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AN UPPER BOUND ON ALGEBRAIC CONNECTIVITY OF GRAPHS WITH MANY CUTPOINTS*

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Abstract. Let G be a graph on n vertices which has k cutpoints. A tight upper bound on the algebraic connectivity of G in terms of n and k for the case that $k > n/2$ is provided; the graphs which yield equality in the bound are also characterized. This completes an investigation initiated by the author in a previous paper, which dealt with the corresponding problem for the case that $k \leq n/2$.

Key words. Laplacian matrix, algebraic connectivity, cutpoint.

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1. Introduction and Preliminaries. Let G be a graph on n vertices. Its *Laplacian matrix* L can be written as $L = D - A$, where A is the $(0, 1)$ adjacency matrix of G, and D is the diagonal matrix of vertex degrees. There is a wealth of literature on Laplacian matrices in general (see, e.g., the survey by Merris [9]), and on their eigenvalues in particular. It is straightforward to see that L is a positive semidefinite singular M-matrix, with the all-ones vector $\mathit{1}$ as a null vector. Further, Fiedler [5] has shown that if G is connected, then the remaining eigenvalues of L are positive. Motivated by this observation, the second smallest eigenvalue of L is known as the *algebraic connectivity of* G; throughout this paper, we denote the algebraic connectivity of G by $\alpha(G)$. The eigenvectors of L corresponding to $\alpha(G)$ have come to be known as *Fiedler vectors* for G.

We list here a few of the well-known properties of algebraic connectivity; these can be found in [5]. Since $\alpha(G)$ is the second smallest eigenvalue of L, it follows that $\alpha(G) = min\{y^T Ly | y^T \mathit{1} = 0, y^T y = 1\}$. Further, if we add an edge into G to form $\tilde{G}$, then $\alpha(G) \leq \alpha(\tilde{G})$. Finally, if G has vertex connectivity $c \leq n-2$, then $\alpha(G) \leq c$. In particular, if G has a *cutpoint* - that is, a vertex whose deletion (along with all edges incident with it) yields a disconnected graph - then we see that $\alpha(G) \leq 1$.

Motivated by this last observation, Kirkland [7] posed the following problem: if G is a graph on n vertices which has k cutpoints, find an attainable upper bound on $\alpha(G)$. In [7], such a bound is constructed for the case that $1 \leq k \leq n/2$, and the graphs attaining the bound are characterized. The present paper is a continuation of the work in [7]; here we give an attainable upper bound on $\alpha(G)$ when $n/2 < k \leq n-2$, and explicitly describe the equality case.

The technique used in this paper relies on the analysis of the various connected components which arise from the deletion of a cutpoint. We now briefly outline that technique. Suppose that G is a connected graph and that v is a cutpoint of G. The *components at* v are just the connected components of $G - v$, the (disconnected) graph

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which is produced when we delete v and all edges incident with it. For a connected component C at v, the *bottleneck matrix for* C is the inverse of the principal submatrix of L induced by the vertices of C. It is straightforward to see that the bottleneck matrix B for C is entrywise positive, and so it has a Perron value, $\rho(B)$, and we occasionally refer to $\rho(B)$ as the *Perron value of* C. If the components at v are $C_1, \cdots, C_m$, then we say that C_j is a *Perron component at* v if its Perron value is maximum amongst those of the connected components at v. We note that there may be several Perron components at a vertex.

The following result, which pulls together several facts established in [4] and [1], shows how the viewpoint of Perron components can be used to describe both $\alpha(G)$ and the corresponding Fiedler vectors. Throughout this paper, J denotes the all-ones matrix, O denotes the zero matrix (possibly a vector), and the orders of both J and O will be apparent from the context. We use $\rho(M)$ to denote the Perron value of any square entrywise nonnegative matrix M, while $\lambda_1(S)$ denotes the largest eigenvalue of any symmetric matrix S. We refer the reader to [3] for the basics on nonnegative matrices, and to [6] for background on symmetric matrices.

PROPOSITION 1.1. *Let G be a connected graph having vertex v as a cutpoint. Suppose that the components at v are $C_1, \cdots, C_m$, with bottleneck matrices $B_1, \cdots, B_m$, respectively. If C_m is a Perron component at v, then there exists a unique $\gamma \geq 0$ such that*

$$\rho\left(\left[\begin{array}{cccc|c} B_1 & O & \cdots & O & O \\ O & B_2 & \cdots & O & O \\ \vdots & & \ddots & \vdots & \vdots \\ O & \cdots & O & B_{m-1} & O \\ \hline O & O & \cdots & O & 0 \end{array}\right] + \gamma J\right) = \lambda_1(B_m - \gamma J) = \frac{1}{\alpha(G)}. \tag{1.1}$$

Further, we have $\gamma = 0$ if and only if there are two or more Perron components at v. Finally, y is a Fiedler vector for G if and only if it can be written as $\left[\begin{array}{c} y_1 \\ \hline y_2 \end{array}\right]$ where y_1 is an eigenvector of $\left[\begin{array}{cccc|c} B_1 & O & \cdots & O & O \\ O & B_2 & \cdots & O & O \\ \vdots & & \ddots & \vdots & \vdots \\ O & \cdots & O & B_{m-1} & O \\ \hline O & O & \cdots & O & 0 \end{array}\right] + \gamma J$ *corresponding to ρ, y_2 is an eigenvector of $B_m - \gamma J$ corresponding to λ_1, and where $1^T y_1 + 1^T y_2 = 0$.*

We emphasize that in both of the partitioned matrices appearing in Proposition 1.1, the last diagonal block is 1×1.

REMARK 1.2. Note that if $\gamma > 0$ in Proposition 1.1, then necessarily the entries of y_1 either all have the same sign, or they are all 0, while the signs of the entries in y_2 depend on the specifics of B_m and γ. If $\gamma = 0$, then the nonzero entries of y_1 correspond to entries in Perron vectors of bottleneck matrices of Perron components at v amongst $C_1, \cdots, C_{m-1}$, while y_2 is either a Perron vector for the bottleneck matrix of the (Perron) component C_m, or is the zero vector.

REMARK 1.3. We observe here that Proposition 1.1 holds even if v is not a cutpoint. In that case $m = 1$, so that the matrix whose Perron value we compute on the left side of (1.1) is the 1×1 matrix $[\gamma]$, while B_m is interpreted as the inverse of the principal submatrix of the Laplacian induced by the vertices of $G - v$.

The next result follows readily from Proposition 1.1; the proof is a variation on that of Theorems 2.4 and 2.5 of [4].

COROLLARY 1.4. *Let G be a connected graph with a cutpoint v, and suppose that there are just two components at v. Let B be the bottleneck matrix of a component C at v which is not the unique Perron component at v. Form a new graph $\tilde{G}$ by replacing the component C at v by another component $\tilde{C}$ such that the corresponding bottleneck matrix satisfies $\rho\left(\left[\begin{array}{c|c} B & O \\ \hline O & 0 \end{array}\right] + \gamma J\right) > \rho\left(\left[\begin{array}{c|c} \tilde{B} & O \\ \hline O & 0 \end{array}\right] + \gamma J\right)$ for all $\gamma \geq 0$. Then $\alpha(G) < \alpha(\tilde{G})$.*

The following result can also be deduced from Proposition 1.1.

COROLLARY 1.5. *Let G be a connected graph with a cutpoint v, and suppose that C is a connected component at v. Let $G - C$ be the graph obtained from G by deleting both C and each edge between v and any vertex of C. Then $\alpha(G) \leq \alpha(G - C)$.*

The following result will be useful in the sequel, and is a recasting of Lemma 6 of [2].

PROPOSITION 1.6. *Let G be a connected graph with a cutpoint v. Suppose that we have two components C_1, C_2 at v with corresponding Perron values ρ_1 and ρ_2, respectively. If $\rho_1 \leq \rho_2$, then $\alpha(G) \leq 1/\rho_1$. Further, if $\alpha(G) = 1/\rho_1$, then $\rho_1 = \rho_2$ and both C_1 and C_2 are Perron components at v.*

We close the section with a result from [4] which helps describe the structure of a bottleneck matrix when the component under consideration contains some cutpoints.

LEMMA 1.7. *Suppose that we have a component C at a vertex v; suppose further that C has p vertices, and let $M = [M_{i,j}]_{1 \leq i,j \leq p}$ be the bottleneck matrix for C. Construct a new component at v as follows: fix some integer $1 \leq k \leq p$ and select vertices $i = 1, \cdots, k$ of C; for each $1 \leq i \leq k$, add a component with bottleneck matrix B_i at vertex i. Then the resulting component at v has bottleneck matrix given by*

$$\left[\begin{array}{cccc|c} B_1 + M_{1,1}J & M_{1,2}J & \cdots & M_{1,k}J & \mathbf{1}e_1^TM \\ M_{2,1}J & B_2 + M_{2,2}J & \cdots & M_{2,k}J & \mathbf{1}e_2^TM \\ \vdots & & \ddots & \vdots & \vdots \\ M_{k,1}J & \cdots & M_{k,k-1}J & B_k + M_{k,k}J & \mathbf{1}e_k^TM \\ \hline Me_1\mathbf{1}^T & Me_2\mathbf{1}^T & \cdots & Me_k\mathbf{1}^T & M \end{array}\right].$$

2. Main Results. In order to construct our bound on algebraic connectivity, we first investigate some special classes of graphs; it will transpire that in fact these graphs are the extremizing ones for the problem at hand. In describing these graphs we will say that a graph G_2 is formed from a graph G_1 by *attaching a path on q*

vertices at vertex v if G_2 differs from G_1 only in the existence of a new connected component at v: a path on q vertices, where v is adjacent to just one vertex in that component, namely to an end point of that path. We will refer to such a component as a *path attached at v*. We remark that the bottleneck matrix for a path on q vertices attached at a vertex v has the form

$$P_q \equiv \begin{bmatrix} q & q-1 & q-2 & \cdots & 2 & 1 \\ q-1 & q-1 & q-2 & \cdots & 2 & 1 \\ \vdots & \vdots & \ddots & & \vdots & \vdots \\ 2 & 2 & \cdots & & 2 & 1 \\ 1 & 1 & \cdots & & 1 & 1 \end{bmatrix}.$$

Given $q, m \in \mathbb{N}$ with $m \geq 2$, we form the following classes of graphs:
i) $E_0(q,m)$ is the graph formed by attaching a path on q vertices to each vertex of the complete graph on m vertices. (By an abuse of terminology, we will sometimes refer to $E_0(q,m)$ as a class of graphs.)
ii) $E_1(q,m)$ denotes the class of graphs formed as follows: start with a graph H on $m+1$ vertices having a special cutpoint labeled v_0 which is adjacent to all other vertices of H, then attach a path on q vertices at each vertex of $H - v_0$.
iii) For each $m \geq l \geq 2, E_l(q,m)$ denotes the class of graphs formed as follows: start with a graph H on m vertices which has at least r vertices of degree $m-1$ for some $m \geq r \geq l$; select r such vertices of degree $m-1$, and at each, attach a path on $q+1$ vertices; at each remaining vertex i (where $1 \leq i \leq m-r$) of H, attach a path on $j_i \leq q$ vertices (possibly $j_i = 0$), subject to the condition that $r + \sum_{i=1}^{m-r}(j_i - q) = l$.

REMARK 2.1. Comparing constructions i) and iii), we see that in fact $E_m(q,m) = E_0(q+1,m)$; occasionally this fact will be notationally convenient in the sequel.

For each l with $0 \leq l \leq m$, consider a graph $G \in E_l(q,m)$, and denote the size of its vertex set by n. We find from constructions i), ii) and iii) above that necessarily the number of cutpoints in G is $k = (qn+l)/(q+1)$.

Next, given $q, m \in \mathbb{N}$ with $m \geq 2$, we define the following quantities, which will turn out to furnish our extremal values for algebraic connectivity:

$$\alpha_{0,q,m} = 1/\rho\left(\left[\begin{array}{c|c} P_q & O \\ \hline O & 0 \end{array}\right] + \frac{1}{m}J\right);$$

$$\alpha_{1,q,m} = 1/\rho\left(P_{q+1}\right);$$

and for each $2 \leq l \leq m$,

$$\alpha_{l,q,m} = 1/\rho\left(\left[\begin{array}{c|c} P_{q+1} & O \\ \hline O & 0 \end{array}\right] + \frac{1}{m}J\right).$$

REMARK 2.2. Observe that $\alpha_{0,q,m} > \alpha_{1,q,m} > \alpha_{2,q,m} = \alpha_{l,q,m}$ for $l \geq 3$, that $\alpha_{l,q,m}$ is strictly decreasing in q, and that $\alpha_{l,q,m}$ is strictly increasing in m for $l \neq 1$.

The following result computes the algebraic connectivity for the graphs in $E_l(q,m)$ for each $l \geq 0$.

PROPOSITION 2.3. i) $\alpha(E_0(q,m)) = \alpha_{0,q,m}$. *Further, if any edge is deleted from* $E_0(q,m)$, *then the resulting graph has algebraic connectivity strictly less than* $\alpha_{0,q,m}$.

ii) *For any graph* $G \in E_1(q,m)$, *we have* $\alpha(G) = \alpha_{1,q,m}$. *Further if any edge incident with the special cutpoint* v_0 *is deleted from* G, *then the resulting graph has algebraic connectivity strictly less than* $\alpha_{1,q,m}$.

iii) *If* $l \geq 2$, *then for any graph* $G \in E_l(q,m)$, *we have* $\alpha(G) = \alpha_{l,q,m}$. *Further if any edge incident with a vertex of degree* m *is deleted from* G, *then the resulting graph has algebraic connectivity strictly less than* $\alpha_{l,q,m}$.

Proof. i) Let u be a vertex of $E_0(q,m)$ which has degree m. Then the non-Perron component at u is the path on q vertices, which has bottleneck matrix P_q. Further, it follows from Lemma 1.7 that the bottleneck matrix for the Perron component at u is given by

$$B = \left[\begin{array}{cccc|c} P_q + \frac{2}{m}J & \frac{1}{m}J & \cdots & \frac{1}{m}J & \frac{1}{m}\mathit{1}e_1^T(I+J) \\ \frac{1}{m}J & P_q + \frac{2}{m}J & \cdots & \frac{1}{m}J & \frac{1}{m}\mathit{1}e_2^T(I+J) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{1}{m}J & \frac{1}{m}J & \cdots & P_q + \frac{2}{m}J & \frac{1}{m}\mathit{1}e_{m-1}^T(I+J) \\ \hline \frac{1}{m}(I+J)e_1\mathit{1}^T & \frac{1}{m}(I+J)e_2\mathit{1}^T & \cdots & \frac{1}{m}(I+J)e_{m-1}\mathit{1}^T & \frac{1}{m}(I+J) \end{array}\right].$$

We find that $B - \dfrac{1}{m}J$ is permutationally similar to a direct sum of $m-1$ copies of $\left[\begin{array}{c|c} P_q & O \\ \hline O & 0 \end{array}\right] + \dfrac{1}{m}J$. It now follows from Proposition 1.1 that $\alpha(E_0(q,m)) = \alpha_{0,q,m}$.

Let w be another vertex of $E_0(q,m)$ of degree m. From Proposition 1.1 we see that the following construction yields a Fiedler vector y of $E_0(q,m)$. Let z be a positive Perron vector of $\left[\begin{array}{c|c} P_q & O \\ \hline O & 0 \end{array}\right] + \dfrac{1}{m}J$. Now let the subvector of y corresponding to the vertices in the Perron component at u, along with u itself, be given by z, let the subvector of y corresponding to the direct summand of $B - \frac{1}{m}J$ which includes vertex w be given by $-z$, and let the remaining entries of y be 0. Note in particular that $y_u > 0 > y_w$. Thus if L is the Laplacian matrix of the graph formed from $E_0(q,m)$ by deleting the edge between u and w, we find that $y^TLy = \alpha_{0,q,m}y^Ty - (y_u - y_w)^2 < \alpha_{0,q,m}y^Ty$, so that the algebraic connectivity of that graph is less than $\alpha_{0,q,m}$.

ii) Consider the graph D_1 formed by attaching m paths on $q+1$ vertices to the single vertex v_0. Evidently $D_1 \in E_1(q,m)$, and it is readily seen from Proposition 1.1 that $\alpha(D_1) = \alpha_{1,q,m}$. Further, since any $G \in E_1(q,m)$ can be formed by adding edges to D_1, we see that $\alpha(G) \geq \alpha_{1,q,m}$. Next, let C be a connected component at v_0 in G. We claim that the Perron value of C is at least $\rho(P_{q+1})$; once the claim is established, an application of Proposition 1.6 will then yield that $\alpha(G) = \alpha_{1,q,m}$. Since adding edges into C can only decrease its Perron value (see, e.g., [8]), we need only establish the claim for the case that the vertices in C adjacent to v_0 induce a complete subgraph, say on $a-1$ vertices. In that case, we find from Lemma 1.7 that the bottleneck matrix for C has the form

$$B = \left[\begin{array}{cccc|c} P_q + \frac{2}{a}J & \frac{1}{a}J & \cdots & \frac{1}{a}J & \frac{1}{a}\mathit{1}e_1^T(I+J) \\ \frac{1}{a}J & P_q + \frac{2}{a}J & \cdots & \frac{1}{a}J & \frac{1}{a}\mathit{1}e_2^T(I+J) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{1}{a}J & \frac{1}{a}J & \cdots & P_q + \frac{2}{a}J & \frac{1}{a}\mathit{1}e_a^T(I+J) \\ \hline \frac{1}{a}(I+J)e_1\mathit{1}^T & \frac{1}{a}(I+J)e_2\mathit{1}^T & \cdots & \frac{1}{a}(I+J)e_a\mathit{1}^T & \frac{1}{a}(I+J) \end{array}\right].$$

Next we observe that B is permutationally similar to

$$\begin{bmatrix} qI + \frac{1}{a}(I+J) & (q-1)I + \frac{1}{a}(I+J) & \cdots & I + \frac{1}{a}(I+J) & \frac{1}{a}(I+J) \\ (q-1)I + \frac{1}{a}(I+J) & (q-1)I + \frac{1}{a}(I+J) & \cdots & I + \frac{1}{a}(I+J) & \frac{1}{a}(I+J) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{1}{a}(I+J) & \frac{1}{a}(I+J) & \cdots & \frac{1}{a}(I+J) & \frac{1}{a}(I+J) \end{bmatrix},$$

where each block is $(a-1)\times(a-1)$. Since the rows in each block of this last matrix sum to the corresponding entry of P_{q+1}, it follows readily that the Perron value of C is $\rho(P_{q+1})$. We thus conclude that $\alpha(G) = \alpha_{1,q,m}$.

Let w be a vertex of G which is adjacent to v_0. From Proposition 1.1 we see that the following construction yields a Fiedler vector y for G. Let z_1 be a positive Perron vector for the bottleneck matrix of the (Perron) component at v_0 containing w, and let z_2 be a negative Perron vector for the bottleneck matrix of some other (Perron) component at v_0, normalized so that $\mathit{1}^T z_1 + \mathit{1}^T z_2 = 0$. Now let the subvectors of y corresponding to those components at v_0 be z_1 and z_2, respectively, and let the remaining entries of y be 0. Note in particular that $y_w > 0 = y_{v_0}$. Thus if L is the Laplacian matrix of the graph formed from G by deleting the edge between v_0 and w, we find that $y^T Ly < \alpha_{1,q,m} y^T y$, so that the algebraic connectivity of that graph is less than $\alpha_{1,q,m}$.

iii) Suppose that $l \geq 2$, and that $G \in E_{l,q,m}$; then G can be constructed by starting with a graph H on m vertices in which vertices $1, \cdots, r$ have degree $m-1$ (where $m \geq r \geq l$), attaching paths of length $q+1$ to vertices $1, \cdots, r$, and attaching paths of length $0 \leq j_i \leq q$ to vertex i, for each $i = r+1, \cdots, m$. Let H_1 be the complete graph on m vertices and construct $G_1 \in E_{l,q,m}$ from H_1 via a procedure parallel to the construction of G. Let H_2 be the graph on m vertices in which vertices $1, \cdots, r$ have degree $m-1$ and vertices $r+1, \cdots, m$ have degree r; now construct $G_2 \in E_{l,q,m}$ from H_2 via a procedure parallel to the construction of G. Observe that G can be formed by adding edges to G_2, or by deleting edges from G_1; we thus find that $\alpha(G_1) \geq \alpha(G) \geq \alpha(G_2)$.

Let u be a vertex of G_1 of degree m. Then the non-Perron component at u is the path on $q+1$ vertices, which has bottleneck matrix P_{q+1}. Further, it follows from

Lemma 1.7 that the bottleneck matrix B_1 for the Perron component at u has the form

$$\left[\begin{array}{cc|c} A_1 & \frac{1}{m}J & U_1 \\ \frac{1}{m}J & A_2 & U_2 \\ \hline \multicolumn{2}{c|}{U_3} & \frac{1}{m}(I+J) \end{array}\right],$$

where

$$A_1 = \begin{bmatrix} P_{q+1}+\frac{2}{m}J & \frac{1}{m}J & \cdots & & \frac{1}{m}J \\ \frac{1}{m}J & \ddots & & & \vdots \\ \vdots & & & & \frac{1}{m}J \\ \frac{1}{m}J & & \cdots & \frac{1}{m}J & P_{q+1}+\frac{2}{m}J \end{bmatrix}, \quad U_1 = \begin{bmatrix} \frac{1}{m}\mathit{1}e_1^T(I+J) \\ \vdots \\ \vdots \\ \frac{1}{m}\mathit{1}e_{r-1}^T(I+J) \end{bmatrix},$$

$$A_2 = \begin{bmatrix} P_{j_1}+\frac{2}{m}J & \frac{1}{m}J & \cdots & & \frac{1}{m}J \\ \frac{1}{m}J & \ddots & & & \vdots \\ \vdots & & & & \frac{1}{m}J \\ \frac{1}{m}J & & \cdots & \frac{1}{m}J & P_{j_{m-r}}+\frac{2}{m}J \end{bmatrix}, \quad U_2 = \begin{bmatrix} \frac{1}{m}\mathit{1}e_r^T(I+J) \\ \vdots \\ \vdots \\ \frac{1}{m}\mathit{1}e_{m-1}^T(I+J) \end{bmatrix}$$

and

$$U_3 = \left[\ \frac{1}{m}(I+J)e_1\mathit{1}^T \quad \cdots \quad \frac{1}{m}(I+J)e_{m-1}\mathit{1}^T\ \right].$$

Note that $B_1 - \dfrac{1}{m}J$ is permutationally similar to a direct sum of $r-1$ copies of $\left[\begin{array}{c|c} P_{q+1} & O \\ \hline O & 0 \end{array}\right] + \dfrac{1}{m}J$, along with the matrices $\left[\begin{array}{c|c} P_{j_i} & O \\ \hline O & 0 \end{array}\right] + \dfrac{1}{m}J$, $1 \le i \le m-r$.

It now follows from Proposition 1.1 that $\alpha(G_1) = \alpha_{l,q,m}$. From Proposition 1.1 we also see that the following construction yields a Fiedler vector y for G_1. Let z_1 be a positive Perron vector for $\left[\begin{array}{c|c} P_{q+1} & O \\ \hline O & 0 \end{array}\right] + \dfrac{1}{m}J$, and let z_2 be a λ_1-eigenvector of $B_1 - \dfrac{1}{m}J$ with all nonpositive entries, normalized so that $\mathit{1}^T z_1 + \mathit{1}^T z_2 = 0$. (Observe that such a z_2 exists, since $B_1 - \dfrac{1}{m}J$ is a direct sum of positive matrices.) Now let the subvector of y corresponding to the vertices in the Perron component at u, along with u itself, be z_2, and let the remaining subvector of y be z_1. In particular, for each vertex w in the Perron component at u in G_1, $y_u > 0 \geq y_w$; it now follows as above that if we delete an edge from G which is incident with u, the resulting graph has algebraic connectivity strictly less than $\alpha_{l,q,m}$.

Next we consider the graph G_2, and again let u be a vertex of G_2 of degree m. As above, the non-Perron component at u is a path on $q+1$ vertices. Let $M = \left[\begin{array}{c|c} \frac{1}{m}(I_{r-1}+J) & \frac{1}{m}J \\ \hline \frac{1}{m}J & \frac{1}{r}I_{m-r} + \frac{r-1}{mr}J \end{array}\right]$. We find from Lemma 1.7 that the bottleneck matrix B_2 for the Perron component at u can be written as

$$B_2 = \left[\begin{array}{cc|c} N_1 & N_3 & V_1 \\ N_3^T & N_2 & V_2 \\ \hline \multicolumn{2}{c|}{V_3} & M \end{array}\right],$$

where

$$N_1 = \begin{bmatrix} P_{q+1} + M_{1,1}J & M_{1,2}J & \dots & M_{1,r-1}J \\ M_{2,1}J & \ddots & & \vdots \\ \vdots & & & M_{r-2,r-1}J \\ M_{r-1,1}J & \cdots & M_{r-1,r-2}J & P_{q+1} + M_{r-1,r-1}J \end{bmatrix},$$

$$N_2 = \begin{bmatrix} P_{j_1} + M_{r,r}J & M_{r,r+1}J & \dots & M_{r,m-1}J \\ M_{r+1,r}J & \ddots & & \vdots \\ \vdots & & & M_{m-2,m-1}J \\ M_{m-1,r} & \cdots & M_{m-1,m-2}J & P_{j_{m-r}} + M_{m-1,m-1}J \end{bmatrix},$$

$$N_3 = \begin{bmatrix} M_{1,r}J & \dots & M_{1,m-1}J \\ \vdots & & \vdots \\ M_{r-1,r}J & \dots & M_{r-1,m-1}J \end{bmatrix}$$

and

$$V_1 = \begin{bmatrix} \mathit{1} e_1^T M \\ \vdots \\ \vdots \\ \mathit{1} e_{r-1}^T M \end{bmatrix}, \quad V_2 = \begin{bmatrix} \mathit{1} e_r^T M \\ \vdots \\ \vdots \\ \mathit{1} e_{m-1}^T M \end{bmatrix}, \quad V_3 = \begin{bmatrix} Me_1\mathit{1}^T & \cdots & Me_{m-1}\mathit{1}^T \end{bmatrix},$$

with $M_{i,j}$ denoting the entry of M in row i and column j. Consequently, $B_2 - \frac{1}{m}J$ is permutationally similar to a direct sum of $r-1$ copies of $\left[\begin{array}{c|c} P_{q+1} & O \\ \hline O & 0 \end{array}\right] + \frac{1}{m}J$, along with the matrix

$$R = \left[\begin{array}{ccccc|c} P_{j_1} + \frac{1}{r}I & O & \cdots & & O & \frac{1}{r}\mathit{1}e_1^T \\ \vdots & \ddots & & & \vdots & \vdots \\ \vdots & & & & O & \vdots \\ O & & \cdots & O & P_{j_{m-r}} + \frac{1}{r}I & \frac{1}{r}\mathit{1}e_{m-r}^T \\ \hline \frac{1}{r}e_1\mathit{1}^T & & \cdots & \cdots & \frac{1}{r}e_{m-r}\mathit{1}^T & \frac{1}{r}I \end{array}\right] - \frac{1}{mr}J.$$

Now $R + \frac{1}{mr}J$ is permutationally similar to a direct sum of the matrices $\left[\begin{array}{c|c} P_{j_i} & O \\ \hline O & 0 \end{array}\right] + \frac{1}{r}J$ for $1 \le i \le m-r$, so we see that

$$\lambda_1(R) \le \lambda_1\left(R + \frac{1}{mr}J\right) < \rho\left(\left[\begin{array}{c|c} P_{q+1} & O \\ \hline O & 0 \end{array}\right] + \frac{1}{m}J\right).$$

In particular, we have

$$\lambda_1\left(B_2 - \frac{1}{m}J\right) = \rho\left(\left[\begin{array}{c|c} P_{q+1} & O \\ \hline O & 0 \end{array}\right] + \frac{1}{m}J\right)$$

and so considering the bottleneck matrices for the components at u, an application of Proposition 1.1 (with $\gamma = 1/m$) shows that $\alpha(G_2) = \alpha_{l,q,m}$. The result now follows from the fact that $\alpha(G_1) \ge \alpha(G) \ge \alpha(G_2)$. □

Remark 2.4. Observe that from the proof of Proposition 2.3, we find that in case ii), each graph in $E_1(q, m)$ has the property that at the special cutpoint v_0, every component is a Perron component, with Perron value equal to $\rho(P_{q+1})$.

The following lemma deals with a special case which arises in the proof of our main result.

Lemma 2.5. *Let G be a connected graph on n vertices having $k > n/2$ cutpoints, such that $k = (qn+l)/(q+1)$ for some $q \ge 1$ and $l \ge 0$. Suppose that at each cutpoint u of G there are exactly two components, that one of those components, say C, is not the unique Perron component at u, and that C is a path attached at u. Then $\alpha(G) \le \alpha_{l,q,n-k}$, and equality holds if and only if $G \in E_l(q, n-k)$.*

Proof. It is straightforward to show by induction on n that since at each cutpoint there are two components, one of which is an attached path, the graph G can be constructed as follows: begin with a graph H on $n-k$ vertices which has no cutpoints, and for some $1 \le m \le n-k$, select m vertices of H, say vertices $1, \cdots, m$; for each $1 \le i \le m$, attach a path of length j_i at vertex i. In order to facilitate notation in the sequel, we will let $j_i = 0$ for $i = m+1, \cdots, n-k$ in the case that $m < n-k$. The graph thus constructed has $k = \sum_{i=1}^{m} j_i = \sum_{i=1}^{n-k} j_i$ cutpoints and $n-k+\sum_{i=1}^{n-k} j_i = \sum_{i=1}^{n-k} (j_i+1) = n$ vertices. From the hypothesis we may also assume without loss of generality that for each $1 \le i \le n-k$, the path on j_i vertices attached at vertex i is not the unique Perron component at vertex i.

If $m = 1$ then $j_1 = k$ and since $n - 2 \ge k = (qn + l)/(q + 1)$, we find that $n \ge 2q + l + 2$. Since $n \ge 2q + l + 2$, we find that $(qn + l)/(q + 1) \ge 2q + l$; further it is clear that if $q \ge 2$ then $2q + l \ge q + 2$, while if $q = 1$ then necessarily $l \ge 1$, since our hypothesis asserts that $n/2 < k = (qn + l)/(q + 1)$, and again we see that $2q+l \ge q+2$. Thus we have $k = (qn+l)/(q+1) \ge 2q+l \ge q+2$. In particular, since the path on k vertices attached at vertex 1 is not the unique Perron component, we have $\alpha(G) \le 1/\rho(P_k) \le 1/\rho(P_{q+2}) < \alpha_{l,q,n-k}$.

Henceforth we assume that $m \ge 2$. Note that as above, if some $j_i \ge q + 2$, then $\alpha(G) < \alpha_{l,q,n-k}$. So henceforth we also suppose that $j_i \le q + 1, i = 1, \cdots, m$. If each j_i is at most q, then note that $mq \ge \sum_{i=1}^{m} j_i = k$, while $n = \sum_{i=1}^{m} j_i + n - k$. Since $(q + 1)k = qn + l$, it follows that $mq \ge \sum_{i=1}^{m} j_i = q(n - k) + l \ge mq + l$. We deduce that $l = 0$, that $m = n - k$ and that each $j_i = q$. Observe now that by adding edges (if necessary) into G, we can construct $E_0(q, n-k)$. The conclusion now follows from Proposition 2.3.

Next we assume that at least one j_i is equal to $q + 1$. If there are $r \ge 2$ such j_i's, $j_1, \cdots, j_r$ say, then note that $l = (q + 1)k - qn = (q + 1)\sum_{i=1}^{n-k} j_i - q \sum_{i=1}^{n-k} (j_i + 1) = r + \sum_{i=r+1}^{n-k} (j_i - q)$. Thus, by adding edges into G (if necessary) we can construct a graph in $E_l(q, n - k)$. The conclusion then follows from Proposition 2.3.

Finally, suppose that just one j_i is equal to $q + 1$, say $j_1 = q + 1$. If some j_i is at most $q-1$, then we see that $(q+1)+(q-1)+(m-2)q \ge \sum_{i=1}^{m} j_i = q(n-k)+l \ge qm+l$. Thus $l = 0$, but then we have $\alpha(G) \le 1/\rho(P_{q+1}) < \alpha_{0,q,n-k}$. On the other hand, if each $j_i = q$ for each $2 \le i \le m$, then we have $mq + 1 = q(n - k) + l$. Note that if $n - k > m$, then $q + l \le 1$, contradicting the fact that $k > n/2$. Thus it must be the case that $n-k = m$, so that $l = 1$. Observing that by adding edges to G if necessary, we can construct a graph in $E_1(q, n-k)$, the conclusion then follows from Proposition 2.3. □

We are now ready to present the main result of this paper.

THEOREM 2.6. *Let G be a connected graph on n vertices which has k cutpoints. Suppose that $k > n/2$, say with $k = (qn + l)/(q + 1)$ for some positive integer q and nonnegative integer l. Then $\alpha(G) \le \alpha_{l,q,n-k}$. Furthermore, equality holds if and only $G \in E_l(q, n - k)$.*

Proof. We proceed by induction on n, and since the proof is somewhat lengthy, we first give a brief outline of our approach. After establishing the base case for the induction, we then assume the induction hypothesis, and deal with the case that at some cutpoint of G, there is a component on at least two vertices containing no cutpoints of G. Next, we cover the case that $l \geq 3$. We follow that by a discussion of the case that $0 \leq l \leq 2$ and that at some cutpoint of G there are at least three components. We then suppose that $0 \leq l \leq 2$, and that at each cutpoint v of G there are exactly two components (note that one of those components is not the unique Perron component at v). We deal with the case that for some cutpoint v of G there is a component which is not the unique Perron component at v, and which is not an attached path. The last remaining case is then covered by Lemma 2.5.

As noted above, we will use induction on n. Note that since $(n+1)/2 \leq k \leq n-2$ we see that the smallest admissible case is $n = 5$. This yields $k = 3$, so we have $q = 1$ and $l = 1$. In that instance, G is the path on 5 vertices, so that $\alpha(G) = 1/\rho(P_2) = \alpha_{1,1,2} = \alpha_{l,q,n-k}$; note also that $G \in E_1(1, 2) = E_l(q, n-k)$ in this case.

Now we suppose that $n \geq 6$ and that the result holds for all graphs on at most $n-1$ vertices. Let v be a cutpoint of G at which there is a component C which contains no cutpoints of G and suppose that C has $n_1 \geq 2$ vertices. We claim that in this case, $\alpha(G) < \alpha_{l,q,n-k}$. To see the claim, note that the graph $G - C$ has at least $k - 1$ cutpoints and exactly $n-n_1$ vertices; since $k-1 = (q(n-n_1)+l-1+q(n_1-1))/(q+1)$, we find from Corollary 1.5 and the induction hypothesis that $\alpha(G) \leq \alpha(G - C) \leq \alpha_{l-1+q(n_1-1),q,n-n_1-k+1}$. Since $q(n_1 - 1) \geq 1$ and $n_1 \geq 2$, we find from Remark 2.2 that $\alpha_{l-1+q(n_1-1),q,n-n_1-k+1} \leq \alpha_{l,q,n-k}$, with strict inequality if either $q(n_1 - 1) > 1$ or $l \neq 1$. Thus it remains only to establish the claim when $q(n_1 - 1) = 1$ and $l = 1$ - i.e. when $n_1 = 2$, $l = 1$ and $q = 1$. From the induction hypothesis, either $\alpha(G - C) < \alpha_{1,1,n-k-1}$, in which case we are done, or $G - C \in E_1(1, n - k - 1)$. In that case, note that at the special cutpoint v_0 of $G-C$, there are at least two Perron components, each of Perron value $\rho(P_2)$. Note also that in G, v cannot be the same as v_0, otherwise G has fewer than $(qn+l)/(q+1) = (n+1)/2$ cutpoints. Thus we see that in G, there is at least one component at v_0 with Perron value $\rho(P_2)$, and another with Perron value larger than $\rho(P_2)$. The claim now follows from Proposition 1.6.

Henceforth we will assume that any component at a cutpoint v which does not contain a cutpoint of G must necessarily consist of a single vertex. Suppose now that $l \geq 3$; select a cutpoint v of G at which one of the components is a single (pendant) vertex, and form $\tilde{G}$ by deleting that pendant vertex. Since $\tilde{G}$ has at least $k - 1$ cutpoints and $n - 1$ vertices, we find as above that $\alpha(G) \leq \alpha(\tilde{G}) \leq \alpha_{l-1,q,n-k} = \alpha_{l,q,n-k}$ (the last since $l \geq 3$), yielding the desired inequality on $\alpha(G)$. Further, if $\alpha(G) = \alpha_{l,q,n-k}$ then necessarily $\tilde{G}$ has exactly $k - 1$ cutpoints (otherwise $\alpha(\tilde{G}) \leq \alpha_{l-1,q,n-k-1} < \alpha_{l,q,n-k}$, the last inequality from Remark 2.2), and $\alpha(\tilde{G}) = \alpha_{l-1,q,n-k}$. Thus by the induction hypothesis, $\tilde{G} \in E_{l-1}(q, n - k)$. Further, G is formed from $\tilde{G}$ by adding a pendant vertex p at one of the pendant vertices of $\tilde{G}$. Consider the construction of $\tilde{G}$ described in iii): if p is added at the end of a path on $j_i \leq q$ vertices, then $G \in E_l(q, n - k)$, and we are done; if p is added at the end of a path on $q + 1$ vertices, then in G there is a vertex u (the root of that path) at which there are two components: one with Perron value $\rho(P_{q+2})$ and the other with Perron value at least $\rho\left(\left[\begin{array}{c|c} P_{q+1} & O \\ \hline O & 0 \end{array}\right] + \frac{1}{n-k}J\right)$. It now follows from Proposition 1.6 that $\alpha(G) < \alpha_{l,q,n-k}$, contrary to our assumption. We have thus established the result for $l \geq 3$.

Henceforth we assume that $0 \leq l \leq 2$. Suppose that at a cutpoint v of G there are $m \geq 3$ components, say $C_1, \cdots, C_m$, where C_i contains n_i vertices and k_i cutpoints of G, $1 \leq i \leq m$. For each such i, we see that $G - C_i$ has $n - n_i$ vertices and $k - k_i$ cutpoints. Suppose that for each $1 \leq i \leq m$ we have $k - k_i \leq (q(n-n_i)+l-1)/(q+1)$. Summing these inequalities, we find that $mk-k+1 \leq (q(mn-n+1)+m(l-1))/(q+1)$, so that $(m-1)k \leq (q(m-1)n+ml-m-1)/(q+1) \leq (m-1)(qn+l-1)/(q+1)$, the last inequality following from the fact that $l \leq 2$. Thus $k < (qn+l)/(q+1)$, contrary to our hypothesis. We conclude that for some i we must have $k-k_i \geq (q(n-n_i)+l)/(q+1)$. But then we have $\alpha(G) \leq \alpha(G - C_i) \leq \alpha_{l,q,n-n_i-k+k_i} \leq \alpha_{l,q,n-k}$, with the last inequality being strict in the case that $l = 0$ or 2 (by Remark 2.2). We thus find that $\alpha(G) \leq \alpha_{l,q,n-k}$. Suppose now that $\alpha(G) = \alpha_{l,q,n-k}$. Then as remarked above, we must have $l = 1$; further, we necessarily have $k - k_i = (q(n-n_i)+l)/(q+1)$ and $G - C_i \in E_1(q, n-k-n_i+k_i)$ by the induction hypothesis. Let v_0 denote the special cutpoint of $G - C_i$, at which each component is a Perron component, having Perron value $\rho(P_{q+1})$. If $v \neq v_0$, then we find that in G, the cutpoint v_0 has one component with Perron value greater than $\rho(P_{q+1})$ and at least one component with Perron value equal to $\rho(P_{q+1})$; from Proposition 1.6, we conclude that $\alpha(G) < \alpha_{1,q,n-k}$, contrary to our assumption. Thus necessarily $v = v_0$ and so the graph $G - C_i$ is constructed as described in ii). In particular, for each $j \neq i$, C_j satisfies $k_j = qn_j/(q+1)$, and so the analysis above also applies to the graph $G - C_j$. Consequently, $G - C_j \in E_l(q, n-k-n_j+k_j)$, from which it follows that $G \in E_1(q, n-k)$, as desired.

Henceforth we assume that at each cutpoint of G, there are just two components. Let u be a cutpoint of G, and suppose that there is a component C at u which is not the unique Perron component at u, and which is not a path attached at u. Consider the subgraph induced by the vertices of $C \cup u$ and let w be a cutpoint of G in that subgraph which is farthest from u (possibly $w = u$) such that at w, there is a component $\hat{C}$ which is not the unique Perron component at w in G, and which is not a path attached at w. Observe that $\hat{C}$ contains at least one cutpoint of G (since we are dealing with the case that a component without any cutpoints is a path on one vertex). Further, at each cutpoint in $\hat{C}$, the component not containing w is an attached path, otherwise there is a cutpoint t farther from u than w, such that at t, there is a component $\hat{C}$ which is not the unique Perron component at t, and which is not a path attached at t, contrary to the fact that w is a cutpoint farthest from u with that property.

We claim that if this is the case, then either $\alpha(G) < \alpha_{l,q,n-k}$ or $l = 1$ and $G \in E_1(q, n-k)$. Since adding edges into G cannot decrease its algebraic connectivity, it is enough to prove the claim in the case that $\hat{C}$ is constructed by taking a complete graph on vertices $1, \cdots, m+x$, attaching a path of length $j_i \geq 1$ at vertex i, $1 \leq i \leq m$ (we admit the possibility that x may be 0), and ensuring that w is adjacent to each of vertices $1, \cdots, m+x$. Observe that necessarily, $m + x \geq 2$, otherwise $\hat{C}$ would be a path attached at w. If some $j_i \geq q+1$, it follows readily that

$$\alpha(G) \leq 1/\rho\left(\left[\begin{array}{c|c} P_{q+1} & O \\ \hline O & 0 \end{array}\right] + \frac{1}{m+x}J\right) < \alpha_{l,q,n-k},$$

where the last inequality holds since $m+x < n-k$. So we suppose that $j_i \leq q$ for $1 \leq i \leq m$. Next, form G' from G by replacing the component $\hat{C}$ at w by a path on j_1 vertices attached at w. Since the bottleneck matrix $\hat{B}$ for $\hat{C}$ satisfies $\rho\left(\left[\begin{array}{c|c} \hat{B} & O \\ \hline O & 0 \end{array}\right] + \gamma J\right) > \rho\left(\left[\begin{array}{c|c} P_{j_1} & O \\ \hline O & 0 \end{array}\right] + \gamma J\right)$ for any nonnegative γ (the strict inequality following from the fact that the order of $\hat{B}$ is strictly greater than j_1) , we find from Corollary 1.4 that $\alpha(G) < \alpha(G')$. Note that G' has $k - 1 - \sum_{i=2}^{m} j_i$ cutpoints and $n - m - x - \sum_{i=2}^{m} j_i$ vertices. Further,

$$k - 1 - \sum_{i=2}^{m} j_i = \left(q(n-m-x-\sum_{i=2}^{m} j_i) + \sum_{i=2}^{m}(q-j_i) + qx - 1 + l\right)/(q+1).$$

In particular, if $x \geq 1$, then by the induction hypothesis and Remark 2.2, $\alpha(G) < \alpha(G') \leq \alpha_{l,q,n-k-m-x+1}$, yielding the desired inequality. If $x = 0$, then necessarily $m \geq 2$ (otherwise $\hat{C}$ is a path) and so if $j_i < q$ for some $2 \leq i \leq m$, we again find that $\alpha(G) < \alpha_{l,q,n-k}$. An analogous argument applies if $x = 0$ and $j_1 < q$, so it remains only to consider the case that $x = 0$ and $j_i = q$ for $1 \leq i \leq m$.

In that case, the bottleneck matrix $\hat{B}$ for $\hat{C}$ can be written as

$$\left[\begin{array}{ccccc} qI + \frac{1}{m+1}(I+J) & (q-1)I + \frac{1}{m+1}(I+J) & \cdots & I + \frac{1}{m+1}(I+J) & \frac{1}{m+1}(I+J) \\ (q-1)I + \frac{1}{m+1}(I+J) & (q-1)I + \frac{1}{m+1}(I+J) & \cdots & I + \frac{1}{m+1}(I+J) & \frac{1}{m+1}(I+J) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{1}{m+1}(I+J) & \frac{1}{m+1}(I+J) & \cdots & \frac{1}{m+1}(I+J) & \frac{1}{m+1}(I+J) \end{array}\right],$$

where each block is $m \times m$. Further, each block of $\hat{B}$ has constant row sums which are equal to the corresponding entry in P_{q+1}, and it then follows that $\rho(\hat{B}) = \rho(P_{q+1})$, while for each positive γ,

$$\rho\left(\left[\begin{array}{c|c} \hat{B} & O \\ \hline O & 0 \end{array}\right] + \gamma J\right) = \rho\left(\left[\begin{array}{c|c} P_{q+1} & O \\ \hline O & 0 \end{array}\right] + m\gamma J\right) > \rho\left(\left[\begin{array}{c|c} P_{q+1} & O \\ \hline O & 0 \end{array}\right] + \gamma J\right).$$

If there are two Perron components at w in G, then an analogous argument on the other Perron component at w (i.e., the component not equal to $\hat{C}$) reveals that either $\alpha(G) < \alpha_{l,q,n-k}$ or that $l = 1$ and $G \in E_1(q, n-k)$. On the other hand, if there is a unique Perron component at w in G, form G'' from G by replacing $\hat{C}$ by a path on $q+1$ vertices; it follows from Proposition 1.1 that $\alpha(G) < \alpha(G'')$. Observe that G''

has $k-(m-1)q$ cutpoints and $n-(m-1)(q+1)$ vertices. Since

$$k-(m-1)q=\frac{q(n-(m-1)(q+1))+l}{q+1},$$

we find from the induction hypothesis that $\alpha(G'') \le \alpha_{l,q,n-k-m+1}$, thus completing the proof of the claim.

From the forgoing, we now need only consider the case that at each cutpoint of G there are just two components, and that for any cutpoint u, there is a component which is not the unique Perron component at u, and which is a path attached at u. The conclusion now follows from Lemma 2.5. $\square$

REMARK 2.7. The hypothesis of Theorem 2.6 is stated for any integers q and l such that $q \ge 1$, $l \ge 0$ and $k=(qn+l)/(q+1)$, but it is straightforward to see that the resulting bound on $\alpha(G)$ is tightest when q is as large as possible and that equality is attainable only in that case. Observe that if $l \ge n-k$, say $l=n-k+i$, then we find that $k=((q+1)n+i)/(q+2)$, so the case that q is as large as possible is equivalent to the case that $l < n-k$. That case is easily seen to correspond to $q=\lfloor k/(n-k)\rfloor$ and $l=k-(n-k)\lfloor k/(n-k)\rfloor$. Thus we see that if G has n vertices and $k > n/2$ cutpoints, then $\alpha(G) \le \alpha_{k-(n-k)\lfloor k/(n-k)\rfloor,\lfloor k/(n-k)\rfloor,n-k}$, with equality if and only if $G \in E_{k-(n-k)\lfloor k/(n-k)\rfloor}(\lfloor k/(n-k)\rfloor, n-k)$.

While Theorem 2.6 gives us the upper bound $\alpha_{l,q,n-k}$ in terms of Perron values, the following result makes the value of $\alpha_{l,q,n-k}$ a little more explicit.

PROPOSITION 2.8. *Suppose that $q \in \mathbb{N}$, and that $m \ge 1$. Then there exists a unique $\theta_0 \in \left[\frac{\pi}{2q+3}, \frac{\pi}{2q+1}\right]$ such that $(m-1)\cos((2q+1)\theta_0/2)+\cos((2q+3)\theta_0/2)=0$. Furthermore,*

$$1/\rho\left(\left[\begin{array}{c|c} P_q & O \\ \hline O & 0\end{array}\right]+\frac{1}{m}J\right)=2(1-\cos(\theta_0)).$$

Proof. It is straightforward to see that the function $(m-1)\cos((2q+1)\theta_0/2)+\cos((2q+3)\theta_0/2)$ is decreasing from $(m-1)\cos((2q+1)\pi/(2(2q+3))) \ge 0$ to $\cos((2q+3)\pi)/(2(2q+1))) < 0$ for $\theta \in \left[\frac{\pi}{2q+3}, \frac{\pi}{2q+1}\right]$, so the existence and uniqueness of θ_0 follows readily.

Further, we have

$$\left(\left[\begin{array}{c|c} P_q & O \\ \hline O & 0\end{array}\right]+\frac{1}{m}J\right)^{-1}=M\equiv\begin{bmatrix} 1 & -1 & 0 & \cdots & 0 & 0\\ -1 & 2 & -1 & 0 & \cdots & 0\\ \vdots & & \ddots & & & \vdots\\ 0 & 0 & \cdots & -1 & 2 & -1\\ 0 & 0 & \cdots & 0 & -1 & m+1\end{bmatrix},$$

so that $1/\rho\left(\left[\begin{array}{c|c} P_q & O \\ \hline O & 0\end{array}\right]+\frac{1}{m}J\right)$ is the smallest eigenvalue of M. Observe that M is an M-matrix. Further, since

$$\cos((i-1)\theta_0+\theta_0/2)+\cos((i+1)\theta_0+\theta_0/2)=2\cos(i\theta_0+\theta_0/2)\cos(\theta_0)$$

for each $i = 0, \cdots, q$, we find that the vector $v = \begin{bmatrix} \cos(\theta_0/2) \\ \cos(3\theta_0/2) \\ \vdots \\ \cos((2q+1)\theta_0/2) \end{bmatrix}$ is an eigenvector of M corresponding to the eigenvalue $2(1-\cos(\theta_0))$. Since v is an eigenvector with all positive entries, it corresponds to the smallest eigenvalue of M, and the result now follows. □

COROLLARY 2.9. *For each* $q \in \mathbb{N}$, $\alpha_{1,q,n-k} = 2(1-\cos(\frac{\pi}{2q+3}))$.

Proof. Since $1/\rho(P_{q+1})$ corresponds to the case $m = 1$ in Proposition 2.8, the conclusion follows. □

REMARK 2.10. The principal results of [7] assert that for a graph G on n vertices with k cutpoints, we have: i) if $k = 1$, then $\alpha(G) \le 1$, with equality if and only if the single cutpoint v_0 is adjacent to all other vertices of G; ii) if $2 \le k \le n/2$, then $\alpha(G) \le 2(n-k)/(n-k+2+\sqrt{(n-k)^2+4})$, with equality if and only if G is constructed by taking a graph on $n-k$ vertices which has k vertices of degree $n-k-1$, and attaching a pendant vertex at each of those vertices of maximum degree.

In the language of the present paper, case i) corresponds to $q = 0$ and $l = 1$, and yields the upper bound $\alpha(G) \le 1/\rho(P_1)$; equality holds if and only if G is formed from a construction analogous to that of the graphs in $E_1(q, n-k)$. Similarly, for $k < n/2$, case ii) corresponds to $q = 0$ and $l = k$. A straightforward computation with the 2×2 matrix $\left[\begin{array}{c|c} P_1 & 0 \\ \hline 0 & 0 \end{array}\right] + \frac{1}{n-k}J$ shows that

$$2(n-k)/(n-k+2+\sqrt{(n-k)^2+4}) = 1/\rho\left(\left[\begin{array}{c|c} P_1 & 0 \\ \hline 0 & 0 \end{array}\right] + \frac{1}{n-k}J\right),$$

so the upper bound can be written as

$$\alpha(G) \le 1/\rho\left(\left[\begin{array}{c|c} P_1 & 0 \\ \hline 0 & 0 \end{array}\right] + \frac{1}{n-k}J\right).$$

Further, equality holds if and only if G is formed from a construction analogous to that of the graphs in $E_l(q, n-k)$. If $k = n/2$, then case ii) corresponds to $q = 1, l = 0$, and again

$$\alpha(G) \le 1/\rho\left(\left[\begin{array}{c|c} P_1 & 0 \\ \hline 0 & 0 \end{array}\right] + \frac{1}{n-k}J\right),$$

with equality holding if and only if G can be constructed in a manner analogous to that in $E_l(q, n-k)$. Thus we see that both the upper bounds and the extremizing graphs in the present paper are natural extensions of the corresponding ones in [7].

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