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THE P_0-MATRIX COMPLETION PROBLEM[*]

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Abstract. In this paper the P_0-matrix completion problem is considered. It is established that every asymmetric partial P_0-matrix has P_0-completion. All 4×4 patterns that include all diagonal positions are classified as either having P_0-completion or not having P_0-completion. It is shown that any positionally symmetric pattern whose graph is an n-cycle with $n \geq 5$ has P_0-completion.

Key words. Matrix completion, P_0-matrix, P-matrix, digraph, n-cycle, asymmetric.

AMS subject classifications. 15A48

1. Introduction. A *partial matrix* is a rectangular array in which some entries are specified while others are free to be chosen. A *completion* of a partial matrix is a specific choice of values for the unspecified entries. A *pattern* for $n \times n$ matrices is a list of positions of an $n \times n$ matrix, that is, a subset of $\{1, \ldots, n\} \times \{1, \ldots, n\}$. A *positionally symmetric* pattern is a pattern with the property that (i, j) is in the pattern if and only if (j, i) is also in the pattern. A partial matrix *specifies a pattern* if its specified entries lie exactly in those positions listed in the pattern. For a particular class, Π, of matrices, we say a pattern *has* Π*-completion* if every partial Π-matrix specifying the pattern can be completed to a Π-matrix. The Π-matrix completion problem for patterns is to determine which patterns have Π-completion. For example, the positive definite completion problem asks: "Which patterns have the property that any partial positive definite matrix specifying the pattern can be completed to a positive definite matrix?" The answer to this question is given in [3] through the use of graph theoretic methods.

A *principal minor* is the determinant of a principal submatrix. For α a subset of $\{1, 2, \ldots n\}$, the *principal submatrix* obtained from A by deleting all rows and columns not in α is denoted by $A(\alpha)$. An $n \times n$ matrix is called a P_0*-matrix* (P*-matrix*) if all of its principal minors are nonnegative (positive). A *partial* P_0-matrix (*partial* P-matrix) is a partial matrix in which all fully specified principal submatrices are P_0-matrices (P-matrices). The P-matrix completion problem is treated in [1, 8]. The main results in [8] include:

- all positionally symmetric patterns for $n \times n$ matrices have P-completion,
- all patterns for 3×3 matrices have P-completion,

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ELA
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- the partial P_0-matrix

$$\begin{bmatrix} 1 & 2 & 1 & x_{14} \\ -1 & 0 & 0 & -2 \\ -1 & 0 & 0 & -1 \\ x_{41} & 1 & 1 & 1 \end{bmatrix} \tag{1.1}$$

(which specifies the positionally symmetric pattern {(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (2, 4), (3, 1), (3, 2), (3, 3), (3, 4), (4, 2), (4, 3), (4, 4)}) does not have P_0-completion. In this article we discuss the P_0-matrix completion problem.

Throughout the paper we denote the entries of a partial matrix A as follows: d_i denotes a specified diagonal entry, a_{ij} a specified off-diagonal entry, and x_{ij} an unspecified entry, $1 \leq i, j \leq n$.

Graph theory has played an important role in the study of matrix completion problems. A positionally symmetric pattern for $n \times n$ matrices that includes all diagonal positions can be represented by means of a *graph* $G = \{V, E\}$ on n vertices. That is, $V = \{1, 2, \ldots, n\}$, and E is the edge set. For $1 \leq i, j \leq n$, the edge $\{i, j\}$ belongs to E if and only if the ordered pair (i, j) is in the pattern (in this case, the ordered pair (j, i) is also in the pattern). A non-symmetric pattern for $n \times n$ matrices that includes all diagonal positions is best described by means of a *digraph* $G = \{V, E\}$ on n vertices. That is, the directed edge or arc, $(i, j), 1 \leq i, j \leq n$, is in the arc set E if and only if the ordered pair (i, j) is in the pattern. Since we consider non-symmetric as well as positionally symmetric patterns, we use digraphs for all patterns in this paper. The partial matrix (1.1) specifies the pattern whose digraph is shown in Figure 1.1(a). We say that a partial matrix that specifies a pattern also specifies the digraph determined by the pattern. We say that a digraph *has* Π-*completion* if the associated pattern has Π-completion. When working with digraphs (Sections 3 and 4) we assume that patterns contain all diagonal positions, and thus we can use digraphs as discussed here (if some diagonal positions were missing, marked digraphs should be used, cf. [5]).

A *subdigraph* of a digraph G is a digraph $G' = \{V', E'\}$, where $V' \subseteq V$ and $E' \subseteq E$ (note that $(u, v) \in E'$ requires $u, v \in V'$, since G' is a digraph). If $W \subseteq V$, the *subdigraph induced by* W is the digraph $\langle W \rangle = \{W, E'\}$, where $(i, j) \in E'$ if and only if $i, j \in W$ and $(i, j) \in E$. A digraph is *complete* if it includes all possible arcs. A *clique* is a complete subdigraph. A *path* is a sequence of arcs $(v_1, v_2), (v_2, v_3), \ldots, (v_{k-1}, v_k)$ in which the vertices are distinct, except possibly $v_1 = v_k$. A digraph is called *strongly connected* if for all $i, j \in V$, there is a path from i to j. A digraph is *connected* if for all $i, j \in V$, there is a semipath (i.e. a path ignoring orientation) from i to j. A *cut-vertex* of a connected digraph is a vertex whose deletion from G disconnects the digraph. A connected digraph is *nonseparable* if it has no cut-vertices. A *block* is a maximal nonseparable subdigraph. A *block clique* digraph is a digraph whose blocks are all cliques. A *symmetric* n-*cycle* is a digraph on n vertices with arc set $E = \{(i, i+1), (i+1, i), (n, 1), (1, n) | i = 1, 2, \ldots, n-1\}$.

Further study of related matrix completion problems appears in [2]. In their paper the authors consider completion problems for several classes of matrices under special symmetry assumptions on the specified entries. One of the main results establishes

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ELA
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that certain classes, Π, of matrices have Π-completion for any pattern whose digraph is block-clique. The authors also establish that a positionally symmetric pattern for a positive P-matrix has positive P-completion if the graph of the pattern is an n-cycle (the digraph is a symmetric n-cycle).

The recent survey article [5] contains a summary of currently known results on a number of matrix completion problems. The article also contains detailed definitions of graph theoretic concepts and an extended bibliography on matrix completion problems. In our discussion, we make use of [5, Theorem 5.8], which reduces the P_0-matrix completion problem to nonseparable strongly connected digraphs; the theorem establishes that a pattern that includes all diagonal positions has P_0-completion if and only if every nonseparable strongly connected induced subdigraph of the pattern's digraph has P_0-completion. We also make use of [5, Example 9.6]

$$\begin{bmatrix} 0 & -1 & x_{13} \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix} \tag{1.2}$$

whose digraph is shown in Figure 1.1 (b) and [5, Example 9.7]

$$\begin{bmatrix} 0 & 1 & x_{13} & 0 \\ 0 & 0 & 1 & x_{24} \\ x_{31} & 0 & 0 & 1 \\ 1 & x_{42} & 0 & 0 \end{bmatrix} \tag{1.3}$$

whose digraph is shown in Figure 1.1 (c), to establish that digraphs containing these digraphs as induced subdigraphs do not have P_0-completion.

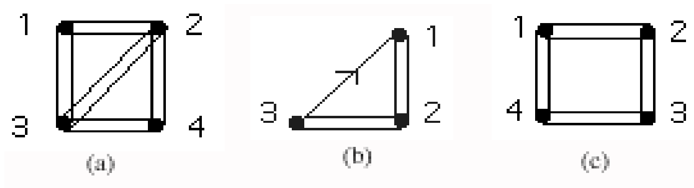

FIG. 1.1. *Digraphs not having P_0-completion.*

In Section 2 of this manuscript we establish that all asymmetric patterns have Π-completion, where Π is either the class of P- or the class of P_0-matrices. In Section 3 we classify all patterns of 4×4 matrices that include all diagonal positions as either having P_0-completion or not having P_0-completion. In Section 4 we show that every symmetric n-cycle has P_0-completion for $n \geq 5$. Finally, Section 5 contains tables that support the results of Sections 3 and 4.

2. Asymmetry. A partial matrix is *asymmetric* if whenever $i \neq j$ and a_{ij} is specified, then a_{ji} is not specified. The diagonal elements of the matrix may or may not be specified.

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ELA
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In this section we show that every asymmetric partial Π-matrix can be completed to a Π-matrix for Π the class of either P- or P_0-matrices.

LEMMA 2.1. *Every real skew-symmetric matrix is a P_0-matrix.*

Proof. Any skew-symmetric matrix, S, has purely imaginary eigenvalues. Since S is real, its complex eigenvalues occur in pairs, and therefore det $S \geq 0$. Also, since any principal submatrix of S is also skew-symmetric, it follows that S is a P_0-matrix. □

THEOREM 2.2. *Every asymmetric partial Π-matrix has Π-completion.*

Proof. Let A be an asymmetric partial Π-matrix. The proof is divided into three cases.

Case 1: A is a partial P_0-matrix with all specified diagonal entries equal to 0.

Complete A to a skew-symmetric matrix. By Lemma 2.1, this completion yields a P_0-matrix.

Case 2: A is a partial P_0-matrix with some positive diagonal entries.

Let the entries of A be as indicated on page 2. Let $\widehat{A}$ be the completion of A obtained by setting all x_{ii}, and all unspecified pairs x_{ij}, x_{ji} to 0. Set all other x_{ij} to $-a_{ji}$, that is

$$\widehat{A} = \begin{bmatrix} \hat{d}_1 & \hat{a}_{12} & \hat{a}_{13} & \cdots & \hat{a}_{1n} \\ -\hat{a}_{12} & \hat{d}_2 & \hat{a}_{23} & \cdots & \hat{a}_{2n} \\ -\hat{a}_{13} & -\hat{a}_{23} & \hat{d}_3 & \cdots & \hat{a}_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\hat{a}_{1n} & -\hat{a}_{2n} & -\hat{a}_{3n} & \cdots & \hat{d}_n \end{bmatrix},$$

where $\hat{d}_i = d_i$ or 0, and $\hat{a}_{ij} = a_{ij}$, $-a_{ji}$ or 0, for $i, j = 1, 2, \ldots, n$. Let $D = \mathrm{diag}(\hat{d}_1, \hat{d}_2, \ldots, \hat{d}_n)$, then $D \geq 0$. We can write $\widehat{A} = A_0 + D$, where A_0 is a skew-symmetric real matrix. By Lemma 2.1, A_0 is a P_0-matrix, and $\widehat{A} = A_0 + D$ is also a P_0-matrix [7].

Case 3: A is a partial P-matrix.

Complete A to $\widehat{A}$, and let $\widehat{A} = A_0 + D$ as in Case 2. If no diagonal entries are specified, let $d = 1$, otherwise let $d = \min\{d_i | d_i \text{ specified}\}$. It follows that $A_1 = A_0 + dI$ is a P-matrix ([7]). Let $D_1 = \mathrm{diag}(f_1, f_2, \ldots, f_n)$, where

$$f_i = \begin{cases} 0 & \text{if the } (i,i)\text{-entry is not specified} \\ d_i - d & \text{if the } (i,i)\text{-entry is specified} \end{cases}.$$

Then $D_1 \geq 0$, and $\widehat{A}_1 = A_1 + D_1$ is a P-matrix ([7]) that completes A. □

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ELA
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3. Classification of Patterns of 4 × 4 Matrices. This section contains a complete classification of the patterns of 4×4 matrices that include all diagonal positions into two categories: those having P_0-completion and those not having P_0-completion. The classification of patterns is carried out by analysis of the corresponding digraphs on four vertices. A list of digraph diagrams on four vertices appears in [4]; all diagrams are numbered by q (the number of edges in the digraph) and n (the diagram number within all digraphs with the same number of edges).

EXAMPLE 3.1. The pattern whose digraph is shown in Figure 3.1 does not have P_0-completion, because the matrix

$$A = \begin{bmatrix} 0 & -1 & x_{13} \\ 0 & 0 & -1 \\ -1 & x_{32} & 0 \end{bmatrix}$$

does not have P_0-completion. A is a partial P_0-matrix because the diagonal entries are nonnegative and the only complete principal submatrix, the $A(\{1,2\})$ submatrix, has determinant 0. However, $\det A = -1$. □

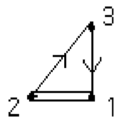
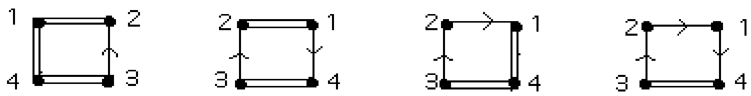


FIG. 3.1. *Digraph not having P_0-completion*

LEMMA 3.2. *The patterns whose digraphs are shown in Figure* 3.2 *have P_0-completion.*



$q = 7, n = 2$ $\quad$ $q = 6, n = 4$ $\quad$ $q = 6, n = 7$ $\quad$ $q = 5, n = 7$

FIG. 3.2. *Digraphs having P_0-completion (identified as per* [4]*).*

Proof. Note that since P_0-matrices are closed under permutation similarity, we are free to label the diagrams as we choose.

Let $A = \begin{bmatrix} d_1 & a_{12} & x_{13} & a_{14} \\ a_{21} & d_2 & x_{23} & x_{24} \\ x_{31} & a_{32} & d_3 & a_{34} \\ a_{41} & x_{42} & a_{43} & d_4 \end{bmatrix}$ be a partial matrix specifying the pattern of the digraph $q = 7, n = 2$. We need to consider three cases: (1) d_1 and d_3 both nonzero, (2) $d_1 = 0$, and (3) $d_3 = 0$. In the first case, since multiplication of a P_0-matrix by a positive diagonal matrix produces a P_0-matrix, without loss of generality assume $d_1 = d_3 = 1$.

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Volume 9, pp. 1-20, February 2002

ELA
http://math.technion.ac.il/iic/ela

Case 1: $d_1 = 1, d_3 = 1$.

Set $x_{23} = 0, x_{13} = 0, x_{42} = w, x_{24} = -w$. If $a_{14}a_{43} = 0$, then set $x_{31} = 0$. If $a_{14}a_{43} \neq 0$, then set $x_{31} = \frac{d_4}{a_{14}a_{43}}$. In both cases, the magnitude of w can be chosen sufficiently large to ensure that this matrix is a P_0-matrix. The principal minors are shown in Table 1-1 in Section 5 and are all nonnegative. If $a_{14}a_{43} = 0$, then either $a_{14} = 0$ or $a_{43} = 0$. If $a_{14} = 0$, then det $A(\{1,3,4\}) = -a_{34}a_{43} + d_4 =$ det $A(\{3,4\}) \geq 0$. If $a_{43} = 0$, then det $A(\{1,3,4\}) = -a_{14}a_{41} + d_4 =$ det $A(\{1,4\}) \geq 0$. If $a_{14}a_{43} \neq 0$, then det $A(\{1,3,4\}) = -a_{14}a_{41} - a_{34}a_{43} + 2d_4 =$ det $A(\{1,4\})+$det $A(\{3,4\}) \geq 0$. All other nonconstant principal minors, including the determinant, are monic polynomials of degree two in w. Therefore they can all be made nonnegative by selecting w of sufficiently large magnitude.

Case 2: $d_1 = 0$.

If $-a_{43}a_{32}a_{21}a_{14} \geq 0$, then A can be completed to a P_0-matrix by setting all unspecified entries to 0. The principal minors are shown in Table 1-2 in Section 5, and it is easy to see that they are all nonnegative (det $A = -a_{43}a_{32}a_{21}a_{14} +$ det $A(\{1,2\}) \cdot$ det $A(\{3,4\}) + d_2 d_3 \cdot$ det $A(\{1,4\}) \geq 0$).

If $-a_{43}a_{32}a_{21}a_{14} < 0$, then a_{43}, a_{32}, a_{21}, and a_{14} are all nonzero. By use of a diagonal similarity (which preserves P_0-matrices), we may assume that $a_{32} = 1$ (thus $a_{43}a_{21}a_{14} > 0$). Then A can be completed to a P_0-matrix as follows: Set $x_{23} = -1, x_{13} = 0$, and $x_{24} = 0$. Set $x_{42} = \frac{a_{43}}{d_3}$ if $d_3 \neq 0$ or $x_{42} = a_{43}$ if $d_3 = 0$ (note sign $x_{42} =$ sign a_{43}). Choose $x_{31} = w$, where w has the same sign as a_{21} and of sufficiently large magnitude. The principal minors are shown in Table 1-2 and are all nonnegative: We have det $A(\{1,2,3\}) = d_3 \cdot$ det $A(\{1,2\}) - a_{12}w \geq 0$ since sign $w =$ sign a_{21}. det $A(\{1,2,4\}) = d_2 \cdot$det $A(\{1,4\}) + d_4 \cdot$det $A(\{1,2\}) + a_{14}a_{21}x_{42} \geq 0$, since $a_{14}a_{21}a_{43} > 0$ and sign $x_{42} =$ sign a_{43}. det $A(\{1,3,4\}) = d_3 \cdot$det $A(\{1,4\}) + a_{14}a_{43}w$. det $A(\{2,3,4\}) = d_2 \cdot$ det $A(\{3,4\}) + d_4 - a_{34}x_{42}$. If $d_3 \neq 0$, $d_4 - a_{24}x_{42} = \left(\frac{1}{d_3}\right) \cdot$ det $A(\{3,4\}) \geq 0$. If $d_3 = 0$, then $d_4 - a_{34}x_{42} = d_4 +$ det $A(\{3,4\}) \geq 0$. det $A = b + a_{14}a_{43}d_2 w - a_{12}d_4 w + a_{14}x_{42}w$, where b is constant. The terms $a_{14}a_{43}d_2 w$ and $-a_{12}d_4 w$ are nonnegative because w has the same sign as a_{21}, and $a_{43}a_{21}a_{14}$, $-a_{12}a_{21}$, d_2,and $d_4 \geq 0$. Finally, the term $a_{14}x_{42}w$ is positive since $a_{14}a_{43}a_{21} > 0$, and x_{42} and w have the appropriate signs. Therefore, we can take w of sufficiently large magnitude to ensure that det $A \geq 0$.

Case 3: $d_3 = 0$.

This case is similar to Case 2, and can be derived from the information in Table 1-3.

Any partial P_0-matrix specifying the pattern of the digraph $q = 6, n = 4$ may be extended to a partial P_0-matrix specifying the digraph $q = 7, n = 2$ by setting the unspecified $(4,1)$-entry equal to 0. The same reasoning applies to the digraphs $q = 6, n = 7$ and $q = 5, n = 7$. Thus these patterns also have P_0-completion. □

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THEOREM 3.3. *(Classification of Patterns of 4×4 Matrices.) Let Q be a pattern for 4×4 matrices that includes all diagonal positions. The pattern Q has P_0-completion if and only if its digraph is one of the following (numbered as in [4], q is the number of edges, n is the diagram number).*

q = 0;
q = 1;
q = 2, n = 1–5;
q = 3, n = 1–13;
q = 4, n = 1–12, 14–27;
q = 5, n = 1–5, 7–10, 14–17, 21–38;
q = 6, n = 1–8, 13, 15, 17, 19, 23, 26, 27, 32, 35, 38–40, 43, 45–48;
q = 7, n = 2, 4, 5, 9, 14, 24, 29, 34, 36;
q = 8, n = 1, 10, 12, 18;
q = 9, n = 8, 11;
q = 12.

Proof. Part 1. Completion

The patterns of the following digraphs have P_0-completion because any asymmetric digraph has P_0-completion (by Theorem 2.2): $q = 1$; $q = 2$, $n = 2$–5; $q = 3$, $n = 4$–13; $q = 4$, $n = 16$–27; $q = 5$, $n = 29$–38; $q = 6$, $n = 45$–48.

The patterns of the following digraphs have P_0-completion because every strongly connected nonseparable induced subdigraph has P_0-completion [5, Theorem 5.8]. (This includes the cases when each component is complete or is block-clique. This list does not include those digraphs that fall under this rule but were already listed in the previous list, although the technique of completing an asymmetric part first may be used, as in $q = 5$, $n = 25$): $q = 0$; $q = 2$, $n = 1$; $q = 3$, $n = 1$, 2, 3; $q = 4$, $n =$ 1–12, 14, 15; $q = 5$, $n = 1$–5, 8–10, 14–17, 21–28; $q = 6$, $n = 1$–3, 5, 6, 8, 13, 15, 17, 19, 23, 26, 27, 32, 35, 38–40, 43; $q = 7$, $n = 4$, 5, 9, 14, 24, 29, 34, 36; $q = 8$, $n = 1$, 10, 12, 18; $q = 9$, $n = 8$, 11; $q = 12$.

The patterns of the following digraphs have P_0-completion, by Lemma 3.2: $q =$ 5, $n = 7$; $q = 6$, $n = 4$, 7; $q = 7$, $n = 2$.

Part 2. No Completion.

The patterns of the following digraphs do not have P_0-completion because each contains [5, Example 9.6], (equation (1.2) within) as an induced subdigraph: $q = 5$, $n = 6$; $q = 6$, $n = 9$, 10, 12, 18, 20, 21; $q = 7$, $n = 1$, 3, 6, 11, 12, 15, 16, 18, 19, 22, 23, 25–28; $q = 8$, $n = 3$–9, 13–15, 20–27; $q = 9$, $n = 1$–7, 12, 13; $q = 10$, $n = 2$–5; q = 11.

The patterns of the following digraphs do not have P_0-completion because each contains Example 3.1 as an induced subdigraph (this list does not include those digraphs that fall under this rule but were already listed in the previous list): $q = 4$, $n = 13$; $q = 5$, $n = 11$, 12, 13, 18, 19, 20; $q = 6$, $n = 11$, 14, 16, 24, 25, 28–31, 33, 34, 36, 41, 42, 44; $q = 7$, $n = 7$, 8, 10, 13, 17, 20, 21, 30–33, 37, 38; $q = 8$, $n = 11$, 16,

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ELA
http://math.technion.ac.il/iic/ela

17, 19; $q = 9$, $n = 9$, 10.

The pattern of the digraph $q = 8$, $n = 2$ does not have P_0-completion because it is [5, Example 9.7], (equation (1.3) within).

The pattern of the digraph $q = 10$, $n = 1$ does not have P_0-completion because it corresponds to the example in [8], (equation (1.1)).

By examination of the partial P_0-matrices below, it can be seen that the patterns of the digraphs $q = 6$, $n = 22$; $q = 6$, $n = 37$ and $q = 7$, $n = 35$ do not have P_0-completion (the digraphs are numbered as shown in Figure 3.3). For $q = 6, n = 22$, det $A(\{1, 3\}) = -x_{13}$, and det $A(\{1, 3, 4\}) = x_{13}$, so $x_{13} = 0$. But then det $A(\{1, 2, 3\}) = -1 + x_{13}x_{21}x_{32} = -1$. The pattern of $q = 6, n = 37$ is the transpose of the pattern of $q = 6, n = 22$, and thus the transpose of the previous partial matrix shows this pattern lacks completion also. For $q = 7, n = 35$, det $A(\{1, 2\}) = x_{21}$, det $A(\{1, 2, 4\}) = -x_{21}$ and det $A(\{1, 2, 3\}) = -1 + x_{21}x_{13}x_{32}$.

$$q = 6, n = 22 : \begin{bmatrix} 0 & 1 & x_{13} & x_{14} \\ x_{21} & 0 & -1 & x_{24} \\ 1 & x_{32} & 0 & 1 \\ 1 & x_{42} & 0 & 0 \end{bmatrix},$$

$$q = 6, n = 37 : \begin{bmatrix} 0 & x_{12} & 1 & 1 \\ 1 & 0 & x_{23} & x_{24} \\ x_{31} & -1 & 0 & 0 \\ x_{41} & x_{42} & 1 & 0 \end{bmatrix},$$

$$q = 7, n = 35 : \begin{bmatrix} 0 & -1 & x_{13} & 1 \\ x_{21} & 0 & 1 & 0 \\ 1 & x_{32} & 0 & x_{34} \\ x_{41} & -1 & -1 & 0 \end{bmatrix}. \quad \square$$

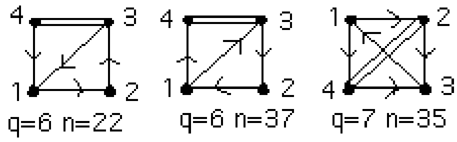

FIG. 3.3. *Digraphs not having P_0-completion (identified as per* [4]*).*

The techniques and examples used in this section also show that all digraphs of order 2 have P_0-completion and all digraphs of order 3, except Example 3.1 and [5, Example 9.6], have P_0-completion.

4. Symmetric n-cycle. Recall that a pattern has P-completion if and only if the principal subpattern determined by the diagonal positions included in the pattern has P-completion [8], but the situation is different for P_0-completion: If a positionally

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ELA
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symmetric pattern has P_0-completion, then each principal subpattern associated with a component of the digraph either includes all diagonal positions or omits all diagonal positions [6]. Any pattern that omits all diagonal positions has completion for any of the classes discussed in this paper. Thus, to determine which positionally symmetric patterns have completion, we need to discuss only patterns that include all diagonal positions.

Cycles often play an important role in the study of the matrix completion problem for a particular class of matrices. We call a pattern that includes the diagonal a *cycle* (or *n-cycle*) pattern if its digraph is a symmetric n-cycle. For example, an n-cycle pattern does not have positive definite completion for $n \geq 4$ [3]. Since every positionally symmetric pattern has P-completion [8], every cycle pattern has P-completion. In [2], induction on the length of the cycle was used to show that every cycle pattern has positive P-completion. In [5, Theorem 8.4] the same method was used to show that every cycle pattern has Π-completion, where Π is any of the classes: $P_{0,1}$-matrices, nonnegative $P_{0,1}$-matrices, nonnegative $P_{0,1}$-matrices. In [2] it was also shown that an n-cycle pattern does not have sign-symmetric $P_{0,1}$- or sign-symmetric P_0-completion for $n \geq 4$. An example was given of a partial sign-symmetric P-matrix specifying a 4-cycle pattern that cannot be completed to a sign-symmetric P-matrix, but the example does not naturally extend to longer cycles. The issue of whether an n-cycle pattern has sign-symmetric P-completion is unresolved for $n \geq 5$. However, it was observed that if for some k, a k-cycle pattern has completion then so does every n-cycle pattern for $n \geq k$, by an induction argument. We find that this situation actually arises for P_0-matrices: It was shown in [5, Example 9.7] that a 4-cycle pattern does not have P_0-completion, but we show in this section that a 5-cycle pattern does have P_0-completion, and by induction, an n-cycle pattern has P_0-completion for all $n \geq 5$.

LEMMA 4.1. *Let A be a partial P_0-matrix that includes all diagonal entries with at least four of these nonzero and let A specify a pattern whose digraph is a symmetric 5-cycle. Then A can be completed to a P_0-matrix.*

Proof. Let A be such a partial P_0-matrix. Without loss of generality, by use of a permutation similarity, and then by multiplication by a positive diagonal matrix, we can assume that the cycle is $1, 2, 3, 4, 5, 1$, and $d_1 = d_2 = d_3 = d_4 = 1$. Furthermore, either (1) det $A(\{3, 4\}) > 0$ or (2) det $A(\{3, 4\}) = 0$. In case (2), by use of a diagonal similarity, without loss of generality we can assume $a_{34} = 1$, which implies $a_{43} = 1$.

The completion is done by the same method used for completions of positionally symmetric partial P-matrices: Choose the unspecified entries in pairs $x_{ij} = t$, $x_{ji} = -t$, in order from the top left to lower right, ensuring that every newly completed principal submatrix has nonnegative determinant. The values of these principal minors are listed in Table 2.

- Choose $x_{13} = u, x_{31} = -u$, u sufficiently large to ensure det $A(\{1, 3\})$, det $A(\{1, 2, 3\}) > 0$. This can be done because each determinant is the sum of u^2 and a linear function of u. The value of u is now fixed.
- Choose $x_{14} = v, x_{41} = -v$, with the sign of v such that $(-a_{45}a_{51} + a_{54}a_{15})v \geq 0$ and $|v|$ sufficiently large to ensure that det $A(\{1, 4\})$, det $A(\{1, 3, 4\}) > 0$, and det $A(\{1, 4, 5\}) \geq 0$. This can be done because each of the first two determinants is the sum of v^2 and a linear function of v and det $A(\{1, 4, 5\}) =$

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ELA
http://math.technion.ac.il/iic/ela

$-a_{45}a_{54} + d_5 - a_{15}a_{51} + (-a_{45}a_{51} + a_{54}a_{15})v + d_5v^2$. If $d_5 > 0$, this expression can be made greater than zero by choice of v; if $d_5 = 0$, then $-a_{45}a_{54} - a_{15}a_{51} = \det A(\{4,5\}) + \det A(\{1,5\}) \geq 0$, so $\det A(\{1,4,5\}) \geq 0$. The value of v is now fixed.

- Choose $x_{24} = w, x_{42} = -w$, with $|w|$ sufficiently large to ensure $\det A(\{2,4\})$, $\det A(\{1,2,4\})$, $\det A(\{2,3,4\})$, $\det A(\{1,2,3,4\}) > 0$. This can be done because each of the first three determinants is the sum of w^2 and a linear function of w and the last determinant is the sum of $w^2(1+u^2)$ and a linear function of w. The value of w is now fixed.
- Choose $x_{35} = y, x_{53} = -y$, with $y > 0$ and sufficiently large to ensure $\det A(\{3,5\})$, $\det A(\{1,3,5\})$, $\det A(\{3,4,5\})$, $\det A(\{1,3,4,5\}) > 0$. This can be done because each of the first three determinants is the sum of y^2 and a linear function of y and the last determinant is the sum of $y^2(1+v^2)$ and a linear function of y. Furthermore, y can be chosen large enough to ensure $-a_{45}a_{54} + a_{23}a_{45}a_{32}a_{54} - a_{23}a_{32}d_5 - a_{23}d_5w + a_{32}d_5w + d_5w^2 - a_{45}y + a_{54}y + a_{45}a_{32}wy + a_{23}a_{54}wy + y^2 + w^2y^2 > 0$. The value of y is now fixed.
- Choose $x_{25} = z, x_{52} = -z$, with the sign of z such that $(-a_{23}a_{45} + a_{32}a_{54} + a_{45}w + a_{54}w + a_{23}y + a_{32}y)z \geq 0$ and z of sufficiently large modulus to ensure that $\det A(\{2,5\})$, $\det A(\{1,2,5\})$, $\det A(\{2,4,5\})$, $\det A(\{2,3,5\})$, $\det A(\{1,2,3,5\})$, $\det A(\{1,2,4,5\})$, $\det A(\{2,3,4,5\})$, $\det A > 0$. This can be done because each of the first four determinants is the sum of z^2 and a linear function of z, $\det A(\{1,2,3,5\})$ is the sum of $z^2(1+u^2)$ term and a linear function of z, $\det A(\{1,2,4,5\})$ is the sum of $z^2(1+v^2)$ and a linear function of z, and $\det A$ is the sum of $z^2 \cdot \det A(\{1,3,4\})$ and a linear function of z. For $\det A(\{2,3,4,5\})$ we need to consider the two cases, (1) $\det A(\{3,4\}) > 0$ or (2) $\det A(\{3,4\}) = 0$, separately: $\det A(\{2,3,4,5\})$ is the sum of $z^2 \cdot \det A(\{3,4\})$ and a linear function of z, so in case (1), z can be chosen sufficiently large to ensure the determinant is positive. In case (2), $\det A(\{2,3,4,5\}) = -a_{45}a_{54} + a_{23}a_{45}a_{32}a_{54} - a_{23}a_{32}d_5 - a_{23}d_5w + a_{32}d_5w + d_5w^2 - a_{45}y + a_{54}y + a_{45}a_{32}wy + a_{23}a_{54}wy + y^2 + w^2y^2 + (-a_{23}a_{45} + a_{32}a_{54} + a_{45}w + a_{54}w + a_{23}y + a_{32}y)z > 0$. Thus A has been completed to a P_0-matrix. □

THEOREM 4.2. *A pattern that includes all diagonal positions and whose digraph is a symmetric* 5*-cycle has* P_0*-completion.*

Proof. Let A be a partial P_0-matrix specifying the symmetric 5-cycle $1,2,3,4,5,1$. By multiplication by a positive diagonal matrix, without loss of generality each diagonal element of A is 0 or 1. Two diagonal entries are called adjacent if the corresponding vertices in the digraph are adjacent. The proof is by cases based on the composition of the diagonal. For each case, the matrix is completed by assigning values to unspecified entries. All minors are evaluated (see Table 3) to verify this results in a P_0 matrix.

Case 1: No adjacent diagonal entries are 1.

By renumbering if necessary, we can assume that $d_1 = d_2 = d_4 = 0$. If $a_{12}a_{23}a_{34}a_{45}a_{51} + a_{21}a_{32}a_{43}a_{54}a_{15} \geq 0$, set all unspecified entries to 0. All principal

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ELA
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minors are clearly non-negative (see Table 3-1). If the product $a_{12}a_{23}a_{34}a_{45}a_{51} + a_{21}a_{32}a_{43}a_{54}a_{15} < 0$, then either $a_{12}a_{23}a_{34}a_{45}a_{51} < 0$ or $a_{21}a_{32}a_{43}a_{54}a_{15} < 0$. Without loss of generality $a_{21}a_{32}a_{43}a_{54}a_{15} < 0$, and by use of a diagonal similarity, without loss of generality $a_{21} = a_{32} = a_{43} = a_{54} = -1$ (which implies $a_{15} < 0$, and $a_{12}, a_{23}, a_{34}, a_{45}, a_{51} \geq 0$. See Table 3-1). Set the $(i, i+2)$-entries all equal to a sufficiently large positive number z and set the $(i+2, i)$-entries equal to 0 (arithmetic of indices modulo 5). All principal minors of size 3×3 or less are are non-negative. Each 4×4 principal minor is a polynomial in z of degree three with positive leading coefficient. det A is a monic polynomial in z of degree five.

Case 2: Two adjacent 1's and two 0's on the diagonal.

By renumbering if necessary, $d_1 = d_2 = 1$. Then $d_4 = d_5 = 0$, or $d_3 = d_5 = 0$ and $d_4 = 1$. Set the $(3, 1)$-entry equal to z and $(1, 3)$-entry equal to $-z$, where z is chosen of sufficiently large magnitude. Choose the sign of z with sign $z =$ sign $(a_{34}a_{45}a_{51} - a_{43}a_{54}a_{15})$. All principal minors are clearly nonnegative or can be made nonnegative by choosing z of sufficiently large magnitude, except det $A(\{1, 3, 4, 5\})$ and det A. These determinants are polynomials in z of degree two with leading coefficient $-a_{45}a_{54}$. If $a_{45}a_{54} = 0$, then each determinant is the sum of a constant and $(a_{34}a_{45}a_{51} - a_{43}a_{54}a_{15})z$. If $(a_{34}a_{45}a_{51} - a_{43}a_{54}a_{15})z$ is 0, the constant is nonnegative.

Case 3: Four diagonal entries are 1.

This is the preceding Lemma. $\square$

THEOREM 4.3. *A pattern that includes all diagonal positions and whose digraph is a symmetric n-cycle has P_0-completion for $n \geq 5$.*

Proof. The proof is by induction on n. Theorem 4.2 establishes the result for $n = 5$. Assume true for $n - 1$. Let A be an $n \times n$ partial P_0-matrix specifying the pattern whose digraph is the symmetric n-cycle $1, 2, \ldots, n, 1$. The general strategy is to complete A to a matrix $\widehat{A}$ in three steps, and then prove $\widehat{A}$ is a P_0-matrix.

Step 1: Choose $x_{2n} = c_{2n}$ and $x_{n2} = c_{n2}$ in an appropriate way ("appropriate" depends on A) so that $\widehat{A}(\{2, n\})$ is a P_0-matrix. Then, the principal submatrix

$$C = \begin{bmatrix} d_2 & a_{23} & x_{24} & \cdots & x_{2,n-1} & c_{2n} \\ a_{32} & d_3 & a_{34} & \cdots & x_{3,n-1} & x_{3n} \\ x_{42} & a_{43} & d_4 & \cdots & x_{4,n-1} & x_{4n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ x_{n-1,2} & x_{n-1,3} & x_{n-1,4} & \cdots & d_{n-1} & a_{n-1,n} \\ c_{n2} & x_{n3} & x_{n4} & \cdots & a_{n,n-1} & d_n \end{bmatrix},$$

obtained by deleting row 1 and column 1, is a partial P_0-matrix that specifies a pattern whose digraph is a symmetric $(n-1)$-cycle.

Step 2: By the induction hypothesis C can be completed to a P_0-matrix, $\widehat{C}$.

The Electronic Journal of Linear Algebra ISSN 1081-3810
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ELA
http://math.technion.ac.il/iic/ela

Step 3: Finish $\widehat{A}$ by specifying the remaining unspecified entries of A in an appropriate way.

Step 4: Show $\widehat{A}$ is a P_0-matrix, i.e., show that det $\widehat{A}(\alpha) \geq 0$ for any $\alpha \subseteq \{1, \ldots, n\}$.

Note that all subscript numbering is mod n, so $n+1$ means 1 and 0 means n. Without loss of generality $d_i = 0$ or 1 for all i. The proof is now divided into cases.

Case 1: For some k, $d_k = d_{k+1} = 1$ and $a_{k,k+1} \neq 0$ or $a_{k+1,k} \neq 0$. Renumber so that $d_1 = d_2 = 1$ and $a_{12} \neq 0$. By use of a diagonal similarity we may assume $a_{12} = 1$.

Case 2: For some k, $d_k = 0$, and $a_{k-1,k} = 0$ or $a_{k,k-1} = 0$. Renumber so $d_2 = 0$ and $a_{12} = 0$ (if $a_{21} = 0$ and $a_{12} \neq 0$, transpose the argument below). If $d_1 = 1$ and $d_n = 1$, we may assume $a_{1n} = 0$ (because if $d_1 = 1$, $d_n = 1$ and $a_{1n} \neq 0$, renumber to obtain Case 1). By use of a diagonal similarity, without loss of generality we may assume $a_{21} \leq 0$.

Case 3: For some k, $d_k = 0$, and $a_{k-1,k} \neq 0$ and $a_{k,k-1} \neq 0$. Renumber so that $d_2 = 0$ and $a_{12} \neq 0$ and $a_{21} \neq 0$. If $d_1 = 1$ and $d_n = 1$, we may assume $a_{1n} = 0$ (because if $d_1 = 1 = d_n$ and $a_{1n} \neq 0$, renumber to obtain Case 1). Without loss of generality assume $a_{12} = 1$ (note that this implies $a_{21} < 0$).

These cases cover all possibilities except a trivial one: If any two adjacent diagonal entries are 1 and either off-diagonal entry in the corresponding 2×2 principal submatrix is nonzero, Case 1 applies. If Case 1 does not apply, then either there is a zero in the diagonal, so Case 2 or Case 3 applies, or all diagonal entries are 1 and all specified off-diagonal entries are 0, in which case, setting all unspecified entries to 0 completes A to the identity matrix.

Step 1: Choose $c_{2n} = a_{1n}$ and

$$c_{n2} = \begin{cases} a_{n1} & \text{Case 1 or 2} \\ -\frac{a_{n1}}{a_{21}} & \text{Case 3} \end{cases}$$

The only fully specified principal submatrices of C are 2×2, and all of these are principal submatrices of A except $C(\{2, n\})$. We show $C(\{2, n\})$ is a P_0-matrix: For Case 1, $C(\{2, n\}) = A(\{1, n\})$. For Cases 2 and 3, $d_2 = 0$, so det $C(\{2, n\}) = -c_{2n}c_{n2} = -a_{1n}c_{n2}$, where $c_{n2} = a_{n1}$ or $-\frac{a_{n1}}{a_{21}}$. If $d_1 = d_n = 1$ and Case 1 does not apply, then $a_{1n} = 0$, so det $C(\{2, n\}) = 0$. If $d_1 = 0$ or $d_n = 0$, then $0 \leq$ det $A(\{1, n\}) = -a_{1n}a_{n1}$. Since c_{n2} has the same sign as a_{n1}, det $C(\{2, n\}) \geq 0$. So in all cases det $C(\{2, n\}) \geq 0$ and C is a partial P_0-matrix.

Step 2: Use the induction hypothesis to complete C to $\widehat{C}$.

Step 3: For $2 < i, j < n$, choose $x_{1j} = c_{2j}$ and

The Electronic Journal of Linear Algebra ISSN 1081-3810
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Volume 9, pp. 1-20, February 2002

$$x_{i1} = \begin{cases} c_{i2} & \text{Case 1 or 2} \\ -c_{i2}a_{21} & \text{Case 3} \end{cases}$$

to obtain the completion $\widehat{A}$ of A.

Step 4: Show $\widehat{A}$ is a P_0-matrix. We must show that det $\widehat{A}(\alpha) \geq 0$ for any $\alpha \subseteq \{1, \ldots, n\}$. For all cases, if $1 \notin \alpha$, $\widehat{A}(\alpha)$ is a principal submatrix of the P_0-matrix $\widehat{C}$, so det $\widehat{A}(\alpha) \geq 0$. Thus, assume $1 \in \alpha$.

Case 1: $d_1 = d_2 = 1$ and $a_{12} = 1$. The proof of this case is the same as [2, Lemma 3.5].

Case 2: $d_2 = a_{12} = 0$, and $a_{21} \leq 0$.

$$\widehat{A} = \left[\begin{array}{c|cccccc} d_1 & 0 & a_{23} & c_{24} & \cdots & c_{2,n-1} & a_{1n} \\ \hline a_{21} & 0 & a_{23} & c_{24} & \cdots & c_{2,n-1} & a_{1n} \\ a_{32} & a_{32} & d_3 & a_{34} & \cdots & c_{3,n-1} & c_{3n} \\ c_{42} & c_{42} & a_{43} & d_4 & \cdots & c_{4,n-1} & c_{4n} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ c_{n-1,2} & c_{n-1,2} & c_{n-1,3} & c_{n-1,4} & \cdots & d_{n-1} & a_{n-1,n} \\ a_{n1} & a_{n1} & c_{n3} & c_{n4} & \cdots & a_{n,n-1} & d_n \end{array}\right].$$

For $2 \notin \alpha$, $\widehat{A}(\alpha) = \widehat{C}((\alpha - \{1\}) \cup \{2\}) + \text{diag}\,(d_1, 0, \ldots, 0)$, so det $\widehat{A}(\alpha) \geq 0$. For $2 \in \alpha$, subtract row 2 from row 1 (which does not change the determinant) so the first row is $(d_1 - a_{21}, 0, \ldots, 0)$. It follows that det $\widehat{A}(\alpha) = (d_1 - a_{21}) \cdot$ det $\widehat{C}(\alpha - \{1\}) \geq 0$.

Case 3: $d_2 = 0, a_{12} = 1$ and $a_{21} < 0$.

$$\widehat{A} = \left[\begin{array}{c|cccccc} d_1 & 1 & a_{23} & c_{24} & \cdots & c_{2,n-1} & a_{1n} \\ \hline a_{21} & 0 & a_{23} & c_{24} & \cdots & c_{2,n-1} & a_{1n} \\ -a_{32}a_{21} & a_{32} & d_3 & a_{34} & \cdots & c_{3,n-1} & c_{3n} \\ -c_{42}a_{21} & c_{42} & a_{43} & d_4 & \cdots & c_{4,n-1} & c_{4n} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -c_{n-1,2}a_{21} & c_{n-1,2} & c_{n-1,3} & c_{n-1,4} & \cdots & d_{n-1} & a_{n-1,n} \\ a_{n1} & -\frac{a_{n1}}{a_{21}} & c_{n3} & c_{n4} & \cdots & a_{n,n-1} & d_n \end{array}\right].$$

For $2 \notin \alpha$, $\widehat{A}(\alpha)$ can be obtained from $\widehat{C}((\alpha - \{1\}) \cup \{2\})$ by multiplying the first column by $-a_{21} > 0$, and adding diag $(d_1, 0, \ldots, 0)$, so det $\widehat{A}(\alpha) \geq (-a_{21})\cdot$ det $\widehat{C}((\alpha - \{1\}) \cup \{2\}) \geq 0$. For $2 \in \alpha$, subtract row 2 from row 1 and then add a_{21} times column 2 to column 1 (which does not change the determinant) to obtain

The Electronic Journal of Linear Algebra ISSN 1081-3810
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Volume 9, pp. 1-20, February 2002

$$\left[\begin{array}{c|cccccc} d_1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ \hline a_{21} & 0 & a_{23} & c_{24} & \cdots & c_{2,n-1} & a_{1n} \\ 0 & a_{32} & d_3 & a_{34} & \cdots & c_{3,n-1} & c_{3n} \\ 0 & c_{42} & a_{43} & d_4 & \cdots & c_{4,n-1} & c_{4n} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & c_{n-1,2} & c_{n-1,3} & c_{n-1,4} & \cdots & d_{n-1} & a_{n-1,n} \\ 0 & -\frac{a_{n1}}{a_{21}} & c_{n3} & c_{n4} & \cdots & a_{n,n-1} & d_n \end{array}\right].$$

$\det \widehat{A}(\alpha) = d_1 \cdot \det \widehat{C}(\alpha - \{1\}) + (-a_{21}) \cdot \det \widehat{C}(\alpha - \{1,2\}) \geq 0.$ □

5. Tables. The following tables are support for the results in Sections 3 and 4. Table 1 is referenced in Lemma 3.2, Table 2 is referenced in Lemma 4.1, and Table 3 is referenced in Theorem 4.2. For those α marked with *, $A(\alpha)$ is a fully specified principal submatrix, and so $\det A(\alpha)$ is assumed to be nonnegative. Note that $d_i \geq 0$ for all i.

Table 1

Table 1-1	Case 1: $d_1 = d_3 = 1$ $a_{14}a_{43} = 0$	$a_{14}a_{43} \neq 0$
α	$x_{23} = x_{13} = x_{31} = 0$ $x_{42} = w,\ x_{24} = -w$	$x_{23} = x_{13} = 0,\ x_{31} = \frac{d_4}{a_{14}a_{43}}$ $x_{42} = w,\ x_{24} = -w$
$\{1,2\}^*$	$-a_{12}a_{21} + d_2$	$-a_{12}a_{21} + d_2$
$\{1,4\}^*$	$-a_{14}a_{41} + d_4$	$-a_{14}a_{41} + d_4$
$\{3,4\}^*$	$-a_{34}a_{43} + d_4$	$-a_{34}a_{43} + d_4$
$\{1,3\}$	1	1
$\{2,3\}$	d_2	d_2
$\{2,4\}$	$d_2d_4 + w^2$	$d_2d_4 + w^2$
$\{1,2,3\}$	$-a_{12}a_{21} + d_2$	$-a_{12}a_{21} + d_2$
$\{1,2,4\}$	$-a_{14}a_{41}d_2 - a_{12}a_{21}d_4 + d_2d_4 + a_{14}a_{21}w - a_{12}a_{41}w + w^2$	$-a_{14}a_{41}d_2 - a_{12}a_{21}d_4 + d_2d_4 + a_{14}a_{21}w - a_{12}a_{41}w + w^2$
$\{1,3,4\}$	$-a_{14}a_{41} - a_{34}a_{43} + d_4$	$-a_{14}a_{41} - a_{34}a_{43} + 2d_4$
$\{2,3,4\}$	$-a_{34}a_{43}d_2 + d_2d_4 - a_{32}a_{43}w + w^2$	$-a_{34}a_{43}d_2 + d_2d_4 - a_{32}a_{34}w + w^2$
$\{1,2,3,4\}$	$-a_{14}a_{21}a_{32}a_{43} + a_{12}a_{21}a_{34}a_{43} - a_{14}a_{41}d_2 - a_{34}a_{43}d_2 - a_{12}a_{21}d_4 + d_2d_4 + a_{14}a_{21}w - a_{12}a_{41}w - a_{32}a_{43}w + w^2$	$-a_{14}a_{21}a_{32}a_{43} + a_{12}a_{21}a_{34}a_{43} - a_{14}a_{41}d_2 - a_{34}a_{43}d_2 - a_{12}a_{21}d_4 + 2d_2d_4 + a_{14}a_{21}w - a_{12}a_{41}w - a_{32}a_{43}w + \frac{a_{12}d_4}{a_{14}}w + w^2$

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Volume 9, pp. 1-20, February 2002

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	Case 2: $d_1 = 0$	
Table 1-2	$-a_{43}a_{32}a_{21}a_{14} \geq 0$	$-a_{43}a_{32}a_{21}a_{14} < 0, a_{32} = 1$
α	$x_{ij} = 0$	$x_{23} = -1$, $x_{13} = x_{24} = 0$, $x_{42} = \frac{a_{43}}{d_3}$ if $d_3 \neq 0$ $x_{42} = a_{43}$ if $d_3 = 0$ $x_{31} = w$, sign$w$ = sign a_{21}
$\{1,2\}^*$	$-a_{12}a_{21}$	$-a_{12}a_{21}$
$\{1,4\}^*$	$-a_{14}a_{41}$	$-a_{14}a_{41}$
$\{3,4\}^*$	$-a_{34}a_{43} + d_3d_4$	$-a_{34}a_{43} + d_3d_4$
$\{1,3\}$	0	0
$\{2,3\}$	d_2d_3	$1 + d_2d_3$
$\{2,4\}$	d_2d_4	d_2d_4
$\{1,2,3\}$	$-a_{12}a_{21}d_3$	$-a_{12}a_{21}d_3 - a_{12}w$
$\{1,2,4\}$	$-a_{14}a_{41}d_2 - a_{12}a_{21}d_4$	$-a_{14}a_{41}d_2 - a_{12}a_{21}d_4 + a_{14}a_{21}x_{42}$
$\{1,3,4\}$	$-a_{14}a_{41}d_3$	$-a_{14}a_{41}d_3 + a_{14}a_{43}w$
$\{2,3,4\}$	$-a_{34}a_{43}d_2 + d_2d_3d_4$	$-a_{34}a_{43}d_2 + d_4 + d_2d_3d_4 - a_{34}x_{42}$
$\{1,2,3,4\}$	$-a_{43}a_{32}a_{21}a_{14} + a_{12}a_{21}a_{34}a_{43} - a_{14}a_{41}d_2d_3 - a_{12}a_{21}d_3d_4$	$-a_{14}a_{41} + a_{12}a_{34}a_{41} - a_{14}a_{21}a_{43} + a_{12}a_{21}a_{34}a_{43} - a_{14}a_{41}d_2d_3 - a_{12}a_{21}d_3d_4 + a_{14}a_{21}d_3x_{42} + a_{14}a_{43}d_2w - a_{12}d_4w + a_{14}x_{42}w$

	Case 3: $d_3 = 0$	
Table 1-3	$-a_{43}a_{32}a_{21}a_{14} \geq 0$	$-a_{43}a_{32}a_{21}a_{14} < 0, a_{32} = 1$
α	$x_{42} = \frac{d_1d_2d_4}{a_{14}a_{21}}$ if $a_{14}a_{21} \neq 0$ $x_{42} = 0$ if $a_{14}a_{21} = 0$ $x_{ij} = 0$ for others	$x_{23} = -1$, $x_{13} = x_{24} = 0$ $x_{31} = \frac{a_{21}}{d_2}$ if $d_2 \neq 0$ $x_{31} = a_{21}$ if $d_2 = 0$ $x_{42} = w$, sign w = sign a_{43}
$\{1,2\}^*$	$-a_{12}a_{21} + d_1d_2$	$-a_{12}a_{21} + d_1d_2$
$\{1,4\}^*$	$-a_{14}a_{41} + d_1d_4$	$-a_{14}a_{41} + d_1d_4$
$\{3,4\}^*$	$-a_{34}a_{43}$	$-a_{34}a_{43}$
$\{1,3\}$	0	0
$\{2,3\}$	0	1
$\{2,4\}$	d_2d_4	d_2d_4
$\{1,2,3\}$	0	$d_1 - a_{12}x_{31}$
$\{1,2,4\}$	$-a_{14}a_{41}d_2 - a_{12}a_{21}d_4 + d_1d_2d_4 + a_{14}a_{21}x_{42}$	$-a_{14}a_{41}d_2 - a_{12}a_{21}d_4 + d_1d_2d_4 + a_{14}a_{21}w$
$\{1,3,4\}$	$-a_{34}a_{43}d_1$	$-a_{34}a_{43}d_1 + a_{14}a_{43}x_{31}$
$\{2,3,4\}$	$-a_{34}a_{43}d_2$	$-a_{34}a_{43}d_2 + d_4 - a_{34}w$
$\{1,2,3,4\}$	$-a_{14}a_{21}a_{32}a_{43} + a_{12}a_{21}a_{34}a_{43} - a_{34}a_{43}d_1d_2$	$-a_{14}a_{41} + a_{12}a_{34}a_{41} - a_{14}a_{21}a_{43} + a_{12}a_{21}a_{34}a_{43} - a_{34}a_{43}d_1d_2 + d_1d_4 - a_{34}d_1w + a_{14}a_{43}d_2x_{31} - a_{12}d_4x_{31} + a_{14}wx_{31}$

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Table 2

Table 2	$d_1 = d_2 = d_3 = d_4 = 1$	
	$a_{34}a_{43} < 1$	$a_{34} = a_{43} = 1$
α	$x_{13} = u, x_{41} = v, x_{24} = w, x_{35} = y, x_{52} = z$ $x_{31} = -u, x_{14} = -v, x_{42} = -w, x_{53} = -y, x_{25} = -z$	
$\{1, 2\}^*$	$1 - a_{12}a_{21}$	
$\{2, 3\}^*$	$1 - a_{23}a_{32}$	
$\{3, 4\}^*$	$1 - a_{34}a_{43}$	
$\{4, 5\}^*$	$d_5 - a_{45}a_{54}$	
$\{1, 5\}^*$	$d_5 - a_{15}a_{51}$	
$\{1,3\}$	$1 + u^2$	
$\{1,4\}$	$1 + v^2$	
$\{2,4\}$	$1 + w^2$	
$\{2,5\}$	$d_5 + z^2$	
$\{3,5\}$	$d_5 + y^2$	
$\{1,2,3\}$	$1 - a_{12}a_{21} - a_{23}a_{32} - a_{12}a_{23}u + a_{21}a_{32}u + u^2$	
$\{1,2,4\}$	$1 - a_{12}a_{21} + v^2 + a_{12}vw + a_{21}vw + w^2$	
$\{1,2,5\}$	$-a_{15}a_{51} + d_5 - a_{12}a_{21}d_5 - a_{12}a_{51}z + a_{21}a_{15}z + z^2$	
$\{1,3,4\}$	$1 - a_{34}a_{43} + u^2 + a_{34}uv + a_{43}uv + v^2$	
$\{1,3,5\}$	$-a_{15}a_{51} + d_5 + d_5u^2 + a_{15}uy + a_{51}uy + y^2$	
$\{1,4,5\}$	$-a_{45}a_{54} - a_{15}a_{51} + d_5 - a_{45}a_{51}v + a_{54}a_{15}v + d_5v^2$	
$\{2,3,4\}$	$1 - a_{23}a_{32} - a_{34}a_{43} - a_{23}a_{34}w + a_{32}a_{43}w + w^2$	
$\{2,3,5\}$	$d_5 - a_{23}a_{32}d_5 + y^2 + a_{23}yz + a_{32}yz + z^2$	
$\{2,4,5\}$	$-a_{45}a_{54} + d_5 + d_5w^2 + a_{45}wz + a_{54}wz + z^2$	
$\{3,4,5\}$	$-a_{45}a_{54} + d_5 - a_{34}a_{43}d_5 - a_{34}a_{45}y + a_{43}a_{54}y + y^2$	
$\{1,2,3,4\}$	$1 - a_{12}a_{21} - a_{23}a_{32} - a_{34}a_{43} + a_{12}a_{21}a_{34}a_{43} - a_{12}a_{23}u + a_{21}a_{32}u + u^2 - a_{12}a_{23}a_{34}v + a_{21}a_{32}a_{43}v + a_{34}uv + a_{43}uv + v^2 - a_{23}a_{32}v^2 - a_{23}a_{34}w + a_{32}a_{43}w + a_{34}a_{21}uw + a_{12}a_{43}uw + a_{12}vw + a_{21}vw + a_{23}uvw - a_{32}uvw + w^2 + u^2w^2$	
$\{1,2,3,5\}$	$-a_{15}a_{51} + a_{23}a_{32}a_{15}a_{51} + d_5 - a_{12}a_{21}d_5 - a_{23}a_{32}d_5 - a_{12}a_{23}d_5u + a_{21}a_{32}d_5u + d_5u^2 - a_{12}a_{23}a_{51}y + a_{21}a_{32}a_{15}y + a_{51}uy + a_{15}uy + y^2 - a_{12}a_{21}y^2 - a_{12}a_{51}z + a_{21}a_{15}z + a_{51}a_{32}uz + a_{23}a_{15}uz + a_{23}yz + a_{32}yz + a_{12}uyz - a_{21}uyz + z^2 + u^2z^2$	
$\{1,2,4,5\}$	$-a_{45}a_{54} + a_{12}a_{21}a_{45}a_{54} - a_{15}a_{51} + d_5 - a_{12}a_{21}d_5 - a_{45}a_{51}v + a_{54}a_{15}v + d_5v^2 - a_{12}a_{45}a_{51}w + a_{21}a_{54}a_{15}w + a_{12}d_5vw + a_{21}d_5vw - a_{15}a_{51}w^2 + d_5w^2 - a_{12}a_{51}z + a_{21}a_{15}z + a_{45}a_{21}vz + a_{12}a_{54}vz + a_{45}wz + a_{54}wz + a_{51}vwz - a_{15}vwz + z^2 + v^2z^2$	

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<table>
<tr><th rowspan="2">Table 2</th><th colspan="2">$d_1 = d_2 = d_3 = d_4 = 1$</th></tr>
<tr><th>$a_{34}a_{43} < 1$</th><th>$a_{34} = a_{43} = 1$</th></tr>
<tr><td>{1,3,4,5}</td><td colspan="2">$-a_{45}a_{54} - a_{15}a_{51} + a_{34}a_{43}a_{15}a_{51} + d_5 - a_{34}a_{43}d_5 - a_{34}a_{45}a_{51}u + a_{43}a_{54}a_{15}u - a_{45}a_{54}u^2 + d_5u^2 - a_{45}a_{51}v + a_{15}a_{54}v + a_{34}d_5uv + a_{43}d_5uv + d_5v^2 - a_{34}a_{45}y + a_{43}a_{54}y + a_{15}uy + a_{51}uy + a_{51}a_{43}vy + a_{34}a_{15}vy + a_{45}uvy - a_{54}uvy + y^2 + v^2y^2$</td></tr>
<tr><td>{2,3,4,5}</td><td>$-a_{45}a_{54} + a_{23}a_{32}a_{45}a_{54} + d_5 - a_{23}a_{32}d_5 - a_{34}a_{43}d_5 - a_{23}a_{34}d_5w + a_{32}a_{43}d_5w + d_5w^2 - a_{34}a_{45}y + a_{43}a_{54}y + a_{45}a_{32}wy + a_{23}a_{54}wy + y^2 + w^2y^2 - a_{23}a_{34}a_{45}z + a_{32}a_{43}a_{54}z + a_{45}wz + a_{54}wz + a_{23}yz + a_{32}yz + a_{34}wyz - a_{43}wyz + z^2 - a_{34}a_{43}z^2$</td><td>$-a_{45}a_{54} + a_{23}a_{32}a_{45}a_{54} - a_{23}a_{32}d_5 - a_{23}d_5w + a_{32}d_5w + d_5w^2 - a_{45}y + a_{54}y + a_{45}a_{32}wy + a_{23}a_{54}wy + y^2 + w^2y^2 - a_{23}a_{45}z + a_{32}a_{54}z + a_{45}wz + a_{54}wz + a_{23}yz + a_{32}yz$</td></tr>
<tr><td>{1,2,3,4,5}</td><td colspan="2">$-a_{15}a_{51} + a_{15}a_{23}a_{32}a_{51} + a_{15}a_{34}a_{43}a_{51} + a_{12}a_{23}a_{34}a_{45}a_{51} + a_{15}a_{21}a_{32}a_{43}a_{54} - a_{45}a_{54} + a_{12}a_{21}a_{45}a_{54} + a_{23}a_{32}a_{45}a_{54} + d_5 - a_{12}a_{21}d_5 - a_{23}a_{32}d_5 - a_{34}a_{43}d_5 + a_{12}a_{21}a_{34}a_{43}d_5 - a_{34}a_{45}a_{51}u + a_{15}a_{43}a_{54}u + a_{12}a_{23}a_{45}a_{54}u - a_{21}a_{32}a_{45}a_{54}u - a_{12}a_{23}d_5u + a_{21}a_{32}d_5u - a_{45}a_{54}u^2 + d_5u^2 - a_{45}a_{51}v + a_{23}a_{32}a_{45}a_{51}v + a_{15}a_{54}v - a_{15}a_{23}a_{32}a_{54}v - a_{12}a_{23}a_{34}d_5v + a_{21}a_{32}a_{43}d_5v + a_{34}d_5uv + a_{43}d_5uv + d_5v^2 - a_{23}a_{32}d_5v^2 + a_{15}a_{23}a_{34}a_{51}w - a_{15}a_{32}a_{43}a_{51}w - a_{12}a_{45}a_{51}w + a_{15}a_{21}a_{54}w - a_{23}a_{34}d_5w + a_{32}a_{43}d_5w + a_{32}a_{45}a_{51}uw + a_{15}a_{23}a_{54}uw + a_{21}a_{34}d_5uw + a_{12}a_{43}d_5uw + a_{12}d_5vw + a_{21}d_5vw + a_{23}d_5uvw - a_{32}d_5uvw - a_{15}a_{51}w^2 + d_5w^2 + d_5u^2w^2 + a_{15}a_{21}a_{32}y - a_{34}a_{45}y + a_{12}a_{21}a_{34}a_{45}y - a_{12}a_{23}a_{51}y + a_{43}a_{54}y - a_{12}a_{21}a_{43}a_{54}y + a_{15}uy + a_{51}uy + a_{15}a_{34}vy + a_{21}a_{32}a_{45}vy + a_{43}a_{51}vy + a_{12}a_{23}a_{54}vy + a_{45}uvy - a_{54}uvy + a_{15}a_{21}a_{34}wy + a_{32}a_{45}wy + a_{12}a_{43}a_{51}wy + a_{23}a_{54}wy + a_{12}a_{45}uwy - a_{21}a_{54}uwy - a_{15}a_{32}vwy + a_{23}a_{51}vwy + a_{15}uw^2y + a_{51}uw^2y + y^2 - a_{12}a_{21}y^2 + v^2y^2 + a_{12}vwy^2 + a_{21}vwy^2 + w^2y^2 + a_{15}a_{21}z - a_{15}a_{21}a_{34}a_{43}z - a_{23}a_{34}a_{45}z - a_{12}a_{51}z + a_{12}a_{34}a_{43}a_{51}z + a_{32}a_{43}a_{54}z + a_{15}a_{23}uz + a_{21}a_{34}a_{45}uz + a_{32}a_{51}uz + a_{12}a_{43}a_{54}uz + a_{15}a_{23}a_{34}vz + a_{21}a_{45}vz + a_{32}a_{43}a_{51}vz + a_{12}a_{54}vz + a_{23}a_{45}uvz - a_{32}a_{54}uvz + a_{45}wz + a_{54}wz - a_{15}a_{43}uwz + a_{34}a_{51}uwz + a_{45}u^2wz + a_{54}u^2wz - a_{15}vwz + a_{51}vwz + a_{23}yz + a_{32}yz + a_{12}uyz - a_{21}uyz + a_{12}a_{34}vyz - a_{21}a_{43}vyz + a_{23}v^2yz + a_{32}v^2yz + a_{34}wyz - a_{43}wyz + z^2 - a_{34}a_{43}z^2 + u^2z^2 + a_{34}uvz^2 + a_{43}uvz^2 + v^2z^2$</td></tr>
</table>

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Table 3

Table 3-1	Case 1: $d_1 = d_2 = d_4 = 0$ $a_{12}a_{23}a_{34}a_{45}a_{51} + a_{21}a_{32}a_{43}a_{54}a_{15} \geq 0$	$a_{12}a_{23}a_{34}a_{45}a_{51} + a_{21}a_{32}a_{43}a_{54}a_{15} < 0$ $a_{21} = a_{32} = a_{43} = a_{54} = -1$, $a_{15} < 0$
α	$x_{ij} = 0$	$x_{14} = x_{25} = x_{31} = x_{42} = x_{53} = 0$ $x_{13} = x_{24} = x_{35} = x_{41} = x_{52} = z > 0$
$\{1,2\}^*$	$-a_{12}a_{21}$	a_{12}
$\{2,3\}^*$	$-a_{23}a_{32}$	a_{23}
$\{3,4\}^*$	$-a_{34}a_{43}$	a_{34}
$\{4,5\}^*$	$-a_{45}a_{54}$	a_{45}
$\{1,5\}^*$	$-a_{15}a_{51}$	$-a_{15}a_{51}$
$\{1,3\}$	0	0
$\{1,4\}$	0	0
$\{2,4\}$	0	0
$\{2,5\}$	0	0
$\{3,5\}$	d_3d_5	d_3d_5
$\{1,2,3\}$	$-a_{12}a_{21}d_3$	$a_{12}d_3 + z$
$\{1,2,4\}$	0	$a_{12}z^2$
$\{1,2,5\}$	$-a_{12}a_{21}d_5$	$a_{12}d_5 - a_{15}z$
$\{1,3,4\}$	0	$a_{34}z^2$
$\{1,3,5\}$	$-a_{15}a_{51}d_3$	$-a_{15}a_{51}d_3 + a_{51}z^2$
$\{1,4,5\}$	0	$-a_{15}z$
$\{2,3,4\}$	0	z
$\{2,3,5\}$	$-a_{23}a_{32}d_5$	$a_{23}d_5 + a_{23}z^2$
$\{2,4,5\}$	0	$a_{45}z^2$
$\{3,4,5\}$	$-a_{45}a_{54}d_3 - a_{34}a_{43}d_5$	$a_{45}d_3 + a_{34}d_5 + z$
$\{1,2,3,4\}$	$a_{12}a_{21}a_{34}a_{43}$	$a_{12}a_{34} - a_{12}a_{23}a_{34}z + a_{12}d_3z^2 + z^3$
$\{1,2,3,5\}$	$a_{15}a_{51}a_{23}a_{32} - a_{12}a_{21}d_3d_5$	$-a_{15}a_{51}a_{23} + a_{12}d_3d_5 - a_{12}a_{23}a_{51}z - a_{15}d_3z + d_5z + z^3$
$\{1,2,4,5\}$	$a_{12}a_{21}a_{45}a_{54}$	$a_{12}a_{45} - a_{12}a_{45}a_{51}z + a_{12}d_5z^2 - a_{15}z^3$
$\{1,3,4,5\}$	$a_{15}a_{51}a_{34}a_{43}$	$-a_{15}a_{51}a_{34} - a_{34}a_{45}a_{51}z - a_{15}d_5z + a_{34}d_5z^2 + z^3$
$\{2,3,4,5\}$	$a_{23}a_{32}a_{54}a_{45}$	$a_{23}a_{45} - a_{23}a_{34}a_{45}z + d_5z + a_{45}d_3z^2 + z^3$
$\{1,2,3,4,5\}$	$a_{12}a_{23}a_{34}a_{45}a_{51} + a_{15}a_{21}a_{32}a_{43}a_{54} + a_{12}a_{21}a_{45}a_{54}d_3 + a_{12}a_{21}a_{34}a_{43}d_5$	$a_{15} + a_{12}a_{23}a_{34}a_{45}a_{51} + a_{12}a_{45}d_3 + a_{12}a_{34}d_5 + a_{12}z - a_{15}a_{23}z - a_{15}a_{34}z + a_{45}z - a_{15}a_{51}z - a_{12}a_{45}a_{51}d_3z - a_{12}a_{23}a_{34}d_5z - a_{12}a_{23}z^2 + a_{15}a_{23}a_{34}z^2 - a_{34}a_{45}z^2 - a_{12}a_{51}z^2 - a_{45}a_{51}z^2 + a_{12}d_3d_5z^2 - a_{15}d_3z^3 + d_5z^3 + z^5$

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Volume 9, pp. 1-20, February 2002

Table 3-2	Case 2: $d_1 = d_2 = 1$ $d_4 = d_5 = 0$	$d_3 = d_5 = 0$, $d_4 = 1$
α	$x_{31} = z$, $x_{13} = -z$, $x_{ij} = 0$ for others sign z = sign $(a_{34}a_{45}a_{51} - a_{43}a_{54}a_{51})$.	
$\{1,2\}^*$	$1 - a_{12}a_{21}$	$1 - a_{12}a_{21}$
$\{2,3\}^*$	$-a_{23}a_{32} + d_3$	$-a_{23}a_{32}$
$\{3,4\}^*$	$-a_{34}a_{43}$	$-a_{34}a_{43}$
$\{4,5\}^*$	$-a_{45}a_{54}$	$-a_{45}a_{54}$
$\{1,5\}^*$	$-a_{15}a_{51}$	$-a_{15}a_{51}$
{1,3}	$d_3 + z^2$	z^2
{1,4}	0	1
{2,4}	0	1
{2,5}	0	0
{3,5}	0	0
{1,2,3}	$-a_{23}a_{32} + d_3 - a_{12}a_{21}d_3 + a_{12}a_{23}z - a_{21}a_{32}z + z^2$	$-a_{23}a_{32}+a_{12}a_{23}z-a_{21}a_{32}z+z^2$
{1,2,4}	0	$1 - a_{12}a_{21}$
{1,2,5}	$-a_{15}a_{51}$	$-a_{15}a_{51}$
{1,3,4}	$-a_{34}a_{43}$	$-a_{34}a_{43} + z^2$
{1,3,5}	$-a_{15}a_{51}d_3$	0
{1,4,5}	$-a_{45}a_{54}$	$-a_{45}a_{54} - a_{15}a_{51}$
{2,3,4}	$-a_{34}a_{43}$	$-a_{34}a_{43} - a_{23}a_{32}$
{2,3,5}	0	0
{2,4,5}	$-a_{45}a_{54}$	$-a_{45}a_{54}$
{3,4,5}	$-a_{45}a_{54}d_3$	0
{1,2,3,4}	$-a_{34}a_{43} + a_{12}a_{21}a_{34}a_{43}$	$-a_{34}a_{43} + a_{12}a_{21}a_{34}a_{43} - a_{23}a_{32}+a_{12}a_{23}z-a_{21}a_{32}z+z^2$
{1,2,3,5}	$a_{15}a_{51}a_{23}a_{32} - a_{15}a_{51}d_3$	$a_{15}a_{51}a_{23}a_{32}$
{1,2,4,5}	$-a_{45}a_{54} + a_{12}a_{21}a_{45}a_{54}$	$-a_{45}a_{54} + a_{12}a_{21}a_{45}a_{54} - a_{15}a_{51}$
{1,3,4,5}	$a_{15}a_{51}a_{34}a_{43} - a_{45}a_{54}d_3 + a_{34}a_{45}a_{51}z - a_{15}a_{43}a_{54}z - a_{45}a_{54}z^2$	$a_{15}a_{51}a_{34}a_{43} + a_{34}a_{45}a_{51}z - a_{15}a_{43}a_{54}z - a_{45}a_{54}z^2$
{2,3,4,5}	$a_{23}a_{32}a_{45}a_{54} - a_{45}a_{54}d_3$	$a_{23}a_{32}a_{45}a_{54}$
{1,2,3,4,5}	$a_{15}a_{51}a_{34}a_{43} + a_{12}a_{23}a_{34}a_{45}a_{51} + a_{15}a_{21}a_{32}a_{43}a_{54} + a_{23}a_{32}a_{45}a_{54} - a_{45}a_{54}d_3 + a_{12}a_{21}a_{45}a_{54}d_3 + a_{34}a_{45}a_{51}z - a_{15}a_{43}a_{54}z - a_{12}a_{23}a_{45}a_{54}z + a_{21}a_{32}a_{45}a_{54}z - a_{45}a_{54}z^2$	$a_{15}a_{51}a_{34}a_{43} + a_{12}a_{23}a_{34}a_{45}a_{51} + a_{15}a_{21}a_{32}a_{43}a_{54} + a_{23}a_{32}a_{45}a_{54}+a_{15}a_{23}a_{32}a_{51}+ a_{34}a_{45}a_{51}z - a_{15}a_{43}a_{54}z - a_{12}a_{23}a_{45}a_{54}z + a_{21}a_{32}a_{45}a_{54}z - a_{45}a_{54}z^2$

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