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ESTIMATION OF THE MAXIMUM MULTIPLICITY OF AN EIGENVALUE IN TERMS OF THE VERTEX DEGREES OF THE GRAPH OF A MATRIX *

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Abstract. The maximum multiplicity among eigenvalues of matrices with a given graph cannot generally be expressed in terms of the degrees of the vertices (even when the graph is a tree). Given are best possible lower and upper bounds, and characterization of the cases of equality in these bounds. A by-product is a sequential algorithm to calculate the exact maximum multiplicity by simple counting.

Key words. Eigenvalues, multiplicity, symmetric matrix, tree, vertex degrees.

AMS subject classifications. 15A18, 15A57, 05C50, 05C05, 05C07

1. Introduction. Throughout let T be a tree on n vertices, whose degrees are $d_1 \geq d_2 \geq \cdots \geq d_k \geq 3 > d_{k+1} \geq \cdots \geq d_n$, and let $\mathcal{S}(T)$ be the set of all real symmetric matrices whose graph is T. The diagonal entries of matrices in $\mathcal{S}(T)$ are unconstrained by T. For any n-by-n matrix A, $m_A(\lambda)$ denotes the (algebraic) multiplicity of λ as an eigenvalue of A, and $m(A) = \max_{\lambda \in \mathbb{C}} m_A(\lambda)$, the maximum multiplicity of an eigenvalue of A. Then, we let $m(T) = \max_{A \in \mathcal{S}(T)} m(A)$, the maximum multiplicity of an eigenvalue among matrices whose graph is T.

It was shown in [2] that $m(T)$ equals the *path cover number*, $p(T)$, of T, the smallest number of vertex disjoint paths of T that cover all the vertices of T. It was also shown in [2] that $m(T) = \max[p - q]$ over all ways in which q vertices may be removed from T so as to leave p paths (i.e. the subgraph of T induced by the other $n-q$ vertices consists of p paths, an isolated vertex counting as a path). However, neither of these formulae for $m(T)$, though polynomially computable from T, is expressible in terms of overt parameters of T, such as the vertex degrees; in fact, there can be no formula for $m(T)$ strictly in terms of the vertex degrees. There are trees with the same degree sequence but different path cover number. For example, the following trees have degree sequence $3, 3, 2, 1, 1, 1, 1$, but, $m(T_1) = 3$ and $m(T_2) = 2$.

T_1 T_2

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Our purpose here is to give simple, tight bounds for $m(T)$ in terms of vertex degrees only, and then to indicate for which trees these bounds are exact. For this purpose, let $H = H(T)$ be the subgraph of T induced by the k vertices of degree at least 3 in T. If H consists of a collection of disjoint vertices (possibly empty), we call T *segregated*; in general $e = e(H)$ is the number of edges present in H, so that $e = 0$ is equivalent to T being segregated.

2. Main results. Our principal result is Theorem 2.1 whose proof consists of the pendant lemmas. Given a graph G we denote by $\nu(G)$ the vertex set of G and by $\deg_G(v)$ the degree of a vertex v in G.

THEOREM 2.1. *Let T be a tree. Then,*

$$1 + \sum_{i=1}^{k}(d_i - 2) - e \leq m(T) \leq 1 + \sum_{i=1}^{k}(d_i - 2).$$

Equality occurs in the right hand inequality if and only if T is segregated. Equality occurs in the left hand inequality if and only if $\nu(H) = \emptyset$ or $\deg_T(v) \geq \deg_H(v) + 2$ for all vertices v of H.

As usual, given a graph G with vertex set $\nu(G)$ and $U \subset \nu(G)$, $G - U$ denotes the subgraph of G induced by the vertex set $\nu(G) \setminus U$. Denote by p_U the number of components of $G - U$ and by q_U the cardinality of U. If $U = \{u\}$ and G is a tree, the components of $G - U$ (or $G - u$) are called *branches* of G at u.

As we will see, the lower bound presented in Theorem 2.1 is $p_{\nu(H)} - q_{\nu(H)}$.

LEMMA 2.2. *Let F be a forest with c components (trees). Let F' be any subtree of F with vertex set $\nu(F') = \{u_1, \ldots, u_q\}$, $q \geq 1$. Then, $F - \nu(F')$ has*

$$c + \sum_{i=1}^{q} [\deg_F(u_i) - \deg_{F'}(u_i)] - 1$$

components.

Proof. For each $u_i \in \nu(F')$, define $\delta(u_i) = \deg_F(u_i) - \deg_{F'}(u_i)$, the number of vertices in F, but not in F', that are adjacent to u_i. Since F is acyclic and F' is a subtree of one component F'' of F, $\sum_{i=1}^{q} \delta(u_i)$ counts the (distinct) branches of F'' that are left as components when $\nu(F')$ is removed from F. Since F'' was a component of F, there are then $\sum_{i=1}^{q} \delta(u_i) - 1$ additional components, $c + \sum_{i=1}^{q} \delta(u_i) - 1$ in all, after the removal of $\nu(F')$ from F. □

LEMMA 2.3. *Let F be a forest with c components. Consider a nonempty subset U of $\nu(F)$ and F' the subgraph of F induced by U. Assume that $U = \{u_1, \ldots, u_q\}$ and F' has $e(F')$ edges. Then, the forest $F - U$ has*

$$p_U = c + \sum_{i=1}^{q}[\deg_F(u_i) - 1] - e(F')$$

components.

Proof. Assume that F' has s components (and hence $q - s$ edges), and that $\{u'_1, \ldots, u'_k\}$ are the vertices of a component F'' of F'. Then by Lemma 2.2, the

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removal of F'' from F will increase the number of components by $\left[\sum_{i=1}^{k}[\deg_F(u_i') - \deg_{F''}(u_i')] - 1\right]$. Thus the removal of all s components of F' from F will increase the number of components to

$$\begin{aligned} p_U &= c + \sum_{i=1}^{q}[\deg_F(u_i) - \deg_{F'}(u_i)] - s \\ &= c + \sum_{i=1}^{q}\deg_F(u_i) - \sum_{i=1}^{q}\deg_{F'}(u_i) - s \\ &= c + \sum_{i=1}^{q}[\deg_F(u_i) - 1] + q - 2e(F') - s. \end{aligned}$$

Since $e(F') = q - s$ the result follows. □

By Lemma 2.3, we have $p_{\nu(H)} = 1 + \sum_{i=1}^{k}(d_i - 1) - e$. Since $q_{\nu(H)} = k$ it follows that

$$p_{\nu(H)} - q_{\nu(H)} = 1 + \sum_{i=1}^{k}(d_i - 2) - e. \tag{2.1}$$

Lemma 2.4. *Let F be a forest with c components. Let p be the number of components when q vertices each of degree at most 2 are removed from F. Then, $p - q \leq c$. Equality occurs if and only if all the vertices removed from F have degree 2 and no two such vertices are adjacent.*

Proof. By Lemma 2.3, the removal of q vertices $u_1, \ldots, u_q$ from F increase the number of components by $\sum_{i=1}^{q}[\deg_F(u_i) - 1] - e(F')$ where F' is the subgraph of F induced by $u_1, \ldots, u_q$. Since all removed vertices from F have degree at most 2, the number of remaining components is $p \leq c + q$, with equality if and only if all the vertices removed from F have degree 2 and no two such vertices are adjacent. □

Given a tree T, let M be a subset of $\nu(T)$ such that $p_M - q_M = \max[p_I - q_I]$ with $I \subseteq \nu(T)$. From Lemma 2.4, there is no vertex of degree 1 in M. Lemma 2.4 also implies that we can assume M contains no vertices of degree 2. In other words, there is a subset M of $\nu(H)$ such that $p_M - q_M = \max[p_I - q_I]$ with $I \subseteq \nu(T)$ i.e., $p_M - q_M = m(T)$.

Lemma 2.5. *Let T be a tree. Suppose that $\nu(H) \neq \emptyset$ and $\deg_T(v) - \deg_H(v) \geq 2$ for all vertices v of H. Then, $p_I - q_I \leq p_{\nu(H)} - q_{\nu(H)}$ for all $I \subseteq \nu(H)$.*

Proof. Suppose to the contrary, i.e., that there exist $I \subset \nu(H)$ such that $p_I - q_I > p_{\nu(H)} - q_{\nu(H)}$. Then, there exist I' and I'' such that $I \subseteq I' \subset I'' \subseteq \nu(H)$, with $I'' = I' \cup \{v\}$ and $v \notin I'$, such that $p_{I'} - q_{I'} > p_{I''} - q_{I''}$. Since $q_{I'} = q_{I''} - 1$, $p_{I'} - q_{I'} > p_{I''} - q_{I''}$ implies that $p_{I'} \geq p_{I''}$. But, since $T - I'$ has $p_{I'}$ components and v is a vertex of one component T' of $T - I'$ with $\deg_{T'}(v) \geq 2$ (remember that $\delta(v) \geq 2$), by Lemma 2.2 the removal of v from T' gives at least 1 additional components. Then, $p_{I''} \geq p_{I'} + 1$ gives a contradiction. □

Lemma 2.6. *Let T be a tree. Then, $m(T) \geq 1 + \sum_{i=1}^{k}(d_i - 2) - e$, with equality if and only if $\nu(H) = \emptyset$ or $\deg_T(v) \geq \deg_H(v) + 2$ for all vertices v of H.*

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Proof. First, observe that if $\nu(H) = \emptyset$ i.e., T is a path, then $k = 0$, $e = 0$ and $1 + \sum_{i=1}^{k}(d_i - 2) - e = 1$, which is the path cover number when T is a path. Suppose now that $\nu(H) \neq \emptyset$ and assume that $\deg_T(v) \geq \deg_H(v) + 2$ for all vertices v of H. Consider a vertex set M such that $p_M - q_M = m(T)$ with $M \subseteq \nu(H)$. Since $M \subseteq \nu(H)$ and $\deg_T(v) \geq \deg_H(v) + 2$ for all vertices v of H then, from Lemma 2.5, $p_M - q_M \leq p_{\nu(H)} - q_{\nu(H)}$. Then, $m(T) = p_{\nu(H)} - q_{\nu(H)}$ and, by (2.1), $p_{\nu(H)} - q_{\nu(H)} = 1 + \sum_{i=1}^{k}(d_i - 2) - e$. Therefore, $m(T) = 1 + \sum_{i=1}^{k}(d_i - 2) - e$. Thus, if $\deg_T(v) \geq \deg_H(v) + 2$ for all vertices v of H, then equality occurs.

It suffices to prove that if there is a vertex v in $\nu(H)$ such that $\delta(v) = \deg_T(v) - \deg_H(v) \leq 1$, then $m(T) > 1 + \sum_{i=1}^{k}(d_i - 2) - e$. Consider the set $H^* = \nu(H) \setminus \{v\}$ of vertices of T. Then, $T - H^*$ has p_{H^*} components and $q_{H^*} = q_{\nu(H)} - 1$. Since $v \notin H^*$, v is a vertex of one component T' of $T - H^*$ and, the removal of v from T' gives $\delta(v) - 1$ "additional" components. Then, $p_{\nu(H)} = p_{H^*} + \delta(v) - 1$ and since $\delta(v) \leq 1$, $p_{H^*} \geq p_{\nu(H)}$. Thus, $p_{H^*} - q_{H^*} \geq p_{\nu(H)} - q_{H^*}$ and since $q_{H^*} = q_{\nu(H)} - 1$, we have $p_{H^*} - q_{H^*} > p_{\nu(H)} - q_{\nu(H)}$. Then, $m(T) > p_{\nu(H)} - q_{\nu(H)}$ and by (2.1), $p_{\nu(H)} - q_{\nu(H)} = 1 + \sum_{i=1}^{k}(d_i - 2) - e$. Therefore, $m(T) > 1 + \sum_{i=1}^{k}(d_i - 2) - e$. $\square$

LEMMA 2.7. *Let T be a tree. Then $m(T) \leq 1 + \sum_{i=1}^{k}(d_i - 2)$, with equality occurring if and only if T is segregated.*

Proof. If $\nu(H) = \emptyset$ then T is a path, and hence a segregated tree. In this case, $k = 0$ and, $1 + \sum_{i=1}^{k}(d_i - 2) = 1$, which is the path cover number when T is a path. Suppose now that $\nu(H) \neq \emptyset$. If T is segregated, for all vertices v of H, $\deg_H(v) = 0$ and $\deg_T(v) \geq 3 > \deg_H(v) + 2$. From Lemma 2.6, since $e = 0$ it follows that $m(T) = 1 + \sum_{i=1}^{k}(d_i - 2)$. Thus, if T is segregated, equality occurs.

It suffices to prove that if T is a tree that is not segregated then, $m(T) < 1 + \sum_{i=1}^{k}(d_i - 2)$. Consider a vertex set M such that $p_M - q_M = m(T)$ and $M \subseteq \nu(H)$. From Lemma 2.3, $p_M - q_M = 1 + \sum_{v \in M}[\deg_T(v) - 2] - e(M)$ where $e(M)$ is the number of edges of the subgraph of T induced by the q_M vertices in M. If $M = \nu(H)$, since T is not segregated, there are, at least, two adjacent vertices of T in M which implies $e(M) > 0$. Therefore $p_M - q_M < 1 + \sum_{v \in \nu(H)}[\deg_T(v) - 2]$ i.e., $p_M - q_M < 1 + \sum_{i=1}^{k}(d_i - 2)$. If $M \subset \nu(H)$, $e(M) \geq 0$ and then, $p_M - q_M \leq 1 + \sum_{v \in M}[\deg_T(v) - 2]$. Since $M \subset \nu(H)$ and $\deg_T(v) \geq 3$ for all $v \in \nu(H)$, $\sum_{v \in M}[\deg_T(v) - 2] < \sum_{v \in \nu(H)}[\deg_T(v) - 2]$, so that $p_M - q_M < 1 + \sum_{i=1}^{k}(d_i - 2)$. $\square$

The proof of Theorem 2.1 is now complete.

3. An algorithm. We close by giving a simple algorithm to compute $m(T)$ by computing the path cover number of T.

LEMMA 3.1. *Let T be a tree and let M be a subset of $\nu(H)$ such that $p_M - q_M = m(T)$. Then, there is no vertex of degree at least 3 in $T - M$.*

Proof. Let $T_{(M)} = T - M$ and suppose that $v \notin M$ and $\deg_{T_{(M)}}(v) \geq 3$. Consider the set $M' = M \cup \{v\}$ of vertices of T and the graph $T_{(M')} = T - M'$ $(= T_{(M)} - v)$. From Lemma 2.2, $T_{(M')}$ has $p_{M'} = p_M + \deg_{T_{(M)}}(v) - 1$ components and $q_{M'} = q_M + 1$. Since $\deg_{T_{(M)}}(v) \geq 3$, $p_{M'} \geq p_M + 2$, and $p_{M'} - q_{M'} \geq p_M + 2 - q_{M'} = p_M - q_M + 1$. Thus, $p_{M'} - q_{M'} > p_M - q_M$ gives a contradiction. $\square$

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LEMMA 3.2. *Let T be a tree and let M be a subset of $\nu(H)$ such that $p_M - q_M = m(T)$. If $v \in \nu(H)$ is such that $\delta(v) = \deg_T(v) - \deg_H(v) \geq 3$, then $v \in M$.*

Proof. Let $T_{(M)} = T - M$ and suppose that $v \notin M$ and $\delta(v) \geq 3$. Remember that $M \subseteq \nu(H)$ thus, $\deg_{T_{(M)}}(v) \geq \delta(v) \geq 3$. From Lemma 3.1, $\deg_{T_{(M)}}(v) \geq 3$ gives a contradiction. Therefore, $v \in M$. □

LEMMA 3.3. *Let T be a tree and let M be a subset of $\nu(H)$ such that $p_M - q_M = m(T)$. For any $v \in \nu(H)$ such that $\delta(v) = \deg_T(v) - \deg_H(v) = 2$, there is a subset M' of $\nu(H)$ such that $M \subseteq M'$, $v \in M'$ and $p_{M'} - q_{M'} = m(T)$.*

Proof. Let $T_{(M)} = T - M$ and suppose that $v \in \nu(H)$ with $\delta(v) = 2$. If $v \in M$, let $M' = M$. Suppose now that $v \notin M$. Recall that $M \subseteq \nu(H)$; then $\deg_{T_{(M)}}(v) \geq \delta(v)$. Consider the set $M' = M \cup \{v\}$ and graph $T_{(M')} = T - M'$ $(= T_{(M)} - v)$. From Lemma 2.2, $T_{(M')}$ has $p_{M'} = p_M + \deg_{T_{(M)}}(v) - 1$ components and $q_{M'} = q_M + 1$. Then, $p_{M'} - q_{M'} = p_M - q_M + \deg_{T_{(M)}}(v) - 2$.

By Lemma 3.1, $\deg_{T_{(M)}}(v) < 3$, hence $\deg_{T_{(M)}}(v) = 2$ in which case $p_{M'} - q_{M'} = p_M - q_M$. Therefore, $v \in M'$ and $p_{M'} - q_{M'} = m(T)$. □

We note that Lemmas 3.2 and 3.3 indicate how to inductively compute a subset M' of $\nu(H)$ such that $p_{M'} - q_{M'} = m(T)$. By Lemma 2.4, given any tree T, there is a subset M of $\nu(H)$ such that $p_M - q_M = m(T)$ and, by Lemmas 3.2 and 3.3, there is always a subset M' of $\nu(H)$, containing all vertices v of T with $\delta(v) \geq 2$, such that $M \subseteq M'$ and $p_{M'} - q_{M'} = m(T)$.

ALGORITHM 3.4. *Given a tree T, consider the subgraph H of T induced by the vertices of degree at least 3. Remove from T all vertices v of H such that $\delta(v) \geq 2$.*

This algorithm is applied to the initial tree T and then to each of the resulting components and so on. Let M' be the set of vertices removed via repeated application of Algorithm 3.4. By Lemmas 3.2 and 3.3, all vertices v of T with $\delta(v) \geq 2$ are in M'. If there is a vertex of degree at least 3 in $T - M'$, again by Lemmas 3.2 and 3.3, the application of the Algorithm 3.4 to each of the components of $T - M'$ (each one is a tree) allows us to include new vertices in M' in route to maximizing $p_{M'} - q_{M'}$. By Lemma 3.1, the process stops when there is no vertex of degree at least 3. The set of removed vertices from T gives M', the cardinality of M' gives $q_{M'}$ and the number of components of $T - M'$ (each of which is a path) gives $p_{M'}$. Finally, the maximum multiplicity of an eigenvalue among matrices whose graph is T, $m(T)$, is equal to $p_{M'} - q_{M'}$.

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