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A SIMPLE PROOF OF THE CLASSIFICATION OF NORMAL TOEPLITZ MATRICES*

AKIO ARIMOTO†

Abstract. An easy proof to show that every complex normal Toeplitz matrix is classified as either of type I or of type II is given. Instead of difference equations on elements in the matrix used in past studies, polynomial equations with coefficients of elements are used. In a similar fashion, it is shown that a real normal Toeplitz matrix must be one of four types: symmetric, skew-symmetric, circulant, or skew-circulant. Here trigonometric polynomials in the complex case and algebraic polynomials in the real case are used.

Key words. Normal matrices, Toeplitz matrices.

AMS subject classifications. 15A57, 47B15, 47B35

1. Introduction: Normal Toeplitz Matrices. Ikramov [3] first showed that a normal Toeplitz matrix over real field must be one of four types: symmetric, skew-symmetric (up to the principal diagonal), circulant, or skew-circulant. Later on, the complex version of the problem was solved by Ikramov et al. [6] and Gel'fgat [2]. The history of the theory was described in [5]. The complex case was also proved by Farenick, Krupnik, Krupnik and Lee [1], and Ito [2], independently of [6]. We give here another proof which, we believe, is the simplest one. The complex case and the real case will be proved in a similar fashion, although trigonometric polynomials will be used in the complex case and algebraic polynomials will be used in the real case. Now denote Toeplitz matrix of order $N+1$ by T_N:

$$T_N = \begin{bmatrix} a_0 & a_{-1} & a_{-2} & \cdots & a_{-N} \\ a_1 & a_0 & a_{-1} & \cdots & a_{-(N-1)} \\ \cdots & \cdots & \cdots & \cdots & \\ a_{N-1} & a_{N-2} & a_{N-3} & \cdots & a_{-1} \\ a_N & a_{N-1} & a_{N-2} & \cdots & a_0 \end{bmatrix}.$$

The matrix T_N is called normal Toeplitz if $T_N T_N^* - T_N^* T_N = 0$, where T_N^* is the transposed conjugate of T_N. We can simply state classifications of normal Toeplitz matrices in the following way. A normal Toeplitz matrix T_N is of type I if and only if for some α_0,with $|\alpha_0| = 1$ and ,

$$\begin{bmatrix} a_{-1} & a_{-2} & \cdots & a_{-N} \end{bmatrix} = \alpha_0 \begin{bmatrix} \bar{a}_1 & \bar{a}_2 & \cdots & \bar{a}_N \end{bmatrix}, \tag{1}$$

whereas T_N is of type II if and only if for some β_0, with $|\beta_0| = 1$,

$$\begin{bmatrix} a_{-1} & a_{-2} & \cdots & a_{-N} \end{bmatrix} = \beta_0 \begin{bmatrix} a_N & a_{N-1} & \cdots & a_1 \end{bmatrix}. \tag{2}$$

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†Department of Electronics and Computer Engineering, Musashi Institute of Technology, Tamazutsumi 1-28-1, Setagaya-ku, Tokyo 158, Japan (arimoto@iname.com).

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Surely the complex case contains the real case, but when all the $a_k^{\prime}s$ are real valued, it will be found that α_0 and β_0 must be either $+1$ or -1, hence we may have the further concrete classification of four types: A normal Toeplitz matrix T_N is symmetric if and only if

$$\begin{bmatrix} a_{-1} & a_{-2} & \cdots & a_{-N} \end{bmatrix} = \begin{bmatrix} a_1 & a_2 & \cdots & a_N \end{bmatrix}, \tag{3}$$

T_N is skew-symmetric if and only if

$$\begin{bmatrix} a_{-1} & a_{-2} & \cdots & a_{-N} \end{bmatrix} = -\begin{bmatrix} a_1 & a_2 & \cdots & a_N \end{bmatrix}, \tag{4}$$

T_N is circulant if and only if

$$\begin{bmatrix} a_{-1} & a_{-2} & \cdots & a_{-N} \end{bmatrix} = \begin{bmatrix} a_N & a_{N-1} & \cdots & a_1 \end{bmatrix}, \tag{5}$$

T_N is skew-circulant if and only if

$$\begin{bmatrix} a_{-1} & a_{-2} & \cdots & a_{-N} \end{bmatrix} = -\begin{bmatrix} a_N & a_{N-1} & \cdots & a_1 \end{bmatrix}. \tag{6}$$

Throughout this paper, we assume a_0 to be zero without loss of generality since a_0 does not play any role in the above classifications.

2. Proof of Theorem in the Complex Case. Now we define trigonometric polynomials $s(x)$ and $t(x)$ with coefficients $a_1, a_2, \cdots, a_N$ and $a_{-1}, a_{-2}, \cdots, a_{-N}$, respectively:

$$\begin{aligned} s\left(x\right) &= a_1 e^{ix} + a_2 e^{2ix} + \cdots + a_N e^{Nix} \\ t\left(x\right) &= a_{-1} e^{-ix} + a_{-2} e^{-2ix} + \cdots + a_{-N} e^{-Nix} \ . \end{aligned}$$

Using these trigonometric polynomials, we can restate conditions (1) and (2) in the previous section as

(1') type I: $t\left(x\right) = \alpha_0 \overline{s\left(x\right)}$, with $\left|\alpha_0\right| = 1$,

(2') type II: $t\left(x\right) = \beta_0 s\left(x\right) e^{-i(N+1)x}$, with $\left|\beta_0\right| = 1$.

THEOREM 2.1. *Every normal Toeplitz matrix is either of type I or of type II.*

Proof. Our proof is based on the expression in [7, p. 998] or [1, p. 1038]. Necessary and sufficient conditions for T_N to be normal in terms of a_n are that

$$a_m \bar{a}_n - \bar{a}_{-m} a_{-n} + \bar{a}_{N+1-m} a_{N+1-n} - a_{-(N+1-m)} \bar{a}_{-(N+1-n)} = 0, \tag{7}$$

$\left(1 \leq n, m \leq N\right)$.

We now rewrite (7) by using polynomials $s(x)$ and $t(x)$. Multiply both sides of (7) by $e^{imx} e^{-iny}$ and sum these new expressions over all m and n to obtain

$$s\left(x\right) \overline{s\left(y\right)} - \overline{t\left(x\right)}\, t\left(y\right) + \overline{s\left(x\right)}\, s\left(y\right) e^{i(N+1)(x-y)} - t\left(x\right) \overline{t\left(y\right)} e^{i(N+1)(x-y)} = 0. \tag{8}$$

It easily seen that (8) is equivalent to (7), and that (1') or (2') implies (8). Hence both (1') and (2') are sufficient conditions for T_N to be normal. Thus it only remains to show the necessity of the condition. When we take $x = y$ in (8), we have

$$s\left(x\right) \overline{s\left(x\right)} - t\left(x\right) \overline{t\left(x\right)} = 0, or \left|s\left(x\right)\right| = \left|t\left(x\right)\right| \tag{9}$$

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for each x. Hence except for trivial case $t(x) \equiv 0$, we can find an x_0 such that $t(x_0) \neq 0$ and there exist α and β such that $s(x_0) = \alpha t(x_0)$ and $t(x_0) = \beta\overline{t(x_0)}$, with $|\alpha| = |\beta| = 1$. By setting $y = x_0$ and then by dividing through by $t(x_0)$, equation (8) becomes

$$s(x)\,\bar{\alpha} - \overline{t(x)}\beta + \overline{s(x)}\alpha\,\beta e^{i(N+1)(x-x_0)} - t(x)\,e^{i(N+1)(x-x_0)} = 0,$$

from which we have

$$\overline{t(x)} = s(x)\,\overline{\alpha\,\beta} + \overline{s(x)}\alpha e^{i(N+1)(x-x_0)} - t(x)\,\bar{\beta}e^{i(N+1)(x-x_0)}. \tag{10}$$

Substituting the right hand side of (10) into $\overline{t(x)}$ of (9), we have

$$\left\{s(x) - t(x)\,\alpha e^{i(N+1)(x-x_0)}\right\}\left\{\overline{s(x)} - t(x)\,\overline{\alpha\beta}\right\} = 0, \tag{11}$$

which implies our required results if we take $\alpha_0 = \alpha\beta$ and $\beta_0 = \bar{\alpha}e^{i(N+1)x_0}$. We should notice here that (11) is the product of polynomials. If one of the polynomials in (11) is nonzero at $x = x_0$, then by continuity the polynomial is nonzero in an open neighborhood $\mathcal{U}$ of x_0, which implies that the other polynomial in the product (11) is identically zero on $\mathcal{U}$. Because the only polynomial with a continuum of roots is the zero polynomial, equation (11) implies equation (1') or (2'). Hence type I or type II must occur. ☐

3. Proof of Theorem in the Real Case.

THEOREM 3.1. *Every real normal Toeplitz matrix is either symmetric, skew-symmetric, circulant or skew-circulant.*

Proof. We define algebraic polynomials $p(x)$ and $q(x)$ such that

$$\begin{aligned} p(x) &= a_1x + a_2x^2 + \cdots + a_Nx^N \\ q(x) &= a_{-1}x + a_{-2}x^2 + \cdots + a_{-N}x^N \ . \end{aligned}$$

Also in this real case we can apply (7) in the previous section as necessary and sufficient conditions for T_N to be normal in terms of real valued a_n:

$$a_ma_n - a_{-m}a_{-n} + a_{N+1-m}a_{N+1-n} - a_{-(N+1-m)}a_{-(N+1-n)} = 0\,(1 \leq n, m \leq N) \tag{12}$$

Now we introduce the reciprocal polynomials to $p(x)$ and $q(x)$

$$\tilde{p}(x) = a_Nx + a_{N-1}x^2 + \cdots + a_1x^N$$

and

$$\tilde{q}(x) = a_{-N}x + a_{-N+1}x^2 + \cdots + a_{-1}x^N,$$

respectively. Multiply both sides of equation (12) by x^my^n and then sum the resulting expressions over all m and n to obtain the following analog of (8):

$$p(x)\,p(y) - q(x)\,q(y) + \tilde{p}(x)\,\tilde{p}(y) - \tilde{q}(x)\,\tilde{q}(y) = 0. \tag{13}$$

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Noticing the relation $\tilde{q}(x) = x^{N+1}q(1/x)$ and $\tilde{p}(x) = x^{N+1}p(1/x)$, and letting $y = 1/x$ in (13), we have $p(x)\tilde{p}(x) = q(x)\tilde{q}(x)$. By setting $x = y$ and then multiplying both sides of (13) by $p^2(x)$, we have from the relation $p^2(x)\tilde{p}^2(x) = q^2(x)\tilde{q}^2(x)$,

$$p^4(x) - p^2(x)q^2(x) + q^2(x)\tilde{q}^2(x) - p^2(x)\tilde{q}^2(x) = 0,$$

which can be factored as the following way:

$$\left(p^2(x) - q^2(x)\right)\left(p^2(x) - \tilde{q}^2(x)\right) = 0.$$

¿From this formula, we obtain four types, symmetric $p(x) = q(x)$, skew-symmetric $p(x) = -q(x)$, circulant $p(x) = \tilde{q}(x)$ and skew-circulant $p(x) = -\tilde{q}(x)$, which are equivalent to (3), (4), (5), and (6), respectively.

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