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PROOF OF ATIYAH'S CONJECTURE FOR TWO SPECIAL TYPES OF CONFIGURATIONS*

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Abstract. To an ordered N-tuple $(x_1, \ldots, x_N)$ of distinct points in the three-dimensional Euclidean space Atiyah has associated an ordered N-tuple of complex homogeneous polynomials $(p_1, \ldots, p_N)$ in two variables x, y of degree $N-1$, each p_i determined only up to a scalar factor. He has conjectured that these polynomials are linearly independent. In this note it is shown that Atiyah's conjecture is true for two special configurations of N points. For one of these configurations, it is shown that a stronger conjecture of Atiyah and Sutcliffe is also valid.

Key words. Atiyah's conjecture, Hopf map, Configuration of N points in the three-dimensional Euclidean space, Complex projective line.

AMS subject classifications. 51M04, 51M16, 70G25

1. Two conjectures. Let $(x_1, \ldots, x_N)$ be an ordered N-tuple of distinct points in the three-dimensional Euclidean space. Each ordered pair (x_i, x_j) with $i \neq j$ determines a point

$$\frac{x_j - x_i}{|x_j - x_i|}$$

on the unit sphere S^2. Identify S^2 with the complex projective line by using a stereographic projection. Hence one obtains a point (u_{ij}, v_{ij}) on this projective line and a complex nonzero linear form $l_{ij} = u_{ij}x + v_{ij}y$ in two variables x and y. Define homogeneous polynomials p_i of degree $N-1$ by

$$p_i = \prod_{j \neq i} l_{ij}(x, y), \quad i = 1, \ldots, N. \tag{1.1}$$

CONJECTURE 1.1. (Atiyah [2]) *The polynomials $p_1, \ldots, p_N$ are linearly independent.*

Atiyah [1], [2] has observed that his conjecture is true if the points $x_1, \ldots, x_N$ are collinear. He has also verified the conjecture for $N = 3$. The case $N = 4$ has been verified by Eastwood and Norbury [4]. For additional information on the conjecture (further conjectures, generalizations, and numerical evidence) see [2], [3].

In order to state the second conjecture, one has to be more explicit. Identify the three-dimensional Euclidean space with $\mathbb{R} \times \mathbb{C}$ and denote the origin by O. Following Eastwood and Norbury [4], we make use of the Hopf map $h : \mathbb{C}^2 \setminus \{O\} \to (\mathbb{R} \times \mathbb{C}) \setminus \{O\}$ defined by

$$h(z, w) = ((|z|^2 - |w|^2)/2, z\bar{w}).$$

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This map is surjective and its fibers are the circles $\{(zu, wu) : u \in S^1\}$, where S^1 is the unit circle. If $h(z, w) = (a, v)$, we say that (z, w) is a *lift* of (a, v). For instance, we can take

$$\lambda^{-1/2}(\lambda, \bar{v}), \quad \lambda = a + \sqrt{a^2 + |v|^2},$$

as the lift of (a, v).

Assume that our points are $x_i = (a_i, z_i)$. For the sake of simplicity assume that if $i < j$ and $z_i = z_j$ then $a_i < a_j$. As the lift of the vector $x_j - x_i$, $i < j$, we choose

$$\frac{1}{\sqrt{\lambda_{ij}}} \left(\lambda_{ij}, \bar{z}_j - \bar{z}_i\right),$$

where

$$\lambda_{ij} = a_j - a_i + \sqrt{(a_j - a_i)^2 + |z_j - z_i|^2}.$$

According to the recipe in [2], [3], [4], we always use the lift $(-\bar{w}, \bar{z})$ for the vector $x_i - x_j$ if (z, w) has been chosen as the lift of $x_j - x_i$. Hence we introduce the linear forms

$$l_{ij}(x, y) = \lambda_{ij} x + (\bar{z}_j - \bar{z}_i) y, \quad i < j;$$
$$l_{ij}(x, y) = (z_j - z_i) x + \lambda_{ji} y, \quad i > j.$$

Define P to be the $N \times N$ coefficient matrix of the binary forms $p_i(x, y)$ defined by (1.1) using the above l_{ij}'s. The second conjecture that we are interested in can now be formulated as follows.

CONJECTURE 1.2. (Atiyah and Sutcliffe [3, Conjecture 2]; see also [4]) *If* $r_{ij} = |x_j - x_i|$, *then*

$$|\det(P)| \geq \prod_{i<j} (2\lambda_{ij} r_{ij}).$$

As $2\lambda_{ij} r_{ij} = \lambda_{ij}^2 + |z_j - z_i|^2$, this conjecture can be rewritten as

$$|\det(P)| \geq \prod_{i<j} \left(\lambda_{ij}^2 + |z_j - z_i|^2\right). \tag{1.2}$$

Obviously, this conjecture is stronger than Conjecture 1.1.

2. Two special cases of Atiyah's conjecture. We shall prove Atiyah's conjecture in the following two cases:

(A) $N - 1$ of the points $x_1, \ldots, x_N$ are collinear.
(B) $N - 2$ of the points $x_1, \ldots, x_N$ are on a line L and the line segment joining the remaining two points has its midpoint on L and is perpendicular to L.

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Let L and M be two perpendicular lines in the three-dimensional Euclidean space intersecting at the origin, O. Let $N = m + n$ and assume that the points $x_1, \ldots, x_m$ are on L and $x_{m+1}, \ldots, x_N$ are on M but not on L. Set $y_j = x_{m+j}$ for $j = 1, \ldots, n$.

Without any loss of generality, we may assume that $L = \mathbb{R}\times\{0\}$ and $M = \{0\}\times\mathbb{R}$. Write $x_i = (a_i, 0)$ for $i = 1, \ldots, m$ and $y_j = (0, b_j)$ for $j = 1, \ldots, n$. We may also assume that $a_1 < a_2 < \cdots < a_m$ and $b_1 < b_2 < \cdots < b_n$.

The lifts of the nonzero vectors $x_j - x_i$, $i, j \in \{1, \ldots, N\}$ are given in Table 2.1, where we have set

$$\lambda_{ij} = a_i + \sqrt{a_i^2 + b_j^2}.$$

Vectors	Index restrictions	Lifts	Linear forms
$x_r - x_i$	$1 \le i < r \le m$	$(2(a_r - a_i))^{1/2}\,(1, 0)$	$2(a_r - a_i)x$
$x_i - x_r$	$1 \le i < r \le m$	$(2(a_r - a_i))^{1/2}\,(0, 1)$	$2(a_r - a_i)y$
$y_s - y_j$	$1 \le j < s \le n$	$(b_s - b_j)^{1/2}(1, 1)$	$(b_s - b_j)(y + x)$
$y_j - y_s$	$1 \le j < s \le n$	$(b_s - b_j)^{1/2}(-1, 1)$	$(b_s - b_j)(y - x)$
$x_i - y_j$	$1 \le i \le m,\ 1 \le j \le n$	$\lambda_{ij}^{-1/2}(\lambda_{ij}, -b_j)$	$\lambda_{ij}x - b_jy$
$y_j - x_i$	$1 \le i \le m,\ 1 \le j \le n$	$\lambda_{ij}^{-1/2}(b_j, \lambda_{ij})$	$b_jx + \lambda_{ij}y$

TABLE 2.1
The lifts of the vectors $x_j - x_i$.

The associated polynomials p_i (up to scalar factors) are given by

$$p_i(x, y) = x^{m-i}y^{i-1}\prod_{j=1}^{n}(b_jx + \lambda_{ij}y), \quad 1 \le i \le m; \tag{2.1}$$

$$p_{m+j}(x, y) = (y + x)^{n-j}(y - x)^{j-1}\prod_{i=1}^{m}(\lambda_{ij}x - b_jy), \quad 1 \le j \le n. \tag{2.2}$$

THEOREM 2.1. *Conjecture 1.1 is valid under the hypothesis* (A).

Proof. In this case we have $n = 1$. Without any loss of generality we may assume that $b_1 = -1$. After dehomogenizing the polynomials p_i (or $-p_i$) by setting $x = 1$, we obtain the polynomials:

$$y^{i-1}(1 - \lambda_i y), \quad 1 \le i \le m;$$
$$\prod_{i=1}^{m}(y + \lambda_i),$$

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where $\lambda_i = \lambda_{i1} > 0$. The coefficient matrix of these polynomials is

$$\begin{bmatrix} 1 & -\lambda_1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & -\lambda_2 & 0 & & 0 & 0 \\ 0 & 0 & 1 & -\lambda_3 & & 0 & 0 \\ \vdots & & & & & & \\ 0 & 0 & 0 & 0 & & 1 & -\lambda_m \\ E_m & E_{m-1} & E_{m-2} & E_{m-3} & & E_1 & 1 \end{bmatrix},$$

where E_k is the k-th elementary symmetric function of $\lambda_1, \dots, \lambda_m$. Its determinant,

$$1 + \lambda_m E_1 + \lambda_{m-1}\lambda_m E_2 + \dots + \lambda_1\lambda_2 \cdots \lambda_m E_m,$$

is positive. □

THEOREM 2.2. *Conjecture 1.1 is valid under the hypothesis* (B).

Proof. In this case $n = 2$ and $b_1 + b_2 = 0$. Without any loss of generality we may assume that $b_1 = -1$. After dehomogenizing the polynomials p_i (or $-p_i$) by setting $x = 1$, we obtain the polynomials:

$$\begin{aligned} &y^{i-1}(1 - \lambda_i^2 y^2), \quad 1 \le i \le m; \\ &(y+1)\prod_{i=1}^{m}(y + \lambda_i), \\ &(y-1)\prod_{i=1}^{m}(y - \lambda_i), \end{aligned}$$

where $\lambda_i = \lambda_{i1} > 0$. The coefficient matrix of these polynomials is

$$\begin{bmatrix} 1 & 0 & -\lambda_1^2 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & 0 & -\lambda_2^2 & & 0 & 0 & 0 \\ \vdots & & & & & & & \\ 0 & 0 & 0 & & & 1 & 0 & -\lambda_m^2 \\ \tilde{E}_{m+1} & \tilde{E}_m & \tilde{E}_{m-1} & & & \tilde{E}_2 & \tilde{E}_1 & 1 \\ (-1)^{m+1}\tilde{E}_{m+1} & (-1)^m\tilde{E}_m & (-1)^{m-1}\tilde{E}_{m-1} & & & \tilde{E}_2 & -\tilde{E}_1 & 1 \end{bmatrix},$$

where $\tilde{E}_k$ is the k-th elementary symmetric function of $1, \lambda_1, \dots, \lambda_m$. Its determinant is $2pq$ where

$$\begin{aligned} p &= 1 + \lambda_m^2 \tilde{E}_2 + \lambda_{m-2}^2 \lambda_m^2 \tilde{E}_4 + \cdots, \\ q &= \tilde{E}_1 + \lambda_{m-1}^2 \tilde{E}_3 + \lambda_{m-3}^2 \lambda_{m-1}^2 \tilde{E}_5 + \cdots, \end{aligned}$$

and thus it is positive. □

3. Atiyah and Sutcliffe conjecture is valid in case (A). In the general setup of the previous section, the Conjecture 1.2 asserts that

$$|\det(P)| \ge 2^{\binom{n}{2}} \prod_{i,j} \left(\lambda_{ij}^2 + b_j^2\right). \tag{3.1}$$

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where P is the coefficient matrix (of order $N = m + n$) of the polynomials (2.1) and (2.2).

In case (A) this inequality takes the form

$$1 + \lambda_m E_1 + \lambda_{m-1}\lambda_m E_2 + \cdots + \lambda_1\lambda_2 \cdots \lambda_m E_m \geq \prod_{i=1}^{m}(1 + \lambda_i^2), \tag{3.2}$$

where, as in the proof of Theorem 2.1, we assume that $b_1 = -1$ and E_k denotes the k-th elementary symmetric function of $\lambda_1, \ldots, \lambda_m$. Thus we have

$$\lambda_i = a_i + \sqrt{1 + a_i^2} > 0$$

and

$$\lambda_1 < \lambda_2 < \cdots < \lambda_m. \tag{3.3}$$

Let $E_k^{(2)}$ denote the k-th elementary symmetric function of $\lambda_1^2, \ldots, \lambda_m^2$. In view of (3.3), we have

$$\lambda_{m-k+1}\lambda_{m-k+2} \cdots \lambda_m E_k \geq E_k^{(2)}, \quad 0 \leq k \leq m.$$

The inequality (3.2) is a consequence of the inequalities just written since

$$\prod_{i=1}^{m}(1 + \lambda_i^2) = \sum_{k=0}^{m} E_k^{(2)}.$$

Hence we have the following result.

THEOREM 3.1. *Conjecture 1.2 is valid in case* (A).

In case (B) the inequality (3.1) takes the form:

$$\left(1 + \lambda_m^2 \tilde{E}_2 + \lambda_{m-2}^2\lambda_m^2 \tilde{E}_4 + \cdots\right) \left(\tilde{E}_1 + \lambda_{m-1}^2 \tilde{E}_3 + \lambda_{m-3}^2\lambda_{m-1}^2 \tilde{E}_5 + \cdots\right)$$
$$\geq \prod_{i=1}^{m}\left(1 + \lambda_i^2\right)^2,$$

where $\tilde{E}_k$ are as in the proof of Theorem 2.2.

It is easy to verify that this inequality holds for $m = 1$, but we were not able to prove it in general. If we set all $\lambda_i = \lambda > 0$, then the above inequality specializes to

$$\left[(1 + \lambda^2)^m + \sum_{k \geq 0} \binom{m}{2k+1}(\lambda^{4k+3} - \lambda^{4k+2})\right] \times$$
$$\left[(1 + \lambda^2)^m - \sum_{k \geq 0} \binom{m}{2k+1}(\lambda^{4k+2} - \lambda^{4k+1})\right] \geq (1 + \lambda^2)^{2m}.$$

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Since

$$\sum_{k\geq 0}\binom{m}{2k+1}(\lambda^{4k+3}-\lambda^{4k+2})=\frac{1}{2}(\lambda-1)\left[(1+\lambda^2)^m-(1-\lambda^2)^m\right],$$

it is easy to verify that the specialized inequality is valid.

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