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SOME INEQUALITIES FOR THE KHATRI-RAO PRODUCT OF MATRICES*

CHONG-GUANG CAO†, XIAN ZHANG†‡, AND ZHONG-PENG YANG§

Abstract. Several inequalities for the Khatri-Rao product of complex positive definite Hermitian matrices are established, and these results generalize some known inequalities for the Hadamard and Khatri-Rao products of matrices.

Key words. Matrix inequalities, Hadamard product, Khatri-Rao product, Tracy-Singh product, Spectral decomposition, Complex positive definite Hermitian matrix.

AMS subject classifications. 15A45, 15A69

1. Introduction.

Consider complex matrices $A = (a_{ij})$ and $C = (c_{ij})$ of order $m \times n$ and $B = (b_{ij})$ of order $p \times q$. Let A and B be partitioned as $A = (A_{ij})$ and $B = (B_{ij})$, where A_{ij} is an $m_i \times n_j$ matrix and B_{kl} is a $p_k \times q_l$ matrix ($\sum m_i = m$, $\sum n_j = n$, $\sum p_k = p$, $\sum q_l = q$). Let $A \otimes B$, $A \circ C$, $A \odot B$ and $A * B$ be the Kronecker, Hadamard, Tracy-Singh and Khatri-Rao products, respectively. The definitions of the mentioned four matrix products are given by Liu in [1]. Additionally, Liu [1, p. 269] also shows that the Khatri-Rao product can be viewed as a generalized Hadamard product and the Kronecker product is a special case of the Khatri-Rao or Tracy-Singh products. The purpose of this present paper is to establish several inequalities for the Khatri-Rao product of complex positive definite matrices, and thereby generalize some inequalities involving the Hadamard and Khatri-Rao products of matrices in [1, Eq. (13) and Theorem 8], [6, Eq. (3), Lemmas 2.1 and 2.2, Theorems 3.1 and 3.2], and [3, Eqs. (2) and (9)].

Let $S(m)$ be the set of all complex Hermitian matrices of order m, and $S^+(m)$ the set of all complex positive definite Hermitian matrices of order m. For M and N in $S(m)$, we write $M \geq N$ in the Löwner ordering sense, i.e., $M - N$ is positive semidefinite. For a matrix $A \in S^+(m)$, we denote by $\lambda_1(A)$ and $\lambda_m(A)$ the largest and smallest eigenvalue of A, respectively. Let B^* be the conjugate transpose matrix of the complex matrix B. We denote the $n \times n$ identity matrix by I_n, also we write I when the order of the matrix is clear.

2. Some Lemmas.

In this section, we give some preliminaries.

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†Department of Mathematics, Heilongjiang University, Harbin, 150080, P. R. of China (caochongguang@163.com). Partially supported by the Natural Science Foundation of China, the Natural Science Foundation of Heilongjiang province under grant No. A01-07, and the N. S. F. of Heilongjiang Education Committee under grant No. 15011014.

‡School of Mechanical and Manufacturing Engineering, The Queen's University of Belfast, Ashby Building, Stranmillis Road, Belfast, BT9 5AH, Northern Ireland, UK (x.zhang@qub.ac.uk).

§Department of Mathematics, Putian College, Putian, Fujian, 351100, P. R. of China (yangzhongpeng@sina.com). Partially supported by NSF of Fujian Education Committee.

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LEMMA 2.1. *There exists an $mp \times \sum m_i p_i$ real matrix Z such that $Z^T Z = I$ and*

$$A * B = Z^T (A \odot B) Z \tag{2.1}$$

for any $A \in S(m)$ and $B \in S(p)$ partitioned as follows:

$$A = \begin{bmatrix} A_{11} & \cdots & A_{1t} \\ \cdots & \cdots & \cdots \\ A_{t1} & \cdots & A_{tt} \end{bmatrix}, \qquad B = \begin{bmatrix} B_{11} & \cdots & B_{1t} \\ \cdots & \cdots & \cdots \\ B_{t1} & \cdots & B_{tt} \end{bmatrix},$$

where $A_{ii} \in S(m_i)$ and $B_{ii} \in S(p_i)$ for $i = 1, 2, \cdots, t$.

Proof. Let

$$Z_i = \begin{bmatrix} O_{i1} & \cdots & O_{i\ i-1} & I_{m_i p_i} & O_{i\ i+1} & \cdots & O_{it} \end{bmatrix}^T, \qquad i = 1, 2, \cdots, t,$$

where O_{ik} is the $m_i p_k \times m_i p_i$ zero matrix for any $k \neq i$. Then $Z_i^T Z_i = I$ and

$$Z_i^T (A_{ij} \odot B) Z_j = Z_i^T (A_{ij} \odot B_{kl})_{kl} Z_j = A_{ij} \otimes B_{ij}, \qquad i, j = 1, 2, \cdots, t.$$

Letting $Z = \begin{bmatrix} Z_1 & & \\ & \ddots & \\ & & Z_t \end{bmatrix}$, the lemma follows by a direct computation. ∎

If $t = 2$ in Lemma 2.1, then Eq. (2.1) becomes Eq. (13) of [1].

COROLLARY 2.2. *There exists a real matrix Z such that $Z^T Z = I$ and*

$$M_1 * \cdots * M_k = Z^T (M_1 \odot \cdots \odot M_k) Z \tag{2.2}$$

for any $M_i \in S(m(i))$ $(1 \leq i \leq k,\ k \geq 2)$ partitioned as

$$M_i = \begin{bmatrix} N_{11}^{(i)} & \cdots & N_{1t}^{(i)} \\ \cdots & \cdots & \cdots \\ N_{t1}^{(i)} & \cdots & N_{tt}^{(i)} \end{bmatrix}, \tag{2.3}$$

where $N_{jj}^{(i)} \in S(m(i)_j)$ for any $1 \leq i \leq k$ and $1 \leq j \leq t$.

Proof. We proceed by induction on k. If $k = 2$, the corollary is true by Lemma 2.1. Suppose the corollary is true when $k = s$, i.e., there exists a real matrix P such that $P^T P = I$ and $M_1 * \cdots * M_s = P^T (M_1 \odot \cdots \odot M_s) P$, we will prove that it is true when $k = s + 1$. In fact,

$$\begin{aligned} & M_1 * \cdots * M_{s+1} = \\ = \ & (M_1 * \cdots * M_s) * M_{s+1} \\ = \ & P^T (M_1 \odot \cdots \odot M_s) P * M_{s+1} \\ = \ & Q^T \left[P^T (M_1 \odot \cdots \odot M_s) P \odot M_{s+1} \right] Q \qquad (Q^T Q = I) \\ = \ & Q^T \left[P^T (M_1 \odot \cdots \odot M_s) P \odot \left(I_{m(s+1)} M_{s+1} I_{m(s+1)} \right) \right] Q \\ = \ & Q^T \left(P^T \odot I_{m(s+1)} \right) \left[(M_1 \odot \cdots \odot M_s) \odot M_{s+1} \right] \left(P \odot I_{m(s+1)} \right) Q. \end{aligned}$$

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Letting $Z = \left(P \odot I_{m(s+1)}\right) Q$, the corollary follows. □

If the Khatri-Rao and Tracy-Singh products are replaced by the the Hadamard and Kronecker products in Corollary 2.2, respectively, then (2.2) becomes Lemma 2.2 in [6].

Lemma 2.3. *Let A and B be compatibly partitioned matrices, then $(A \odot B)^* = A^* \odot B^*$.*

Proof.

$$\begin{aligned}
(A \odot B)^* = &\ \left((A_{ij} \odot B)_{ij}\right)^* = \left(((A_{ij} \otimes B_{kl})_{kl})_{ij}\right)^* = \left(((A_{ij} \otimes B_{kl})_{kl})^*\right)_{ji} \\
= &\ \left(((A_{ij} \otimes B_{kl})^*)_{lk}\right)_{ji} = \left((A_{ij}^* \otimes B_{kl}^*)_{lk}\right)_{ji} = \left(A_{ij}^* \odot B^*\right)_{ji} \\
= &\ A^* \odot B^*. \quad \square
\end{aligned}$$

Definition 2.4. *Let the spectral decomposition of A ($\in S^+(m)$) be*

$$A = U_A^* D_A U_A = U_A^* \mathrm{diag}(d_1,\ \cdots,\ d_m) U_A,$$

where $d_i > 0$ for all i. For any $c \in \mathbf{R}$, we define the power of matrix A as follows

$$A^c = U_A^* D_A^c U_A = U_A^* \mathrm{diag}(d_1^c,\ \cdots,\ d_m^c) U_A.$$

Lemma 2.5. *Let $A \in S^+(m)$, $B \in S^+(p)$ and $c \in \mathbf{R}$, then*
i) $A \odot B \in S^+(mp)$, $\lambda_1(A \odot B) = \lambda_1(A)\lambda_1(B)$, and $\lambda_{mp}(A \odot B) = \lambda_m(A)\lambda_p(B)$;
ii) $(A \odot B)^c = A^c \odot B^c$.

Proof. Let $A = U_A^* D_A U_A$ and $B = U_B^* D_B U_B$ be the spectral decompositions of A and B, respectively. From Lemma 2.3 and [1, Theorem 1(a)], we derive

$$(U_A \odot U_B)^*(U_A \odot U_B) = (U_A^* \odot U_B^*)(U_A \odot U_B) = (U_A^* U_A) \odot (U_B^* U_B) = I_{mp} \tag{2.4}$$

$$\begin{aligned}
A \odot B = &\ (U_A^* D_A U_A) \odot (U_B^* D_B U_B) = (U_A^* \odot U_B^*)(D_A \odot D_B)(U_A \odot U_B) \\
= &\ (U_A \odot U_B)^*(D_A \odot D_B)(U_A \odot U_B).
\end{aligned} \tag{2.5}$$

The lemma follows from (2.4), (2.5), and the definitions of $A \odot B$ and $(A \odot B)^c$. □

If the Tracy-Singh product is placed by the Kronecker product in Lemma 2.5, then ii) of Lemma 2.5 becomes Lemma 2.1 in [6].

Corollary 2.6. *Let $M_i \in S^+(m(i))$ for $i = 1, 2 \cdots, k$, $n = \prod_{i=1}^{k} m(i)$ and $c \in \mathbf{R}$, then*
i) $M_1 \odot \cdots \odot M_k \in S^+(n)$, $\quad \lambda_1(M_1 \odot \cdots \odot M_k) = \prod_{i=1}^{k} \lambda_1(M_i)$ and
$\lambda_n(M_1 \odot \cdots \odot M_k) = \prod_{i=1}^{k} \lambda_{m(i)}(M_i)$;
ii) $(M_1 \odot \cdots \odot M_k)^c = M_1^c \odot \cdots \odot M_k^c$.

Proof. Using Lemma 2.5, the corollary follows by induction. □

If the Tracy-Singh product is replaced by the Kronecker product in Corollary 2.6, then ii) of Corollary 2.6 becomes Eq. (3) in [6].

Lemma 2.7. [4], [5] *Let $H \in S^+(n)$ and V be a complex matrix of order $n \times m$ such that $V^*V = I_m$, then*

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i) $(V^*H^rV)^{1/r} \le (V^*H^sV)^{1/s}$, *where r and s are two real numbers such that $s > r$, and either $s \notin (-1,1)$ and $r \notin (-1,1)$ or $s \ge 1 \ge r \ge \frac{1}{2}$ or $r \le -1 \le s \le -\frac{1}{2}$;*
ii) $(V^*H^sV)^{1/s} \le \overline{\Delta}(s,r)(V^*H^rV)^{1/r}$,
where r and s are two real numbers such that $s > r$ and either $s \notin (-1,1)$ or $r \notin (-1,1)$, $\overline{\Delta}(s,r) = \left\{\frac{r(\delta^s-\delta^r)}{(s-r)(\delta^r-1)}\right\}^{1/s}\left\{\frac{s(\delta^r-\delta^s)}{(r-s)(\delta^s-1)}\right\}^{-1/r}$, $W = \lambda_1(H)$, $w = \lambda_n(H)$ and $\delta = \frac{W}{w}$.
iii) $(V^*H^sV)^{1/s} - (V^*H^rV)^{1/r} \le \Delta(s,r)I$, *where* $\Delta(s,r) = \max_{\theta\in[0,\ 1]} \left\{[\theta W^s + (1-\theta)w^s]^{1/s} - [\theta W^r + (1-\theta)w^r]^{1/r}\right\}$, *and r, s, W, w and δ are as in* ii).

3. Main results. In this section, we establish some inequalities for the Khatri-Rao product of matrices.

THEOREM 3.1. *Let $M_i \in S^+(m(i))$ $(1 \le i \le k)$ be partitioned as in* (2.3) *and $n = \prod_{i=1}^k m(i)$, then*
(i) $(M_1^s * \cdots * M_k^s)^{1/s} \ge (M_1^r * \cdots * M_k^r)^{1/r}$, *where r and s are as in* i) *of Lemma 2.7;*
(ii) $(M_1^s * \cdots * M_k^s)^{1/s} \le \overline{\Delta}(s,r)(M_1^r * \cdots * M_k^r)^{1/r}$, *where $W = \prod_{i=1}^k \lambda_1(M_i)$ and $w = \prod_{i=1}^k \lambda_{m(i)}(M_i)$, and r, s, δ and $\overline{\Delta}(s,r)$ are as in* ii) *of Lemma 2.7;*
(iii) $(M_1^s * \cdots * M_k^s)^{1/s} - (M_1^r * \cdots * M_k^r)^{1/r} \le \Delta(s,r)I$, *where $W = \prod_{i=1}^k \lambda_1(M_i)$ and $w = \prod_{i=1}^k \lambda_{m(i)}(M_i)$, and r, s, δ and $\Delta(s,r)$ is as in* iii) *of Lemma 2.7.*

Proof. Let $H = M_1 \odot \cdots \odot M_k$, then $H \in S^+(n)$, $\lambda_1(H) = \prod_{i=1}^k \lambda_1(M_i)$ and $\lambda_n(H) = \prod_{i=1}^k \lambda_{m(i)}(M_i)$ from i) of Corollary 2.6. Therefore, using ii) of Corollary 2.6, Corollary 2.2, and Lemma 2.7,

$$\begin{aligned}
(M_1^r * \cdots * M_k^r)^{1/r} &= \left(Z^T(M_1^r \odot \cdots \odot M_k^r)Z\right)^{1/r} \\
&= \left(Z^T(M_1 \odot \cdots \odot M_k)^r Z\right)^{1/r} \\
&\le \left(Z^T(M_1 \odot \cdots \odot M_k)^s Z\right)^{1/s} \\
&= \left(Z^T(M_1^s \odot \cdots \odot M_k^s)Z\right)^{1/s} \\
&= (M_1^s * \cdots * M_k^s)^{1/s},
\end{aligned}$$

$$\begin{aligned}
(M_1^s * \cdots * M_k^s)^{1/s} &= \left(Z^T(M_1^s \odot \cdots \odot M_k^s)Z\right)^{1/s} \\
&= \left(Z^T(M_1 \odot \cdots \odot M_k)^s Z\right)^{1/s} \\
&\le \overline{\Delta}(s,r)\left(Z^T(M_1 \odot \cdots \odot M_k)^r Z\right)^{1/r} \\
&= \overline{\Delta}(s,r)\left(Z^T(M_1^r \odot \cdots \odot M_k^r)Z\right)^{1/r} \\
&= \overline{\Delta}(s,r)\,(M_1^r * \cdots * M_k^r)^{1/r},
\end{aligned}$$

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$$
\begin{aligned}
& (M_1^s * \cdots * M_k^s)^{1/s} - (M_1^r * \cdots * M_k^r)^{1/r} = \\
= \ & \left(Z^T(M_1 \odot \cdots \odot M_k)^s Z\right)^{1/s} - \left(Z^T(M_1 \odot \cdots \odot M_k)^r Z\right)^{1/r} \\
\leq \ & \Delta(s,r)I. \quad \square
\end{aligned}
$$

If the Khatri-Rao and Tracy-Singh products are replaced by the Hadamard and Kronecker products in Theorem 3.1, respectively, then (i) becomes Theorem 3.1 in [6], and (ii) and (iii) become Theorem 3.2 in [6].

THEOREM 3.2. *Let $M_i \in S^+(m(i))$ $(1 \leq i \leq k)$ be partitioned as in* (2.3)*, then*

$$(M_1 * \cdots * M_k)^{-1} \leq M_1^{-1} * \cdots * M_k^{-1}, \tag{3.1}$$

$$M_1^{-1} * \cdots * M_k^{-1} \leq \frac{(W+w)^2}{4Ww}(M_1 * \cdots * M_k)^{-1}, \tag{3.2}$$

$$M_1 * \cdots * M_k - (M_1^{-1} * \cdots * M_k^{-1})^{-1} \leq (\sqrt{W} - \sqrt{w})^2 I, \tag{3.3}$$

$$(M_1 * \cdots * M_k)^2 \leq M_1^2 * \cdots * M_k^2, \tag{3.4}$$

$$M_1^2 * \cdots * M_k^2 \leq \frac{(W+w)^2}{4Ww}(M_1 * \cdots * M_k)^2, \tag{3.5}$$

$$(M_1 * \cdots * M_k)^2 - M_1^2 * \cdots * M_k^2 \leq \frac{1}{4}(W-w)^2 I, \tag{3.6}$$

$$M_1 * \cdots * M_k \leq (M_1^2 * \cdots * M_k^2)^{1/2}, \tag{3.7}$$

$$(M_1^2 * \cdots * M_k^2)^{1/2} \leq \frac{W+w}{2\sqrt{Ww}}(M_1 * \cdots * M_k), \tag{3.8}$$

$$(M_1^2 * \cdots * M_k^2)^{1/2} - M_1 * \cdots * M_k \leq \frac{(W-w)^2}{4(W+w)} I, \tag{3.9}$$

where W and w are as in Theorem 3.1.

Proof. Noting that $G \geq H > O$ if and only if $H^{-1} \geq G^{-1} > O$ [2], we obtain (3.1), (3.2) and (3.3) by choosing $r = -1$ and $s = 1$ in Theorem 3.1. Similarly, (3.7), (3.8) and (3.9) can be obtained by choosing $r = 1$ and $s = 2$ in Theorem 1. Thereby, using that $G \geq H > 0$ implies $G^2 \geq H^2 > 0$, we derive that (3.4) and (3.5) hold.

Liu and Neudecker [3] show that

$$V^* A^2 V - (V^* A V)^2 \leq \frac{1}{4}\left(\lambda_1(A) - \lambda_m(A)\right)^2 I \tag{3.10}$$

for $A \in S^+(m)$ and $V^*V = I$. Replacing A by $M_1 \odot \cdots \odot M_k$ and V by Z in (3.10), we obtain (3.6). $\square$

If we replace the Khatri-Rao product by the Hadamard product in (3.1), (3.2), (3.3), (3.4), (3.7), (3.8) and (3.9), then we obtain some inequalities in [6]. If choosing $t = 2$ and considering the real positive definite matrices in Theorem 3.2, then Theorem 3.2 becomes Theorem 8 in [1]. If choosing $t = 2$ and replacing the Khatri-Rao product by the Hadamard product in (3.6) and (3.8), respectively, then we obtain Eqs. (2) and (9) of [3].

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