $n-k-1$, and attaching a pendant vertex at each of those vertices of maximum degree.

In the language of the present paper, case i) corresponds to $q = 0$ and $l = 1$, and yields the upper bound $\alpha(G) \le 1/\rho(P_1)$; equality holds if and only if $G$ is formed from a construction analogous to that of the graphs in $E_1(q, n-k)$. Similarly, for $k < n/2$, case ii) corresponds to $q = 0$ and $l = k$. A straightforward computation with the $2 \times 2$ matrix $\left[\begin{array}{c|c} P_1 & 0 \\ \hline 0 & 0 \end{array}\right] + \frac{1}{n-k}J$ shows that

$$2(n-k)/(n-k+2+\sqrt{(n-k)^2+4}) = 1/\rho\left(\left[\begin{array}{c|c} P_1 & 0 \\ \hline 0 & 0 \end{array}\right] + \frac{1}{n-k}J\right),$$

so the upper bound can be written as

$$\alpha(G) \le 1/\rho\left(\left[\begin{array}{c|c} P_1 & 0 \\ \hline 0 & 0 \end{array}\right] + \frac{1}{n-k}J\right).$$

Further, equality holds if and only if $G$ is formed from a construction analogous to that of the graphs in $E_l(q, n-k)$. If $k = n/2$, then case ii) corresponds to $q = 1, l = 0$, and again

$$\alpha(G) \le 1/\rho\left(\left[\begin{array}{c|c} P_1 & 0 \\ \hline 0 & 0 \end{array}\right] + \frac{1}{n-k}J\right),$$

with equality holding if and only if $G$ can be constructed in a manner analogous to that in $E_l(q, n-k)$. Thus we see that both the upper bounds and the extremizing graphs in the present paper are natural extensions of the corresponding ones in [7].

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