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ON A STRONG FORM OF A CONJECTURE OF BOYLE AND HANDELMAN *

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Abstract. Let $\rho_{r,m}(x,\lambda) := (x-\lambda)^r \sum_{i=0}^{m} \binom{r+i-1}{i} x^{m-i}\lambda^i$. In this paper it is shown that if $\lambda_1, \ldots, \lambda_n$ are complex numbers such that $\lambda_1 = \lambda_2 = \ldots = \lambda_r > 0$ and $0 \leq \sum_{i=1}^{n} \lambda_i^k \leq n\lambda_1^k$, for $1 \leq k \leq m := n - r$, then

$$\prod_{i=1}^{n}(\lambda - \lambda_i) \ \leq \ \rho_{r,m}(\lambda, \lambda_1), \quad \text{for all} \ \ \lambda \geq 6.75\lambda_1. \ \ (*)$$

Moreover, if $r \geq m$, then $(*)$ holds for all $\lambda \geq \lambda_1$, while if $r < m$, but r is close to m, and n is large, one can lower the constant of 6.75 in the inequality $(*)$. The inequality $(*)$ is inspired by, and related to, a conjecture of Boyle and Handelman on the nonzero spectrum of a nonnegative matrix.

Key words. Nonnegative matrices, M-matrices, Inverse eigenvalue problem.

AMS subject classifications. 15A48, 15A18, 11C08

1. Introduction and background. Suppose that $\lambda_1, \ldots, \lambda_n$ are complex numbers satisfying that

$$\mathcal{S}_k \ := \ \sum_{i=1}^{n} \lambda_i^k \ \geq \ 0, \quad k \geq 1. \tag{1.1}$$

The condition (1.1) on $\lambda_1, \ldots, \lambda_n$ is a well–known necessary condition for n numbers to be the eigenvalues of an $n \times n$ nonnegative matrix (see, for example, Berman and Plemmons [3]). Furthermore, from a result due to Friedland [6, Theorem 1], it is known that (1.1) implies that one of the λ_i's is nonnegative and majorizes the moduli of the remaining numbers. Assume for the moment, without loss of generality, that $\lambda_1 = \max_{1 \leq i \leq n} |\lambda_i|$.

In a celebrated result due to Boyle and Handelman [4], the following claim, which is stated here in a special case, is proved:

THEOREM 1.1. ([4, Subtuple Theorem, Theorem 5.1]) *Suppose* $\Delta = (\lambda_1, \ldots, \lambda_r)$ *is an* r*–tuple of nonzero complex numbers with the following properties:*
(i) *The polynomial in the variable* λ *given by* $\prod_{i=1}^{r}(t - \lambda_i)$ *has all its coefficients in* $\mathbb{R}$.
(ii) $\lambda_1 = |\lambda_1| > |\lambda_i|$, $i = 2, \ldots, r$.
(iii) *The condition* (1.1) *holds for all* $k \geq 1$ *and when* $\mathcal{S}_k > 0$*, then* $\mathcal{S}_{\ell k} > 0$*, for all* $\ell \geq 1$.

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Then there exists a nonnegative matrix B such that for some positive integer q, B^q is a matrix with positive entries and such that its nonzero spectrum is $\lambda_1, \ldots, \lambda_r$.

We comment that the nonzero numbers $\lambda_1, \ldots, \lambda_n$ satisfying Theorem 1.4 may have to be augmented by a very large number of 0's so that, altogether, they form the spectrum of a nonnegative matrix. As an example we cite the 5–tuple $(3 + \epsilon, 3, -2, -2, -2)$ which, for sufficiently small $\epsilon > 0$, is not the spectrum of a 5×5 nonnegative matrix, but which for sufficiently large n is the nonzero spectrum of some $n \times n$ nonnegative matrix, see Boyle and Handelman [4].

In the papers by Keilson and Styan [8], Fiedler [5], and Ashley [2] the authors show independently and using different methods of proof that if the λ_i's are the eigenvalues of a nonnegative matrix A, then

$$\prod_{i=1}^{n}(\lambda - \lambda_i) \;\le\; \lambda^n - \lambda_1^n, \quad \forall \lambda \ge \lambda_1, \tag{1.2}$$

with equality for any $\lambda > \lambda_1$, if and only if A is the simple cycle matrix.

In Boyle and Handelman, [4, Question, p.311], the authors conjecture a stronger result than (1.2), namely, if $\lambda_1, \ldots, \lambda_r$ are all the nonzero eigenvalues of A, then

$$\prod_{i=1}^{r}(\lambda - \lambda_i) \;\le\; \lambda^r - \lambda_1^r, \quad \forall \lambda \ge \lambda_1. \tag{1.3}$$

Consider the following conjectures:

CONJECTURE 1.2 (WEAK CONJECTURE). *Let λ_i, $1 \le i \le n$, be complex numbers satisfying:*

(i) $|\lambda_i| \le \lambda_1$, *for all* $1 \le i \le n$.

(ii) *The moments* $\mathcal{S}_k := \sum_{i=1}^{n} \lambda_i^k \ge 0$, *for all positive integers* k.

Then for every $\lambda \ge \lambda_1$,

$$\prod_{i=1}^{n}(\lambda - \lambda_i) \;\le\; \lambda^n - \lambda_1^n. \tag{1.4}$$

Firstly, note that if Conjecture 1.2 is true, then it implies that the Boyle and Handelman conjecture ((1.3) with n instead of r) is true as we can take $\lambda_1, \ldots, \lambda_n$ to be the nonzero spectrum of the nonnegative matrix in question. In relation to this we remark that the conditions of Conjecture 1.2 are not sufficient for the λ_i's to be the nonzero eigenvalues of a square nonnegative matrix, even when appended by any number of zeros. As an example we give the 5–tuple $(3, 3, -2, -2, -2)$; see [3, p.88]. Secondly, in [9], Koltracht, Neumann, and Xiao show that (i) the conjecture is true for $n \le 5$, (ii) the conjecture is true when all the λ_i's are real, and (iii) in general there is a sequence of numbers $c_n > \lambda_1$, with $c_n \to \lambda_1$ as $n \to \infty$, such that (1.4) holds for all $\lambda \ge c_n$. In [1], Ambikkumar and Drury prove that if $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of a nonnegative matrix, with $\lambda_{r+1} = \ldots = \lambda_n = 0$, then (1.4) is true for $r \ge n/2$.

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Given Conjecture 1.2, most of our present paper is motivated by what can be said about the following stronger conjecture:

CONJECTURE 1.3 (STRONG CONJECTURE). *Let* λ_i, $1 \le i \le n$, *be complex numbers satisfying:*
(i) $\lambda_1 \ge |\lambda_i|$, *for all* $1 \le i \le n$.
(ii) $\mathcal{S}_k := \sum_{i=1}^n \lambda_i^k \ge 0$, *for* $1 \le k \le n-1$.
Then for every $\lambda \ge \lambda_1$,

$$\prod_{i=1}^n (\lambda - \lambda_i) \ \le \ \lambda^n - \lambda_1^n. \tag{1.5}$$

Our main difficulty in proving Conjecture 1.3 is that we do not know how to use the part of condition (i) which requires that $|\lambda_i| \le \lambda_1$ in conjunction with the conditions in (ii). We believe that knowing how to use the two conditions together is the key to proving the conjecture. Therefore in the present paper we shall assume on $\lambda_1, \ldots, \lambda_n$ that the following conditions hold:
(I) $\lambda_1 > 0$.
(II) $0 \le \mathcal{S}_k \le n\lambda_1^k$, for $1 \le k \le n-r$, for some $1 \le r \le n$.

Subsequently, r in (II) will be the multiplicity of λ_1 among the λ_i's. It is not hard to see that assumptions (I) and (II) alone are not sufficient to imply the Strong Conjecture (Conjecture 1.3), as a counterexample that we shall give in the next section shows (see Example 2.6). Nevertheless, in our main result we show that assumptions (I) and (II) imply that (1.5) is true for $\lambda \ge 6.75\lambda_1$. In the course of proving this result it will become clear that if the multiplicity of λ_1 among the λ_i's increases, then a stronger result than (1.5) is available.

In order to state our main result, some further notations are needed. For any two positive integers r and m, let

$$\rho_{r,m}(x,\lambda) \ = \ (x-\lambda)^r \sum_{i=0}^m \tbinom{r+i-1}{i} x^{m-i}\lambda^i. \tag{1.6}$$

It will become evident in Section 2 that this polynomial has the property that if $\delta_1, \ldots, \delta_n$ $(n = r+m)$ are the roots of $\rho_{r,m}$, then the moments $\sum_{i=1}^n \delta_i^k = 0$, for all $1 \le k \le m$. Note that $\rho_{1,n-1}(x,\lambda) = x^n - \lambda^n$. We are now ready to state our main result:

THEOREM 1.4. *Let* λ_i, $1 \le i \le n$, *be complex numbers satisfying* $\lambda_1 = \lambda_2 = \ldots = \lambda_r > 0$ *and* $0 \le \sum_{i=1}^n \lambda_i^k \le n\lambda_1^k$, *for* $1 \le k \le m := n-r$. *Then the inequality*

$$\prod_{i=1}^n (\lambda - \lambda_i) \ \le \ \rho_{r,m}(\lambda,\lambda_1), \quad \textit{for all} \ \ \lambda \ge 6.75\lambda_1, \tag{1.7}$$

holds. Moreover, if $r \ge n/2$, *then* (1.7) *is valid for all* $\lambda \ge \lambda_1$.

We note that as r increases, the right hand side of (1.7), where $\rho_{r,m}(\lambda,\lambda_1)$ is given in (1.6), decreases for all $\lambda \ge \lambda_1$, that is:

$$\rho_{r+1,m-1}(\lambda,\lambda_1) \ \le \ \rho_{r,m}(\lambda,\lambda_1), \quad \text{for all } \lambda \ge \lambda_1 \tag{1.8}$$

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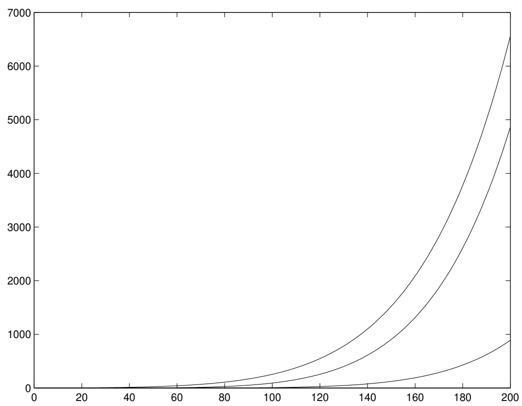
which easily follows from well known properties of the binomial coefficients. As an example for (1.8) consider the case where $n = 8$ and where $\lambda_1 = \ldots = \lambda_5; \lambda_6, \lambda_7, \lambda_8$, which are given by:

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ -0.1633 \\ 0.03495 + 0.06490i \\ 0.03495 - 0.06490i \end{bmatrix},$$

come from the eigenvalues of some 8×8 nonnegative matrix. The figure below represent a plot in the interval $[1, 3]$ using 200 equally spaced points of the graphs of:
(i) $\prod_{i=1}^{8}(x - \lambda_i)$ (the lower curve in the figure).
(ii) $\rho_{5,3}(x, 1) = (x - 1)^5 \sum_{i=0}^{3} \binom{5+i-1}{i} x^{3-i}$ (the middle curve in the figure).
(iii) $\rho_{1,7}(x, 1) = x^8 - 1$ (the top curve in the figure).



In Section 2 we develop some preliminary results which we shall need in the proof of Theorem 1.4. We devote Section 3 to proving Theorem 1.4. Section 4 discusses how can we lower the lower bound $6.75\lambda_1$ when $r < n/2$, but r is still fairly large. See Proposition 4.1 for the precise statement. Two special cases that we can derive from that proposition are as follows: If n is large enough and if $r > 0.41n$, then Theorem 1.4 holds for $\lambda \geq 5\lambda_1$. If n is large enough and if $r > 0.48n$ then Theorem 1.4 holds for $\lambda \geq 4\lambda_1$.

Theorem 1.4 and our work here leads us to conjecture the following:

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CONJECTURE 1.5 (CONJECTURE ST(r)). *Let λ_i, $1 \le i \le n$, be complex numbers satisfying that:*
(i) $\lambda_1 = \ldots = \lambda_r$ *and* $|\lambda_i| \le \lambda_1$, *for all* $i = 1, \ldots, n$.
(ii) $\mathcal{S}_k := \sum_{i=1}^n \lambda_i^k \ge 0$, *for* $1 \le k \le m := n - r$.
Then (1.7) *holds for all* $\lambda \ge \lambda_1$.

2. Preliminaries. Consider the formal equation in the indeterminates $s_1, s_2, \ldots$; $a_1, a_2, \ldots$; and t:

$$\exp\left(-\sum_{n=1}^{\infty} \frac{s_n t^n}{n}\right) \;=\; 1 + \sum_{n=1}^{\infty} a_n t^n \tag{2.1}$$

The left hand side has a meaning in the ring $\mathbb{Q}[s_1, s_2, \ldots][[t]]$ and the right hand side has a meaning in the ring $\mathbb{Q}[a_1, a_2, \ldots][[t]]$. On equating coefficients of t^n on both sides of (2.1), we see that on the one hand $a_n = A_n(s_1, \ldots, s_n)$ is a weighted homogeneous polynomial of degree n in $s_1, \ldots, s_n$, where for each $1 \le i \le n$, s_i has weight i. On the other hand one can express the s_n's as functions of the a_n's as follows: By taking logarithms, (2.1) is equivalent to:

$$-\sum_{n=1}^{\infty} \frac{s_n t^n}{n} \;=\; \log\left(1 + \sum_{n=1}^{\infty} a_n t^n\right). \tag{2.2}$$

In particular, $s_n = S_n(a_1, \ldots, a_n)$ is a weighted homogeneous polynomial of degree n in $a_1, \ldots, a_n$, where for each $1 \le i \le n$, a_i has weight i. Write $\underline{A} = (A_1, \ldots A_r)$ and $\underline{S} = (S_1, \ldots, S_r)$. Then it is clear from the discussion above that:

LEMMA 2.1. $\underline{A} \circ \underline{S} = \underline{S} \circ \underline{A} = Id$.

We shall next determine $\partial A_i / \partial s_j$, for all $1 \le j \le i$ and for $i = 1, 2, \ldots$. Differentiating both sides of (2.1) with respect to s_j, the following lemma is obtained:

LEMMA 2.2.

$$\frac{\partial}{\partial s_j}\left(1 + A_1 t + \ldots + A_n t^n\right) \;=\; -\frac{t^j}{j}\left(1 + A_1 t + \ldots + A_{n-j} t^{n-j}\right). \tag{2.3}$$

The following corollary to Lemma 2.2 can also be derived from Newton's identities which can be found, for example, in Householder [7, p.36]:

COROLLARY 2.3. *Under the above notation, we have that:*

$$A_k \;=\; \sum_{\substack{i_1 + 2i_2 + \cdots n i_n = k \\ i_1, \ldots, i_k \ge 0}} (-1)^{i_1 + i_2 + \cdots + i_n} \frac{s_1^{i_1} s_2^{i_2} \cdots s_n^{i_n}}{1^{i_1} 2^{i_2} \cdots n^{i_n} \cdot i_1! i_2! \cdots i_n!}, \quad k \ge 1. \tag{2.4}$$

Proof. Expand A_k in a Taylor series about $(s_1, \ldots, s_n) = (0, \ldots, 0)$ and apply Lemma 2.2. Alternatively, this can be seen from expanding the infinite product $\prod_{i=1}^{\infty} \exp(-s_n t^n / n)$. $\square$

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We now consider an application of Lemma 2.1 to polynomials. Let $z_1, \ldots z_n$ be complex numbers and consider the polynomial

$$f(z) \;=\; \prod_{i=1}^{n}(z-z_i) \;=\; z^n + a_1(f)z^{n-1} + \ldots + a_n(f).$$

Put

$$s_k(f) \;:=\; z_1^k + \ldots + z_n^k, \quad k = 1, 2, \ldots$$

Then $s_k(f)$ depends only on f and we have the following observation:

LEMMA 2.4. *For all $k = 1, \ldots, n$,*

$$a_k(f) \;=\; A_k(s_1(f), \ldots, s_k(f)) \text{ and } s_k(f) \;=\; S_k(a_1(f), \ldots, a_k(f)). \tag{2.5}$$

Proof. We have the following equalities from the formal power series in t:

$$\log\left(1 + a_1(f)t + \ldots a_n(f)t^n\right) \;=\; \log \textstyle\prod_{i=1}^{n}(1 - z_i t)$$

$$=\; -\sum_{k=1}^{\infty}\left(\sum_{i=1}^{n} z_i^k\right)\frac{t^k}{k} \;=\; -\sum_{k=1}^{\infty}\frac{s_k(f)}{k}t^k.$$

The equalities in (2.5) now follows from (2.2). ☐

For convenience we introduce the following notation. Let $P_n(t;\underline{s}) = 1 + A_1(\underline{s})t + \ldots + A_n(\underline{s})t^n$, where $\underline{s} = (s_1, s_2, \ldots)$. Notice that $P_n(t, \underline{s})$ depends only on $s_1, \ldots, s_n$. For a given number μ write $\underline{\mu}$ for $(\mu, \mu, \ldots)$. Notice that the for $k < n$, $P_k(t;\underline{s})$ is the truncation at t^k of $P_n(t;\underline{s})$.

EXAMPLES 2.5.

(i) $P_n(t;\underline{-1}) = 1 + t + t^2 + \ldots + t^n$. This is easy to see by noticing that this polynomial factors as $\prod_{j=1}^{n}(1 - \zeta_{n+1}^{j} t)$. Clearly $\sum_{j=1}^{n}\zeta_{n+1}^{kj} = -1$, for all $k = 1, 2, \ldots, n$.

(ii) $P_n(t;\underline{0}) = 1$.

(iii) If μ is a positive integer, then for $n \geq \mu$, $P_n(t;\underline{\mu}) = (1-t)^\mu$. If $n < \mu$, then truncating $(1-t)^\mu$ at t^n yields that $P_n(t;\underline{\mu})$.

(iv) More generally, for any integer μ, $P_n(t;\underline{\mu})$ is the truncation at t^n of the Taylor series of $(1-t)^\mu$. This is an easy consequence of (2.2); just expand $\log(1-t)^\mu$.

(v) $P_n(t;(\sigma, 0, 0, \ldots)) \;=\; 1 - \sigma t + (\sigma t)^2/2! - (\sigma t)^3/3! + \ldots + (-1)^n(\sigma t)^n/n!$. This follows immediately from the Taylor series of $e^{-\sigma t}$ and using (2.1).

We further define an involution on the set $\mathcal{P}_n(x)$ of polynomials of degree at most n in x, $f \mapsto \tilde{f}$ given by:

$$\tilde{f}(x) \;=\; x^n f(1/x).$$

If n is not specified, we take n to be equal to $\deg(f)$. Under this involution we have that if $P = P_n(t;\underline{s}) \in \mathcal{P}_n(t)$ is considered as a polynomial in t, where $\underline{s} = (\sigma_1, \sigma_2, \ldots)$ are given numbers, then

$$s_i(\tilde{P}) \;=\; \sigma_i, \quad i = 1, \ldots, n.$$

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EXAMPLE 2.6. Consider the polynomial $f_{11}(x) = (x-1)\sum_{i=0}^{10} \frac{(-9x)^{10-i}}{i!}$ $\in \mathcal{P}_{11}(x)$. Then $\widetilde{f_{11}}(t) = (1-t)P_{10}(t,(9,0,0,\ldots))$ and in particular $0 \le s_k(f_{11}) \le 10$, for $k = 1,2,\ldots,10$. On the other hand it is easy to see that $f_{11}(x) - (x^{11}-1)$ has a root $x_0 \approx 1.62314637$ and is positive for $1 < x < x_0$. Let $\lambda_1 = 1, \lambda_2, \ldots, \lambda_{11}$ be the roots $f_{11}(x)$. Then $\lambda_1, \ldots, \lambda_{11}$ satisfy Conditions (I) and (II) (which appear after (1.5)). This example shows that the Strong Conjecture (Conjecture 1.3) will not be true if we suppress the condition that all $|\lambda_i| \le \lambda_1$. However, in the next section we show that this condition is not necessary if $\lambda \ge 6.75\lambda_1$.

Finally, for every two integers $r > 0$ and $m \ge 0$, we define the polynomial:

$$\pi_{r,m}(x) \;=\; \sum_{i=0}^{m} \binom{r+i-1}{i} x^{m-i}. \tag{2.6}$$

It is easy to see using Taylor's formula for the function $\phi(x) = (1+x)^{-r}$, that $\widetilde{\pi_{r,m}}(x) = P_m(x, \underline{-r})$.

3. Proof of Theorem 1.4. In this section we prove Theorem 1.4. The statement of the theorem is homogeneous in $\lambda_1, \ldots, \lambda_n$. For this reason, there will be no loss of generality if we assume that $\lambda_1 = \ldots = \lambda_r = 1$. The theorem itself is a direct corollary of the following theorem:

THEOREM 3.1. *Let m and r be two positive integers. Suppose that $f(x)$ is a monic polynomial of degree m, which satisfies that $-r \le s_k(f) \le m$, for all $1 \le k \le m$. Then for every $x \ge 6.75$,*

$$f(x) \;\le\; \pi_{r,m}(x), \tag{3.1}$$

where $\pi_{r,m}(x)$ is the polynomial of degree m given in (2.6). *Moreover, if $r \ge m$, the above inequality holds for all $x \ge 1$.* With Theorem 3.1 at hand, Theorem 1.4 easily follows by letting $f(x) = \prod_{i=r+1}^{n}(x-\lambda_i)$ and multiplying both sides of (3.1) by $(x-1)^r$

Before we can prove Theorem 3.1, some estimates are required. Write $l = \max(r,m)$. Let T be the set of all monic polynomials $f(x)$ of degree at most m such that $-r \le s_k(f) \le m$, for all $1 \le k \le n$. Let T' be the set of all monic polynomials $f(x)$ of degree at most m such that $-l \le s_k(f) \le l$, for all $1 \le k \le m$. Clearly $T \subset T'$.

LEMMA 3.2. *For every $f \in T'$ and for any $1 \le q \le m$,*

$$|a_q(f)| \;\le\; \binom{l+q-1}{q} < \binom{l+q}{q}.$$

Proof. It is clear from equation (2.4) that

$$|A_q(s_1, s_2 \ldots)| \;\le\; A_q(-|s_1|, -|s_2|, \ldots).$$

It thus suffices to maximize a_q under the restrictions $-l \le s_k \le 0$ and $1 \le k \le m$. Under these restrictions, equation (2.4) shows that A_q is monotonically decreasing in

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each s_i. Consequently, $a_q(f) \leq A_q(-l, \ldots, -l)$. Now, by Example 2.5(iv), the Taylor polynomial of degree m of the function $(1-t)^{-l}$, $g(t)$, has $s_k(\tilde{g}) = -l$. Therefore,

$$\begin{aligned} A_q(-l, \ldots, -l) &= \frac{1}{q!}\frac{d^q}{dt^q}(1-t)^{-l} \mid_{t=0} \;=\; \frac{l(l+1)\cdots(l+q-1)}{q!} \\ &= \binom{l+q-1}{q} \;<\; \binom{l+q}{q}. \end{aligned}$$

This proves the lemma. □

Lemma 3.2 yields the following corollary:

Corollary 3.3. *Suppose that r and m, with $r \geq m$, are two positive integers and let $f(x)$ be monic polynomial of degree m which satisfies that $-r \leq s_k(f) \leq m$, for all $1 \leq k \leq m$. Then for all $x \geq 1$,*

$$f(x) \;\leq\; \pi_{r,m}(x).$$

Proof. If $r \geq m$, then $l = r$. Hence, by Lemma 3.2,

$$|a_q(f)| \;\leq\; \binom{r+q-1}{q} \;=\; a_q(\pi_{r,m})$$

and the proof follows. □

Corollary 3.3 establishes Theorem 3.1 for the case that $r \geq m$. From now on we will assume that $r < m$ and, in particular, that $l = \max\{r, m\} = m$. We have the following technical lemma:

Lemma 3.4. *For all integers $n \geq 1$ and $q > n/2$,*

$$\binom{n+q}{q} \;\leq\; (1.5)^n \cdot 3^q/2.$$

Proof. We first show by induction on n that for any two positive integers n and q,

$$\binom{n+q}{q} \;\leq\; \frac{(n+q)^{n+q}}{2n^n q^q}. \tag{3.2}$$

Denote the left hand side of this inequality by $a_{n,q}$ and the right hand side by $b_{n,q}$. Then for $n = 1$, $a_{1,q} = 1+q$, while $b_{1,q} = (q+1)(1+1/q)^q/2 \geq a_{1,q}$. Testing successive quotients yields that

$$\frac{a_{n+1,q}}{a_{n,q}} \;=\; \frac{n+q+1}{n+1} \;\leq\; \frac{n+q+1}{n+1} \cdot \frac{(1+1/(n+q))^{n+q}}{(1+1/n)^n} \;=\; \frac{b_{n+1,q}}{b_{n,q}}.$$

Apply now the induction hypothesis.

Assume now that $q > n/2$ and consider the function $\psi(x) = (1+x)^{1+x}/x^x$, for $x \geq 1/2$. It is easily checked that $-\log\psi(x)$ is convex. Using the linear approximation at $x = 1/2$, it follows that for all $x \geq 1/2$:

$$\log\psi(x) \;\leq\; \log\psi(1/2) + \log 3 \cdot (x - 1/2),$$

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which implies that $\psi(x) \leq 1.5 \cdot 3^x$. On the other hand, (3.2) reads $\binom{n+q}{q} < \psi(q/n)^n/2$. The lemma now readily follows. □

We are now ready to prove Theorem 3.1.

Proof. [of Theorem 3.1] Fix a number $x \geq 1$. Consider T as a compact subset of $\mathcal{P}_m(z)$ with respect to the Euclidean topology. The function $g \mapsto g(x)$ is a continuous real function on T. Hence there exists an $f \in T$, such that:

$$f(x) \;=\; \max_{g \in T} g(x). \tag{3.3}$$

We want to show that necessarily $s_k(f) = -r$, for all $1 \leq k \leq m$, which is equivalent to $f(x) = \pi_{r,m}(x)$. Clearly we have that

$$f(x) \;\geq\; \pi_{r,m}(x). \tag{3.4}$$

Consider any $g \in T$ as a polynomial in x as well as in the indeterminates $s_1, \ldots, s_m$. We will show that for any $1 \leq i \leq m$, the numbers

$$\xi_i := \frac{\partial g}{\partial s_i} \quad \text{evaluated at } x \text{ and } s_j = s_j(f),\ 1 \leq j \leq m,$$

are negative. Since f has the maximum property (3.3), if follows that $s_j(f) = -r$, for all $1 \leq j \leq m$, and the theorem will be proved.

Using Lemma 2.2 above, we have that

$$\xi_i \;=\; -\frac{f(x) - a_m(f) - a_{m-1}(f)x - \ldots - a_{m-i+1}(f)x^{i-1}}{ix^i}. \tag{3.5}$$

By (3.4), it suffices to show that

$$a_m(f) + a_{m-1}(f)x + \ldots + a_{m-i+1}(f)x^{i-1} \;\leq\; \pi_{r,m}(x),$$

as this would imply that $\xi_i < 0$.

Let $\mu = \lfloor m/2 \rfloor$ and consider first the case $i \leq m/2$. By a combination of Lemma 3.2 with Lemma 3.4 we have that:

$$a_m(f) + a_{m-1}(f)x + \ldots + a_{m-i+1}(f)x^{i-1} \tag{3.6}$$

$$\leq \sum_{q=\mu+1}^{m} (1.5)^m 3^q x^{m-q}/2 \;=\; \frac{(1.5)^m 3^m}{2} \cdot \frac{(x/3)^{m-\mu} - 1}{x/3 - 1}. \tag{3.7}$$

Assume for the moment that x satisfies that:

$$\frac{(1.5)^m 3^m}{2} \cdot \frac{(x/3)^{m-\mu} - 1}{x/3 - 1} \;\leq\; \pi_{r,m}(x). \tag{3.8}$$

Then by (3.6), (3.8), (3.5), and (3.4), it follows that $\xi_i < 0$ whenever $i \leq m/2$, and so $s_i(f) = -r$ for all $i \leq m/2$. Consequently,

$$a_i(f) \;=\; \binom{r+i-1}{i}, \quad \text{for all } i \leq m/2. \tag{3.9}$$

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Suppose now that $i > m/2$. Then using (3.9) together with (2.3) shows that,

$$-i\xi_i \;=\; x^{m-i} + \sum_{j=1}^{m-i} \tbinom{r+j-1}{j} x^{m-i-j},$$

which shows that $\xi_i < 0$ also for $i > m/2$. This proves that the inequality (3.1) is true for every $x \geq 1$ that satisfies (3.8).

It remains to show that every $x \geq 6.75$ satisfies (3.8). Indeed, for $x \geq 6.75$ we simply have that

$$\frac{(1.5)^m 3^m}{2} \cdot \frac{(x/3)^{m-\mu} - 1}{x/3 - 1} \;\leq\; \frac{(4.5)^m (x/3)^{m-\mu}}{2.5} \;\leq\; x^m \;\leq\; \pi_{r,m}(x). \quad (3.10)$$

The proof of the theorem is now complete. ☐

4. Improving the lower bound $6.75\lambda_1$ when the multiplicity of λ_1 is large. Using the notation of Theorem 3.1, we will show now that the constant 6.75 in (1.7) can be reduced when r is large. Recall that in the proof of Theorem 3.1 we have observed that the inequality (1.7) is true whenever $x \geq 1$ and satisfies (3.8). For the purpose of proving Theorem 3.1 it was sufficient to bound from above the left hand side of (3.8) by x^m. When r is large, $\pi_{r,m}(x)$ is significantly larger than x^m and we can take advantage of this fact to establish (3.8) (and thus (1.7)) for values of x smaller than 6.75. Recall that when $r \geq m$, Theorem 3.1 holds for all $x \geq 1$. We will thus assume from now on that $r < m$. We first need an estimate on $\pi_{r,m}(x)$.

Let $f = \widetilde{\pi_{r,m}}$ so that $f(t)$ is the mth Taylor polynomial of $(1-t)^{-r}$ at $t = 0$. Write $(1-t)^{-r} = \sum_{k=0}^{\infty} a_k t^k$ so that $a_k = \binom{k+r-1}{r}$. Suppose now that $t = \frac{1}{2} - \frac{1}{2}\epsilon$, where $\epsilon > 0$. Then, using the fact that for $k \geq m > r$, $a_{k+1}/a_k < 2$, we have that:

$$|(1+t)^{-r} - f(t)| \;=\; \sum_{k=m+1}^{\infty} a_k t^k \;\leq\; a_{m+1} t^{m+1} \sum_{k=0}^{\infty} (2t)^k \;=\; \frac{a_{m+1} t^{m+1}}{\epsilon}. \quad (4.1)$$

Letting $t = 1/x$, (3.8) can be rewritten as

$$g(t) \;:=\; \frac{1.5^m}{2} \cdot (3t)^{\mu+1} \cdot \frac{1 - (3t)^{m-\mu}}{1 - 3t} \;\leq\; f(t).$$

Using the estimate (4.1) on $f(t)$, it suffices for t to satisfy that

$$h(t) \;:=\; g(t) + \frac{a_{m+1} t^{m+1}}{\epsilon} \;\leq\; (1-t)^{-r}. \quad (4.2)$$

From now on assume that $t < 1/3$ and $m \geq 5$. In particular then $\epsilon \geq 1/3 > 1/6$. Since $a_{m+1} < 2^{m+r-1} \leq 2^{2m-1}$, this shows that $a_{m+1} t^{m+1}/\epsilon \leq 3(4t)^m$.

On the other hand, $[1 - (3t)^{m-\mu}]/(1-3t)$ can be bounded above by $(m-\mu)$ and $1.5^m (3t)^{\mu+1} > (4t)^m$. Now,

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$$h(t) \;=\; \frac{1.5^m}{2}\cdot(3t)^{\mu+1}\cdot\frac{1-(3t)^{m-\mu}}{1-3t}+\frac{a_{m+1}t^{m+1}}{\epsilon} \;\le\; \frac{m-\mu}{2}1.5^m(3t)^{\mu+1}+3(4t)^m$$
$$\le\; \left(\frac{m-\mu}{2}+3\right)1.5^m(3t)^{\mu+1} \;\le\; m\cdot 1.5^m(3t)^{m/2},$$

where in the last inequality we have used the assumption that $m \ge 5$. It thus suffices for t to satisfy that:

$$m\cdot 1.5^m(3t)^{m/2} \;\le\; (1-t)^{-r}.$$

Taking logarithms of both sides we obtain that

$$m\psi(t) - \frac{\log m}{\log(1-t)} \;\le\; r, \tag{4.3}$$

where

$$\psi(t) \;=\; \frac{\log(1.5)+\frac{1}{2}\log(3t)}{-\log(1-t)}.$$

The function $\psi(t)$ satisfies that $\psi(1/6.75) = 0$, $\psi(1/3) = 1$, and $\psi(t) < 0$, for $t < 1/6.75$. Moreover, $\psi(t)$ is monotonically increasing on the interval $[1/6.75, 1/3]$. To see this, notice that

$$\phi(t) \;:=\; \log^2(1-t)\psi'(t) \;=\; \frac{-\log(1-t)}{2t} - \frac{\log 1.5 + 0.5\log(3t)}{1-t}$$

which shows that $\phi(1/3) = \psi'(1/3) = 0$, while

$$\phi'(t) \;=\; \frac{\log(1-t)}{2t^2} - \frac{\log 1.5 + 0.5\log(3t)}{(1-t)^2}$$

is clearly negative in $[1/6.75, 1/3]$. Consequently, $1/3$ is the only zero of $\phi(t)$ and $\psi'(t)$ in this interval and so $\psi(t)$ is monotone.

Suppose now that $1/6.75 \le t \le 1/3$. Then $\log(1-t) \le -1/7$. By (4.3) it suffices for t and r to satisfy that:

$$m\psi(t) + 7\log m \;\le\; r. \tag{4.4}$$

Equation (4.4) and the ensuing discussion following equation (4.3) show the following:

PROPOSITION 4.1. *Let $\theta(x) = \psi(1/x)$ be defined on the interval $[3, \infty)$. Then:*

(i) *$\theta(x)$ is monotone decreasing in $[3, 6.75]$ with $\theta(3) = 1$ and $\theta(6.75) = 0$. Moreover, when $x > 6.75$, $\theta(x)$ is negative.*

(ii) *Given $3 \le x_0 \le 6.75$, suppose that $r \ge m\theta(x_0) + 7\log m$ and $m \ge 5$. Then Theorem 3.1 and, in particular, inequality (1.7) hold for all $x \ge x_0$.* □

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Proof. (ii) For $x \geq 6.75$ the conclusion follows from Theorem 3.1. When $x_0 \leq x < 6.75$, note that

$$r \;\geq\; m\theta(x_0) + 7\log m \;\geq\; m\theta(x) + 7\log m.$$

Now equation (4.4) is satisfied for $t = 1/x$ and thus (1.7) holds for x. □

COROLLARY 4.2. *Let* $x_0 \geq 3$. *Then for any* $\delta > 0$*, there exists an integer* $m_0 = m_0(\delta)$ *such that if* $m > m_0(\delta)$ *and* $r \geq m[\theta(x_0) + \delta]$*, then Theorem* 3.1 *and, in particular, inequality* (1.7) *hold for all* $x \geq x_0$. □

We now recall that Theorem 3.1 is true for $x \geq x_0$, if and only if Theorem 1.4 is true for all $\lambda \geq x_0\lambda_0$. Thus we can apply Proposition 4.1 and Corollary 4.2 to the setting of Theorem 1.4 to yield the following two conclusions on recalling that $n = r + m$:

EXAMPLES 4.3.

(i) *If* n *is sufficiently large and if* $r \geq 0.4047n$*, then Theorem* 1.4 *holds for* $\lambda \geq 5\lambda_1$.

(ii) *If* n *is sufficiently large and if* $r \geq 0.4764n$*, then Theorem* 1.4 *holds for* $\lambda \geq 4\lambda_1$.

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