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# ESTIMATION OF THE MAXIMUM MULTIPLICITY OF AN EIGENVALUE IN TERMS OF THE VERTEX DEGREES OF THE GRAPH OF A MATRIX *

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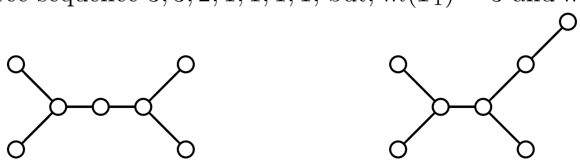
**Abstract.** The maximum multiplicity among eigenvalues of matrices with a given graph cannot generally be expressed in terms of the degrees of the vertices (even when the graph is a tree). Given are best possible lower and upper bounds, and characterization of the cases of equality in these bounds. A by-product is a sequential algorithm to calculate the exact maximum multiplicity by simple counting.

**Key words.** Eigenvalues, multiplicity, symmetric matrix, tree, vertex degrees.

**AMS subject classifications.** 15A18, 15A57, 05C50, 05C05, 05C07

**1. Introduction.** Throughout let $T$ be a tree on $n$ vertices, whose degrees are $d_1 \geq d_2 \geq \cdots \geq d_k \geq 3 > d_{k+1} \geq \cdots \geq d_n$, and let $\mathcal{S}(T)$ be the set of all real symmetric matrices whose graph is $T$. The diagonal entries of matrices in $\mathcal{S}(T)$ are unconstrained by $T$. For any $n$-by-$n$ matrix $A$, $m_A(\lambda)$ denotes the (algebraic) multiplicity of $\lambda$ as an eigenvalue of $A$, and $m(A) = \max_{\lambda\in\mathbb{C}} m_A(\lambda)$, the maximum multiplicity of an eigenvalue of $A$. Then, we let $m(T) = \max_{A\in\mathcal{S}(T)} m(A)$, the maximum multiplicity of an eigenvalue among matrices whose graph is $T$.

It was shown in [2] that $m(T)$ equals the *path cover number*, $p(T)$, of $T$, the smallest number of vertex disjoint paths of $T$ that cover all the vertices of $T$. It was also shown in [2] that $m(T) = \max[p - q]$ over all ways in which $q$ vertices may be removed from $T$ so as to leave $p$ paths (i.e. the subgraph of $T$ induced by the other $n-q$ vertices consists of $p$ paths, an isolated vertex counting as a path). However, neither of these formulae for $m(T)$, though polynomially computable from $T$, is expressible in terms of overt parameters of $T$, such as the vertex degrees; in fact, there can be no formula for $m(T)$ strictly in terms of the vertex degrees. There are trees with the same degree sequence but different path cover number. For example, the following trees have degree sequence $3, 3, 2, 1, 1, 1, 1$, but, $m(T_1) = 3$ and $m(T_2) = 2$.



$T_1$ $T_2$

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Our purpose here is to give simple, tight bounds for $m(T)$ in terms of vertex degrees only, and then to indicate for which trees these bounds are exact. For this purpose, let $H = H(T)$ be the subgraph of $T$ induced by the $k$ vertices of degree at least 3 in $T$. If $H$ consists of a collection of disjoint vertices (possibly empty), we call $T$ *segregated*; in general $e = e(H)$ is the number of edges present in $H$, so that $e = 0$ is equivalent to $T$ being segregated.

**2. Main results.** Our principal result is Theorem 2.1 whose proof consists of the pendant lemmas. Given a graph $G$ we denote by $\nu(G)$ the vertex set of $G$ and by $\deg_G(v)$ the degree of a vertex $v$ in $G$.

THEOREM 2.1. *Let $T$ be a tree. Then,*

$$1 + \sum_{i=1}^{k}(d_i - 2) - e \leq m(T) \leq 1 + \sum_{i=1}^{k}(d_i - 2).$$

*Equality occurs in the right hand inequality if and only if $T$ is segregated. Equality occurs in the left hand inequality if and only if $\nu(H) = \emptyset$ or $\deg_T(v) \geq \deg_H(v) + 2$ for all vertices $v$ of $H$.*

As usual, given a graph $G$ with vertex set $\nu(G)$ and $U \subset \nu(G)$, $G - U$ denotes the subgraph of $G$ induced by the vertex set $\nu(G) \setminus U$. Denote by $p_U$ the number of components of $G - U$ and by $q_U$ the cardinality of $U$. If $U = \{u\}$ and $G$ is a tree, the components of $G - U$ (or $G - u$) are called *branches* of $G$ at $u$.

As we will see, the lower bound presented in Theorem 2.1 is $p_{\nu(H)} - q_{\nu(H)}$.

LEMMA 2.2. *Let $F$ be a forest with $c$ components (trees). Let $F'$ be any subtree of $F$ with vertex set $\nu(F') = \{u_1, \ldots, u_q\}$, $q \geq 1$. Then, $F - \nu(F')$ has*

$$c + \sum_{i=1}^{q} [\deg_F(u_i) - \deg_{F'}(u_i)] - 1$$

*components.*

*Proof.* For each $u_i \in \nu(F')$, define $\delta(u_i) = \deg_F(u_i) - \deg_{F'}(u_i)$, the number of vertices in $F$, but not in $F'$, that are adjacent to $u_i$. Since $F$ is acyclic and $F'$ is a subtree of one component $F''$ of $F$, $\sum_{i=1}^{q} \delta(u_i)$ counts the (distinct) branches of $F''$ that are left as components when $\nu(F')$ is removed from $F$. Since $F''$ was a component of $F$, there are then $\sum_{i=1}^{q} \delta(u_i) - 1$ additional components, $c + \sum_{i=1}^{q} \delta(u_i) - 1$ in all, after the removal of $\nu(F')$ from $F$. □

LEMMA 2.3. *Let $F$ be a forest with $c$ components. Consider a nonempty subset $U$ of $\nu(F)$ and $F'$ the subgraph of $F$ induced by $U$. Assume that $U = \{u_1, \ldots, u_q\}$ and $F'$ has $e(F')$ edges. Then, the forest $F - U$ has*

$$p_U = c + \sum_{i=1}^{q}[\deg_F(u_i) - 1] - e(F')$$

*components.*

*Proof.* Assume that $F'$ has $s$ components (and hence $q - s$ edges), and that $\{u'_1, \ldots, u'_k\}$ are the vertices of a component $F''$ of $F'$. Then by Lemma 2.2, the

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removal of $F''$ from $F$ will increase the number of components by $\left[\sum_{i=1}^{k}[\deg_F(u_i') - \deg_{F''}(u_i')] - 1\right]$. Thus the removal of all $s$ components of $F'$ from $F$ will increase the number of components to

$$\begin{aligned} p_U &= c + \sum_{i=1}^{q}[\deg_F(u_i) - \deg_{F'}(u_i)] - s \\ &= c + \sum_{i=1}^{q}\deg_F(u_i) - \sum_{i=1}^{q}\deg_{F'}(u_i) - s \\ &= c + \sum_{i=1}^{q}[\deg_F(u_i) - 1] + q - 2e(F') - s. \end{aligned}$$

Since $e(F') = q - s$ the result follows. ∎

By Lemma 2.3, we have $p_{\nu(H)} = 1 + \sum_{i=1}^{k}(d_i - 1) - e$. Since $q_{\nu(H)} = k$ it follows that

$$p_{\nu(H)} - q_{\nu(H)} = 1 + \sum_{i=1}^{k}(d_i - 2) - e. \tag{2.1}$$

Lemma 2.4. *Let $F$ be a forest with $c$ components. Let $p$ be the number of components when $q$ vertices each of degree at most 2 are removed from $F$. Then, $p - q \leq c$. Equality occurs if and only if all the vertices removed from $F$ have degree 2 and no two such vertices are adjacent.*

*Proof.* By Lemma 2.3, the removal of $q$ vertices $u_1, \ldots, u_q$ from $F$ increase the number of components by $\sum_{i=1}^{q}[\deg_F(u_i) - 1] - e(F')$ where $F'$ is the subgraph of $F$ induced by $u_1, \ldots, u_q$. Since all removed vertices from $F$ have degree at most 2, the number of remaining components is $p \leq c + q$, with equality if and only if all the vertices removed from $F$ have degree 2 and no two such vertices are adjacent. ∎

Given a tree $T$, let $M$ be a subset of $\nu(T)$ such that $p_M - q_M = \max[p_I - q_I]$ with $I \subseteq \nu(T)$. From Lemma 2.4, there is no vertex of degree 1 in $M$. Lemma 2.4 also implies that we can assume $M$ contains no vertices of degree 2. In other words, there is a subset $M$ of $\nu(H)$ such that $p_M - q_M = \max[p_I - q_I]$ with $I \subseteq \nu(T)$ i.e., $p_M - q_M = m(T)$.

Lemma 2.5. *Let $T$ be a tree. Suppose that $\nu(H) \neq \emptyset$ and $\deg_T(v) - \deg_H(v) \geq 2$ for all vertices $v$ of $H$. Then, $p_I - q_I \leq p_{\nu(H)} - q_{\nu(H)}$ for all $I \subseteq \nu(H)$.*

*Proof.* Suppose to the contrary, i.e., that there exist $I \subset \nu(H)$ such that $p_I - q_I > p_{\nu(H)} - q_{\nu(H)}$. Then, there exist $I'$ and $I''$ such that $I \subseteq I' \subset I'' \subseteq \nu(H)$, with $I'' = I' \cup \{v\}$ and $v \notin I'$, such that $p_{I'} - q_{I'} > p_{I''} - q_{I''}$. Since $q_{I'} = q_{I''} - 1$, $p_{I'} - q_{I'} > p_{I''} - q_{I''}$ implies that $p_{I'} \geq p_{I''}$. But, since $T - I'$ has $p_{I'}$ components and $v$ is a vertex of one component $T'$ of $T - I'$ with $\deg_{T'}(v) \geq 2$ (remember that $\delta(v) \geq 2$), by Lemma 2.2 the removal of $v$ from $T'$ gives at least 1 additional components. Then, $p_{I''} \geq p_{I'} + 1$ gives a contradiction. ∎

Lemma 2.6. *Let $T$ be a tree. Then, $m(T) \geq 1 + \sum_{i=1}^{k}(d_i - 2) - e$, with equality if and only if $\nu(H) = \emptyset$ or $\deg_T(v) \geq \deg_H(v) + 2$ for all vertices $v$ of $H$.*

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*Proof.* First, observe that if $\nu(H) = \emptyset$ i.e., $T$ is a path, then $k = 0$, $e = 0$ and $1 + \sum_{i=1}^{k}(d_i - 2) - e = 1$, which is the path cover number when $T$ is a path. Suppose now that $\nu(H) \neq \emptyset$ and assume that $\deg_T(v) \geq \deg_H(v) + 2$ for all vertices $v$ of $H$. Consider a vertex set $M$ such that $p_M - q_M = m(T)$ with $M \subseteq \nu(H)$. Since $M \subseteq \nu(H)$ and $\deg_T(v) \geq \deg_H(v) + 2$ for all vertices $v$ of $H$ then, from Lemma 2.5, $p_M - q_M \leq p_{\nu(H)} - q_{\nu(H)}$. Then, $m(T) = p_{\nu(H)} - q_{\nu(H)}$ and, by (2.1), $p_{\nu(H)} - q_{\nu(H)} = 1 + \sum_{i=1}^{k}(d_i - 2) - e$. Therefore, $m(T) = 1 + \sum_{i=1}^{k}(d_i - 2) - e$. Thus, if $\deg_T(v) \geq \deg_H(v) + 2$ for all vertices $v$ of $H$, then equality occurs.

It suffices to prove that if there is a vertex $v$ in $\nu(H)$ such that $\delta(v) = \deg_T(v) - \deg_H(v) \leq 1$, then $m(T) > 1 + \sum_{i=1}^{k}(d_i - 2) - e$. Consider the set $H^* = \nu(H) \setminus \{v\}$ of vertices of $T$. Then, $T - H^*$ has $p_{H^*}$ components and $q_{H^*} = q_{\nu(H)} - 1$. Since $v \notin H^*$, $v$ is a vertex of one component $T'$ of $T - H^*$ and, the removal of $v$ from $T'$ gives $\delta(v) - 1$ "additional" components. Then, $p_{\nu(H)} = p_{H^*} + \delta(v) - 1$ and since $\delta(v) \leq 1$, $p_{H^*} \geq p_{\nu(H)}$. Thus, $p_{H^*} - q_{H^*} \geq p_{\nu(H)} - q_{H^*}$ and since $q_{H^*} = q_{\nu(H)} - 1$, we have $p_{H^*} - q_{H^*} > p_{\nu(H)} - q_{\nu(H)}$. Then, $m(T) > p_{\nu(H)} - q_{\nu(H)}$ and by (2.1), $p_{\nu(H)} - q_{\nu(H)} = 1 + \sum_{i=1}^{k}(d_i - 2) - e$. Therefore, $m(T) > 1 + \sum_{i=1}^{k}(d_i - 2) - e$. ◻

Lemma 2.7. *Let $T$ be a tree. Then $m(T) \leq 1 + \sum_{i=1}^{k}(d_i - 2)$, with equality occurring if and only if $T$ is segregated.*

*Proof.* If $\nu(H) = \emptyset$ then $T$ is a path, and hence a segregated tree. In this case, $k = 0$ and, $1 + \sum_{i=1}^{k}(d_i - 2) = 1$, which is the path cover number when $T$ is a path. Suppose now that $\nu(H) \neq \emptyset$. If $T$ is segregated, for all vertices $v$ of $H$, $\deg_H(v) = 0$ and $\deg_T(v) \geq 3 > \deg_H(v) + 2$. From Lemma 2.6, since $e = 0$ it follows that $m(T) = 1 + \sum_{i=1}^{k}(d_i - 2)$. Thus, if $T$ is segregated, equality occurs.

It suffices to prove that if $T$ is a tree that is not segregated then, $m(T) < 1 + \sum_{i=1}^{k}(d_i - 2)$. Consider a vertex set $M$ such that $p_M - q_M = m(T)$ and $M \subseteq \nu(H)$. From Lemma 2.3, $p_M - q_M = 1 + \sum_{v \in M}[\deg_T(v) - 2] - e(M)$ where $e(M)$ is the number of edges of the subgraph of $T$ induced by the $q_M$ vertices in $M$. If $M = \nu(H)$, since $T$ is not segregated, there are, at least, two adjacent vertices of $T$ in $M$ which implies $e(M) > 0$. Therefore $p_M - q_M < 1 + \sum_{v \in \nu(H)}[\deg_T(v) - 2]$ i.e., $p_M - q_M < 1 + \sum_{i=1}^{k}(d_i - 2)$. If $M \subset \nu(H)$, $e(M) \geq 0$ and then, $p_M - q_M \leq 1 + \sum_{v \in M}[\deg_T(v) - 2]$. Since $M \subset \nu(H)$ and $\deg_T(v) \geq 3$ for all $v \in \nu(H)$, $\sum_{v \in M}[\deg_T(v) - 2] < \sum_{v \in \nu(H)}[\deg_T(v) - 2]$, so that $p_M - q_M < 1 + \sum_{i=1}^{k}(d_i - 2)$. ◻

The proof of Theorem 2.1 is now complete.

**3. An algorithm.** We close by giving a simple algorithm to compute $m(T)$ by computing the path cover number of $T$.

Lemma 3.1. *Let $T$ be a tree and let $M$ be a subset of $\nu(H)$ such that $p_M - q_M = m(T)$. Then, there is no vertex of degree at least 3 in $T - M$.*

*Proof.* Let $T_{(M)} = T - M$ and suppose that $v \notin M$ and $\deg_{T_{(M)}}(v) \geq 3$. Consider the set $M' = M \cup \{v\}$ of vertices of $T$ and the graph $T_{(M')} = T - M'$ $(= T_{(M)} - v)$. From Lemma 2.2, $T_{(M')}$ has $p_{M'} = p_M + \deg_{T_{(M)}}(v) - 1$ components and $q_{M'} = q_M + 1$. Since $\deg_{T_{(M)}}(v) \geq 3$, $p_{M'} \geq p_M + 2$, and $p_{M'} - q_{M'} \geq p_M + 2 - q_{M'} = p_M - q_M + 1$. Thus, $p_{M'} - q_{M'} > p_M - q_M$ gives a contradiction. ◻

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LEMMA 3.2. *Let $T$ be a tree and let $M$ be a subset of $\nu(H)$ such that $p_M - q_M = m(T)$. If $v \in \nu(H)$ is such that $\delta(v) = \deg_T(v) - \deg_H(v) \geq 3$, then $v \in M$.*

*Proof.* Let $T_{(M)} = T - M$ and suppose that $v \notin M$ and $\delta(v) \geq 3$. Remember that $M \subseteq \nu(H)$ thus, $\deg_{T_{(M)}}(v) \geq \delta(v) \geq 3$. From Lemma 3.1, $\deg_{T_{(M)}}(v) \geq 3$ gives a contradiction. Therefore, $v \in M$. □

LEMMA 3.3. *Let $T$ be a tree and let $M$ be a subset of $\nu(H)$ such that $p_M - q_M = m(T)$. For any $v \in \nu(H)$ such that $\delta(v) = \deg_T(v) - \deg_H(v) = 2$, there is a subset $M'$ of $\nu(H)$ such that $M \subseteq M'$, $v \in M'$ and $p_{M'} - q_{M'} = m(T)$.*

*Proof.* Let $T_{(M)} = T - M$ and suppose that $v \in \nu(H)$ with $\delta(v) = 2$. If $v \in M$, let $M' = M$. Suppose now that $v \notin M$. Recall that $M \subseteq \nu(H)$; then $\deg_{T_{(M)}}(v) \geq \delta(v)$. Consider the set $M' = M \cup \{v\}$ and graph $T_{(M')} = T - M'$ ($= T_{(M)} - v$). From Lemma 2.2, $T_{(M')}$ has $p_{M'} = p_M + \deg_{T_{(M)}}(v) - 1$ components and $q_{M'} = q_M + 1$. Then, $p_{M'} - q_{M'} = p_M - q_M + \deg_{T_{(M)}}(v) - 2$.

By Lemma 3.1, $\deg_{T_{(M)}}(v) < 3$, hence $\deg_{T_{(M)}}(v) = 2$ in which case $p_{M'} - q_{M'} = p_M - q_M$. Therefore, $v \in M'$ and $p_{M'} - q_{M'} = m(T)$. □

We note that Lemmas 3.2 and 3.3 indicate how to inductively compute a subset $M'$ of $\nu(H)$ such that $p_{M'} - q_{M'} = m(T)$. By Lemma 2.4, given any tree $T$, there is a subset $M$ of $\nu(H)$ such that $p_M - q_M = m(T)$ and, by Lemmas 3.2 and 3.3, there is always a subset $M'$ of $\nu(H)$, containing all vertices $v$ of $T$ with $\delta(v) \geq 2$, such that $M \subseteq M'$ and $p_{M'} - q_{M'} = m(T)$.

ALGORITHM 3.4. *Given a tree $T$, consider the subgraph $H$ of $T$ induced by the vertices of degree at least 3. Remove from $T$ all vertices $v$ of $H$ such that $\delta(v) \geq 2$.*

This algorithm is applied to the initial tree $T$ and then to each of the resulting components and so on. Let $M'$ be the set of vertices removed via repeated application of Algorithm 3.4. By Lemmas 3.2 and 3.3, all vertices $v$ of $T$ with $\delta(v) \geq 2$ are in $M'$. If there is a vertex of degree at least 3 in $T - M'$, again by Lemmas 3.2 and 3.3, the application of the Algorithm 3.4 to each of the components of $T - M'$ (each one is a tree) allows us to include new vertices in $M'$ in route to maximizing $p_{M'} - q_{M'}$. By Lemma 3.1, the process stops when there is no vertex of degree at least 3. The set of removed vertices from $T$ gives $M'$, the cardinality of $M'$ gives $q_{M'}$ and the number of components of $T - M'$ (each of which is a path) gives $p_{M'}$. Finally, the maximum multiplicity of an eigenvalue among matrices whose graph is $T$, $m(T)$, is equal to $p_{M'} - q_{M'}$.

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