The Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 9, pp. 282-289, October 2002

ELA
http://math.technion.ac.il/iic/ela

NON-TRIVIAL SOLUTIONS TO CERTAIN MATRIX EQUATIONS*

AIHUA LI† AND DUANE RANDALL†

Abstract. The existence of non-trivial solutions X to matrix equations of the form $F(\mathbf{X}, \mathbf{A}_1, \mathbf{A}_2, \cdots, \mathbf{A}_s) = G(\mathbf{X}, \mathbf{A}_1, \mathbf{A}_2, \cdots, \mathbf{A}_s)$ over the real numbers is investigated. Here F and G denote monomials in the $(n \times n)$-matrix $\mathbf{X} = (x_{ij})$ of variables together with $(n \times n)$-matrices $\mathbf{A}_1, \mathbf{A}_2, \cdots, \mathbf{A}_s$ for $s \geq 1$ and $n \geq 2$ such that F and G have different total positive degrees in $\mathbf{X}$. An example with $s = 1$ is given by $F(\mathbf{X}, \mathbf{A}) = \mathbf{X}^2\mathbf{A}\mathbf{X}$ and $G(\mathbf{X}, \mathbf{A}) = \mathbf{A}\mathbf{X}\mathbf{A}$ where $deg(F) = 3$ and $deg(G) = 1$. The Borsuk-Ulam Theorem guarantees that a non-zero matrix $\mathbf{X}$ exists satisfying the matrix equation $F(\mathbf{X}, \mathbf{A}_1, \mathbf{A}_2, \cdots, \mathbf{A}_s) = G(\mathbf{X}, \mathbf{A}_1, \mathbf{A}_2, \cdots, \mathbf{A}_s)$ in $(n^2 - 1)$ components whenever F and G have different total odd degrees in $\mathbf{X}$. The Lefschetz Fixed Point Theorem guarantees the existence of special orthogonal matrices $\mathbf{X}$ satisfying matrix equations $F(\mathbf{X}, \mathbf{A}_1, \mathbf{A}_2, \cdots, \mathbf{A}_s) = G(\mathbf{X}, \mathbf{A}_1, \mathbf{A}_2, \cdots, \mathbf{A}_s)$ whenever $deg(F) > deg(G) \geq 1$, $\mathbf{A}_1, \mathbf{A}_2, \cdots, \mathbf{A}_s$ are in $SO(n)$, and $n \geq 2$. Explicit solution matrices $\mathbf{X}$ for the equations with $s = 1$ are constructed. Finally, nonsingular matrices $\mathbf{A}$ are presented for which $\mathbf{X}^2\mathbf{A}\mathbf{X} = \mathbf{A}\mathbf{X}\mathbf{A}$ admits no non-trivial solutions.

Key words. Polynomial equation, Matrix equation, Non-trivial solution.

AMS subject classifications. 39B42, 15A24, 55M20, 47J25, 39B72

1. Matrix equations involving special monomials.

Given monomials $F(\mathbf{X}, \mathbf{A}_1, \mathbf{A}_2, \cdots, \mathbf{A}_s)$ and $G(\mathbf{X}, \mathbf{A}_1, \mathbf{A}_2, \cdots, \mathbf{A}_s)$ in the $(n \times n)$-matrix $\mathbf{X} = (x_{ij})$ of variables with $n \geq 2$ and with total degrees $deg(F) > deg(G) \geq 1$ in $\mathbf{X}$, we investigate the existence of non-trivial solutions $\mathbf{X}$ to the matrix equation

$$F(\mathbf{X}, \mathbf{A}_1, \mathbf{A}_2, \cdots, \mathbf{A}_s) = G(\mathbf{X}, \mathbf{A}_1, \mathbf{A}_2, \cdots, \mathbf{A}_s). \tag{1.1}$$

For example, $\mathbf{X}^2\mathbf{A}\mathbf{X} = \mathbf{A}\mathbf{X}\mathbf{A}$ is such an equation. We note that in this equation, $F(\mathbf{X}, \mathbf{A}) = \mathbf{X}^2\mathbf{A}\mathbf{X}$ and $G(\mathbf{X}, \mathbf{A}) = \mathbf{A}\mathbf{X}\mathbf{A}$ both contain products $\mathbf{A}\mathbf{X}$ and $\mathbf{X}\mathbf{A}$. We first record a sufficient condition for non-trivial solutions to the equation (1.1).

PROPOSITION 1.1. *Suppose that the monomials $F(\mathbf{X}, \mathbf{A}_1, \mathbf{A}_2, \cdots, \mathbf{A}_s)$ and $G(\mathbf{X}, \mathbf{A}_1, \mathbf{A}_2, \cdots, \mathbf{A}_s)$ both contain the product $\mathbf{A}_i\mathbf{X}$ or both contain $\mathbf{X}\mathbf{A}_i$, for some i with $1 \leq i \leq s$. Whenever $\mathbf{A}_i$ is a singular matrix, the matrix equation* (1.1) *admits non-trivial solutions $\mathbf{X}$.*

Proof. Let $\mathbf{X}$ be any non-zero $(n \times n)$-matrix whose columns belong to the null space of $\mathbf{A}_i$ whenever both F and G contain $\mathbf{A}_i\mathbf{X}$. Similarly, let $\mathbf{X}$ be any non-zero matrix whose rows belong to the null space of $\mathbf{A}_i^T$ in case both F and G contain $\mathbf{X}\mathbf{A}_i$. □

Our principal result affirms the existence of non-trivial solutions $\mathbf{X}$ to matrix equations $F(\mathbf{X}, \mathbf{A}_1, \mathbf{A}_2, \cdots, \mathbf{A}_s) = G(\mathbf{X}, \mathbf{A}_1, \mathbf{A}_2, \cdots, \mathbf{A}_s)$ whenever $\mathbf{A}_1, \mathbf{A}_2, \cdots, \mathbf{A}_s$ belong to the special orthogonal group $SO(n)$ for any integer $n \geq 2$. We first construct explicit non-trivial solutions for such matrix equations with $s = 1$.

*Received by the editors on 2 November 2001. Final version accepted for publication on 6 September 2002. Handling editor: Daniel B. Szyld.

†Department of Mathematics and Computer Science, Loyola University New Orleans, New Orleans, LA 70118, USA (ali@loyno.edu, randall@loyno.edu). The first author was supported by the BORSF under grant LEQSF(2000-03)RD-A-24.

The Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 9, pp. 282-289, October 2002

ELA
http://math.technion.ac.il/iic/ela

PROPOSITION 1.2. *Every matrix equation $F(\mathbf{X},\mathbf{A}) = G(\mathbf{X},\mathbf{A})$ for monomials F and G with different total odd degrees in $\mathbf{X}$ admits a non-trivial solution $\mathbf{X}$ of the form $\mathbf{A}^{p/q}$ whenever $\mathbf{A}$ belongs to $SO(n)$ for $n \geq 2$.*

Proof. We may assume that $deg(F) > deg(G) \geq 1$. We seek a solution $\mathbf{X} = \mathbf{A}^{p/q}$ to the matrix equation $F(\mathbf{X},\mathbf{A}) \cdot (G(\mathbf{X},\mathbf{A}))^{-1} = \mathbf{I}_n$. The classical Spectral Theorem for $SO(n)$ in [3] affirms that $\mathbf{A} = \mathbf{C}^{-1}\mathbf{B}\mathbf{C}$ for matrices $\mathbf{B}$ and $\mathbf{C}$ in $SO(n)$ where $\mathbf{B}$ consists of blocks of non-trivial rotations $R(\theta_i) = \begin{bmatrix} \cos\theta_i & -\sin\theta_i \\ \sin\theta_i & \cos\theta_i \end{bmatrix}$ along the diagonal together with an identity submatrix $\mathbf{I}_l$. A solution $\mathbf{X}$ commuting with powers of $\mathbf{A}$ reduces the matrix equation $F(\mathbf{X},\mathbf{A}) \cdot (G(\mathbf{X},\mathbf{A}))^{-1} = \mathbf{I}_n$ to $\mathbf{X}^{deg(F)-deg(G)} = \mathbf{A}^p$ for some integer p. Setting $q = deg(F) - deg(G)$, we obtain $\mathbf{X} = \mathbf{A}^{p/q} = \mathbf{C}^{-1}\mathbf{B}^{p/q}\mathbf{C}$ where $\mathbf{B}^{p/q}$ consists of blocks of rotations $R(p\theta_i/q)$ along the diagonal together with $\mathbf{I}_l$. □

We now establish the existence of non-trivial solutions to many matrix equations via the Lefschetz Fixed Point Theorem. For example, the matrix equation $\mathbf{X}^2\mathbf{A}_1\mathbf{A}_2^2\mathbf{X}\mathbf{A}_2^3\mathbf{A}_1^2 = \mathbf{A}_1^3\mathbf{A}_2\mathbf{A}_1^2\mathbf{X}\mathbf{A}_2^3$ admits rotation matrices as solutions whenever $\mathbf{A}_1$ and $\mathbf{A}_2$ belong to $SO(n)$ for any $n \geq 2$.

THEOREM 1.3. *There is a solution $\mathbf{X}$ in $SO(n)$ to any matrix equation $F(\mathbf{X},\mathbf{A}_1,\mathbf{A}_2,\cdots,\mathbf{A}_s) = G(\mathbf{X},\mathbf{A}_1,\mathbf{A}_2,\cdots,\mathbf{A}_s)$, i.e., equation (1.1), with $deg(F) > deg(G) \geq 1$ and $n \geq 2$ whenever the $(n\times n)$-matrices $\mathbf{A}_i$ belong to $SO(n)$ for $1 \leq i \leq s$.*

Proof. Solutions $\mathbf{X}$ in $SO(n)$ to the matrix equation (1.1) are precisely the fixed points of the continuous function $H : SO(n) \longrightarrow SO(n)$ defined by $H(\mathbf{X}) = \mathbf{X} \cdot F(\mathbf{X},\mathbf{A}_1,\mathbf{A}_2,\cdots,\mathbf{A}_s) \cdot [G(\mathbf{X},\mathbf{A}_1,\mathbf{A}_2,\cdots,\mathbf{A}_s)]^{-1}$. The existence of fixed points for the map H follows from its non-zero Lefschetz number $L(H)$. We affirm that $L(H) = (deg(G) - deg(F))^m$ where $n = 2m$ or $n = 2m+1$.

Brown in [1, p.49], calculated the Lefschetz number $L(\rho_k)$ for the k^{th} power map $\rho_k : G \longrightarrow G$ defined by $\rho_k(g) = g^k$ on any compact connected topological group G which is an ANR (absolute neighborhood retract). He proved that $L(\rho_k) = (1-k)^\lambda$ where λ denotes the number of generators for the primitively generated exterior algebra $H^*(G\,;\mathbb{Q})$. For $G = SO(n)$, $\lambda = m$ where $n = 2m$ or $n = 2m+1$; see [4, p.956]. It suffices to show that H is homotopic to $\rho_k : SO(n) \longrightarrow SO(n)$ where $k = deg(F) - deg(G) + 1$.

For each i with $1 \leq i \leq s$, let $g_i : [0,1] \longrightarrow SO(n)$ denote any path in $SO(n)$ from $\mathbf{A}_i = g_i(0)$ to the identity matrix $\mathbf{I}_n = g_i(1)$. Replacing each matrix $\mathbf{A}_i$ by the function g_i in $H : SO(n) \longrightarrow SO(n)$ produces a homotopy $H_t : SO(n) \longrightarrow SO(n)$ for $0 \leq t \leq 1$ with $H_0 = H$ and $H_1 = \rho_k$. Thus $L(H) = (1-k)^m = (deg(G)-deg(F))^m \neq 0$ so H has a fixed point. □

We now establish the existence of non-trivial solutions $\mathbf{X}$ to all matrix equations of the form (1.1) in any (n^2-1) components whenever F and G have different odd degrees in $\mathbf{X}$ for any $s \geq 1$ and $n \geq 1$. For example, given any $(n\times n)$-matrix $\mathbf{A}$, there is a non-zero matrix $\mathbf{X}$ such that $\mathbf{X}^2\mathbf{A}\mathbf{X} = \mathbf{A}\mathbf{X}\mathbf{A}$ in at least (n^2-1)-components. This is a best possible result, since we shall construct matrices $\mathbf{A}$ for which $\mathbf{X}^2\mathbf{A}\mathbf{X} = \mathbf{A}\mathbf{X}\mathbf{A}$ admits only the trivial solution. We use the Borsuk-Ulam Theorem following the paper of Lam [2] to prove the following.

The Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 9, pp. 282-289, October 2002

ELA
http://math.technion.ac.il/iic/ela

THEOREM 1.4. *Given any monomials* $F(\mathbf{X}, \mathbf{A}_1, \mathbf{A}_2, \cdots, \mathbf{A}_s)$ *and* $G(\mathbf{X}, \mathbf{A}_1, \mathbf{A}_2, \cdots, \mathbf{A}_s)$ *in the* $(n \times n)$*-matrix* $\mathbf{X} = (x_{ij})$ *together with arbitrary matrices* $\mathbf{A}_1, \mathbf{A}_2, \cdots, \mathbf{A}_s$ *in* $\mathrm{M}_n(\mathbb{R})$ *for* $n \geq 2$ *such that deg*(F) *and deg*(G) *are different odd integers, the matrix equation* (1.1) *admits a non-trivial solution* $\mathbf{X}$ *in* (n^2-1) *components.*

Proof. Set each component of the matrix $F(\mathbf{X}, \mathbf{A}_1, \mathbf{A}_2, \cdots, \mathbf{A}_s) - G(\mathbf{X}, \mathbf{A}_1, \mathbf{A}_2, \cdots, \mathbf{A}_s)$ equal to zero, except for one fixed component. We obtain n^2-1 polynomial equations in the n^2 variables x_{ij}. Now each component of $F(\mathbf{X}, \mathbf{A}_1, \mathbf{A}_2, \cdots, \mathbf{A}_s)$ and $G(\mathbf{X}, \mathbf{A}_1, \mathbf{A}_2, \cdots, \mathbf{A}_s)$ is a homogeneous polynomial whose degree is given by *deg*(F) or *deg*(G) respectively. Consequently, every monomial in the (n^2-1) polynomial equations has an odd degree, either *deg*(F) or *deg*(G). Suppose that the system of n^2-1 polynomial equations in the n^2 variables had no non-zero solution. As $\mathbf{X}$ ranges over the unit sphere S^{n^2-1} in $\mathbb{R}^{n^2}$, normalization of the non-zero vectors $F(\mathbf{X}, \mathbf{A}_1, \mathbf{A}_2, \cdots, \mathbf{A}_s) - G(\mathbf{X}, \mathbf{A}_1, \mathbf{A}_2, \cdots, \mathbf{A}_s) \in \mathbb{R}^{n^2-1}$ produces a continuous function $P : S^{n^2-1} \longrightarrow S^{n^2-2}$. Since *deg*$(F)$ and *deg*(G) are distinct odd integers, P commutes with the antipodal maps on the spheres. But the classical Borsuk-Ulam Theorem [5, p.266] affirms that no such function P can exist. $\square$

2. The special matrix equation $\mathbf{X}^2\mathbf{A}\mathbf{X} - \mathbf{A}\mathbf{X}\mathbf{A} = \mathbf{0}$. Given any non-zero $(n \times n)$-matrix $\mathbf{A}$, consider the matrix equation

$$\mathbf{X}^2\mathbf{A}\mathbf{X} - \mathbf{A}\mathbf{X}\mathbf{A} = \mathbf{0}\ . \tag{2.1}$$

In this section we discuss solution types of the equation (2.1). We list a few obvious facts about solutions.

LEMMA 2.1.
1. *If* $\mathbf{X} \in \mathrm{M}_n(\mathbb{R})$ *is a solution to* (2.1)*, then* $-\mathbf{X}$ *is a solution too;*
2. *If* $|\mathbf{A}| < 0$*, then* (2.1) *has no nonsingular solutions.*
3. *If* $\mathbf{A} = \mathbf{B}^2$ *for some* $\mathbf{B} \in \mathrm{M}_n(\mathbb{R})$*, then* $\mathbf{X} = \mathbf{B}$ *is a non-trivial solution.*
4. *If* $\mathbf{A}^m = \mathbf{I}_n$ *and* m *is odd, then* $\mathbf{X} = \mathbf{A}^{\frac{m+1}{2}}$ *is a non-trivial solution.*
5. *If* $\mathbf{A}^3 = \mathbf{0}$*, then* $\mathbf{X} = k\mathbf{A}$ *is a solution to* (2.1) *for all* $k \in \mathbb{R}$.
6. *Suppose* $\mathbf{P}$ *is a nonsingular matrix and* $\mathbf{B} = \mathbf{P}\mathbf{A}\mathbf{P}^{-1}$*. Then a matrix* $\mathbf{X}$ *satisfies the equation* $\mathbf{X}^2\mathbf{A}\mathbf{X} - \mathbf{A}\mathbf{X}\mathbf{A} = \mathbf{0}$ *if and only if* $\mathbf{Y} = \mathbf{P}\mathbf{X}\mathbf{P}^{-1}$ *satisfies* $\mathbf{Y}^2\mathbf{B}\mathbf{Y} - \mathbf{B}\mathbf{Y}\mathbf{B} = \mathbf{0}$.

By Lemma 2.1(6.), when the matrix $\mathbf{A}$ is diagonalizable, the equation (2.1) can be reduced to the diagonal case. We first characterize all solutions for scalar matrices $\mathbf{A}$.

THEOREM 2.2. *Let* $\mathbf{A} = a\mathbf{I}_n \in \mathrm{M}_n(\mathbb{R})$*, where* $n > 1$ *and* $a \neq 0$*. Then the equation* (2.1) *has non-trivial solutions. Furthermore, the solution set (over the real numbers) consists of matrices in* $\mathrm{M}_n(\mathbb{R})$ *of the form*

$$\mathbf{X} = \mathbf{Q}^{-1} \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix} \mathbf{Q},$$

where $\mathbf{Q}$ *is a nonsingular matrix with complex entries and* $\lambda_i = 0$*,* $\sqrt{a}$*, or* $-\sqrt{a}$ *for* $i = 1, 2, \ldots, n$*. In particular, nonsingular solutions are those with* $\lambda_1\lambda_2\cdots\lambda_n$ *not*

The Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 9, pp. 282-289, October 2002

ELA
http://math.technion.ac.il/iic/ela

equal to zero. In summary,
1. *If* $a^n > 0$ *with* $n > 2$*, then* (2.1) *has both singular solutions and nonsingular solutions;*
2. *If* $a^n < 0$ *and* $n > 2$*, then* (2.1) *has only singular solutions;*
3. *In case of* $a < 0$ *and* $n = 2$*, there are nonsingular solutions, but no non-trivial singular solutions to* (2.1).

Proof. Suppose $\mathbf{X}$ is a solution to (2.1). Then

$$\mathbf{X}^2\mathbf{A}\mathbf{X} - \mathbf{A}\mathbf{X}\mathbf{A} = a\mathbf{X}^3 - a^2\mathbf{X} = \mathbf{0} \Longleftrightarrow \mathbf{X}^3 = a\mathbf{X}.$$

Every matrix $\mathbf{X}$ satisfying $\mathbf{X}^3 = a\mathbf{X}$ is diagonalizable over the complex numbers. Suppose $\mathbf{X}$ is similar to a diagonal matrix $\mathbf{D} = diag(\lambda_i)$, then $\mathbf{X}^3 = a\mathbf{X} \Longleftrightarrow \mathbf{D}^3 = a\mathbf{D}$. This implies $\lambda_i^2 = a$ or $\lambda_i = 0$ for $i = 1, 2, \ldots, n$. Thus all the solutions to (2.1) are the real matrices similar to these diagonal matrices. Claim 1. is obvious by choosing appropriate (real) λ_i's. For 2., $|\mathbf{A}| < 0$. By Lemma 2.1(2.), equation (2.1) has no nonsingular solutions. The existence of singular solutions over the real numbers is based on the fact that every 2×2 diagonal matrix of the form $\begin{bmatrix} \lambda & 0 \\ 0 & -\lambda \end{bmatrix}$, where λ is a non-real complex number, can be realized by a complex nonsingular matrix $\mathbf{Q}$. Assume $\lambda = \sqrt{-a} \cdot i$, one can check that $\mathbf{Q} = \begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix}$ gives $\mathbf{Q}^{-1} \begin{bmatrix} \sqrt{-a} \cdot i & 0 \\ 0 & -\sqrt{-a} \cdot i \end{bmatrix} \mathbf{Q} = \begin{bmatrix} 0 & \sqrt{-a} \\ -\sqrt{-a} & 0 \end{bmatrix} \in \mathrm{M}_2(\mathbb{R})$. Since $n > 2$, we always can choose at least one diagonal block of $\mathbf{D}$ to be $\begin{bmatrix} \sqrt{-a} \cdot i & 0 \\ 0 & -\sqrt{-a} \cdot i \end{bmatrix}$ and extend it to a singular solution by choosing at least one zero diagonal element. In case of $a < 0$ and $n = 2$, nonsingular solutions are similar to $\begin{bmatrix} 0 & \sqrt{-a} \\ -\sqrt{-a} & 0 \end{bmatrix}$. We show by contradiction that in this case (2.1) has no non-trivial singular solutions. Assume $\mathbf{0} \neq \mathbf{X} = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}$ is a non-trivial solution to (2.1) and $|\mathbf{X}| = 0$. Then $\mathbf{X}^2 = (x_1 + x_4)\mathbf{X} \Longrightarrow (x_1 + x_4)^2\mathbf{X} = a\mathbf{X} \Longrightarrow a = (x_1 + x_4)^2 \geq 0$, a contradiction. $\square$

By Lemma 2.1(6.), if $\mathbf{A}$ is diagonalizable, we only need to consider the solvability of the equation (2.1) for the similar diagonal matrix. Now let us treat diagonal matrices.

THEOREM 2.3. *Suppose* $\mathbf{A}$ *is a non-zero diagonal matrix which has at least one positive entry. Then the equation* $\mathbf{X}^2\mathbf{A}\mathbf{X} - \mathbf{A}\mathbf{X}\mathbf{A} = \mathbf{0}$ *has non-trivial solutions.*

Proof. Let $\mathbf{A} = diag(\lambda_i)$. Without loss of generality, let $\lambda_1 > 0$. Then the diagonal matrix $\mathbf{X} = diag(\alpha_i)$ will give non-trivial solutions, where $\alpha_1 = \sqrt{\lambda_1}$ and for $i > 1$, $\alpha_i = 0$ or $\sqrt{\lambda_i}$ if $\lambda_i > 0$. When $\lambda_i \geq 0$ for all i, we obtain non-trivial solutions $\mathbf{X} = diag(\sqrt{\lambda_i})$. $\square$

COROLLARY 2.4. *For* $n > 1$*, the equation* (2.1) *has non-trivial solutions for all* $n \times n$ *positive definite and all positive semidefinite matrices* $\mathbf{A}$.

We end this section with the following proposition.

PROPOSITION 2.5. *Suppose* $\mathbf{A} \in \mathrm{M}_n(\mathbb{R})$ *is similar to a block matrix, i.e., there*

The Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 9, pp. 282-289, October 2002
ELA
http://math.technion.ac.il/iic/ela

exists a nonsingular matrix $\mathbf{P}$ *such that*

$$\mathbf{PAP}^{-1} = \begin{bmatrix} \mathbf{A}_1 & & & \\ & \mathbf{A}_2 & & \\ & & \ddots & \\ & & & \mathbf{A}_m \end{bmatrix},$$

where each $\mathbf{A}_i$ *is a square matrix. Suppose* $\mathbf{Y}_i$ *satisfies* $\mathbf{Y}_i^2\mathbf{A}_i\mathbf{Y}_i - \mathbf{A}_i\mathbf{Y}_i\mathbf{A}_i = \mathbf{0}$*, for* $i = 1, 2, \cdots, m$*. Then the matrix* $\mathbf{X} = \mathbf{P}^{-1}\mathbf{BP}$ *is a solution to* $\mathbf{X}^2\mathbf{AX} - \mathbf{AXA} = \mathbf{0}$*, where* $\mathbf{B}$ *is a block matrix with blocks* $\mathbf{B}_i = \mathbf{Y}_i$ *or* $\mathbf{0}$*. Thus, if at least one of the solutions* $\mathbf{Y}_i$*'s is not zero, we can extend it to non-trivial solutions for the equation* $\mathbf{X}^2\mathbf{AX} = \mathbf{AXA}$.

THEOREM 2.6. *Let* $\mathbf{A}$ *be a real* $n \times n$ *matrix with distinct negative eigenvalues. Then the equation* $\mathbf{X}^2\mathbf{AX} = \mathbf{AXA}$ *admits only the trivial solution.*

Proof. Suppose first that $\mathbf{X}$ is an invertible solution. Then we have

$$\mathbf{A}^{-1}\mathbf{X}^2\mathbf{A} = \mathbf{XAX}^{-1}.$$

Thus the eigenvalues of $\mathbf{X}^2$ are the same as those of $\mathbf{A}$. Since the eigenvalues of $\mathbf{A}$ are negative and distinct, the eigenvalues of $\mathbf{X}$ are all pure imaginary and of distinct modulus. This is impossible.

If $\mathbf{X}$ is a singular solution, let $\mathbf{v}$ be a null vector of $\mathbf{X}$ and observe that $\mathbf{0} = \mathbf{AXAv} = \mathbf{XAv}$. Thus the null space of $\mathbf{X}$ is $\mathbf{A}$-invariant. Then there exists an invertible matrix $\mathbf{B}$ such that

$$\mathbf{X} = \mathbf{B}\begin{bmatrix} \mathbf{Y} & \mathbf{0} \\ \mathbf{C} & \mathbf{0} \end{bmatrix}\mathbf{B}^{-1} \quad \text{and} \quad \mathbf{A} = \mathbf{B}\begin{bmatrix} \mathbf{P} & \mathbf{0} \\ \mathbf{D} & \mathbf{E} \end{bmatrix}\mathbf{B}^{-1}.$$

By Lemma 2.1(6.),

$$\begin{bmatrix} \mathbf{Y} & \mathbf{0} \\ \mathbf{C} & \mathbf{0} \end{bmatrix}^2 \begin{bmatrix} \mathbf{P} & \mathbf{0} \\ \mathbf{D} & \mathbf{E} \end{bmatrix} \begin{bmatrix} \mathbf{Y} & \mathbf{0} \\ \mathbf{C} & \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathbf{P} & \mathbf{0} \\ \mathbf{D} & \mathbf{E} \end{bmatrix} \begin{bmatrix} \mathbf{Y} & \mathbf{0} \\ \mathbf{C} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{P} & \mathbf{0} \\ \mathbf{D} & \mathbf{E} \end{bmatrix}.$$

This yields $\mathbf{Y}^2\mathbf{PY} = \mathbf{PYP}$ and by induction $\mathbf{Y} = \mathbf{0}$. (See Theorem 3.3 for the 2×2 case.) This means that

$$\begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{C} & \mathbf{0} \end{bmatrix}^2 = \mathbf{0} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{ECP} & \mathbf{0} \end{bmatrix},$$

which gives $\mathbf{ECP} = \mathbf{0}$. Since $\mathbf{E}$ and $\mathbf{P}$ are invertible, $\mathbf{C} = \mathbf{0}$, so $\mathbf{X}$ is the trivial solution. ∎

3. The special case $n = 2$**.** In this section, we focus on the equation (2.1) for 2×2 matrices. Denote

$$\mathbf{A} = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \quad \text{and} \quad \mathbf{X} = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}.$$

The Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 9, pp. 282-289, October 2002

ELA
http://math.technion.ac.il/iic/ela

We first consider the existence of non-trivial solutions to (2.1) when $\mathbf{A}$ is an orthogonal matrix. When $\mathbf{A}$ is orthogonal with $|\mathbf{A}| = 1$, the existence of a non-trivial (orthogonal) solution $\mathbf{X} = \mathbf{A}^{1/2}$ is given in Proposition 1.2.

PROPOSITION 3.1. *Let* $\mathbf{A}$ *be an orthogonal matrix in* $\mathrm{M}_2(\mathbb{R})$ *with* $|\mathbf{A}| = -1$. *A non-trivial singular solution to (2.1) is given by* $\mathbf{X} = \frac{1}{2}\begin{bmatrix} 1+a_1 & a_2 \\ a_2 & 1-a_1 \end{bmatrix}$.

Proof. When $|\mathbf{A}| = -1$, $\mathbf{A}$ is a symmetric matrix with two distinct eigenvalues 1 and -1. Thus $\mathbf{A}$ is diagonalizable to the matrix $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. By Lemma 2.1(6.) and Theorem 2.3, (2.1) has a non-trivial solution. A matrix of the form $\mathbf{X} = \mathbf{P}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\mathbf{P}^{-1}$ is a non-trivial singular solution to (2.1) when $\mathbf{P}$ satisfies $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. The solution $\mathbf{X} = \frac{1}{2}\begin{bmatrix} 1+a_1 & a_2 \\ a_2 & 1-a_1 \end{bmatrix}$ is obtained by finding such a matrix $\mathbf{P}$ made of two linearly independent eigenvectors of $\mathbf{A}$ via linear algebra (refer to the proof of Theorem 2.2). ◻

Now we discuss more general cases. In the next theorem, we show constructively that the equation (2.1) has non-trivial solutions for a large group of two by two matrices $\mathbf{A}$ (over the real numbers).

THEOREM 3.2. *Consider* $\mathbf{0} \neq \mathbf{A} \in \mathrm{M}_{\mathbf{2}}(\mathbb{R})$. *The equation* (2.1) *has non-trivial solutions in the following cases:*
1. $\mathbf{A}$ *has two distinct real eigenvalues, not both negative.*
2. $\mathbf{A}$ *is a scalar matrix.*
3. $\mathbf{A}$ *is a non-scalar matrix with a repeated non-negative eigenvalue.*

Proof. By Lemma 2.1 and Theorem 2.3, the first is true. The second claim is from Theorem 2.2. For the third, without loss of generality, we may assume

$$\mathbf{A} = \begin{bmatrix} a_1 & 0 \\ a_3 & a_1 \end{bmatrix},$$

where $0 \leq a_1$ and $a_3 \neq 0$. If $a_1 = 0$, the matrix $\mathbf{X} = \begin{bmatrix} 0 & 0 \\ x_3 & 0 \end{bmatrix}$ gives a non-trivial solution to (2.1) for any real number $x_3 \neq 0$. If $a_1 \neq 0$, the lower triangular matrix $\mathbf{X} = \begin{bmatrix} \sqrt{a_1} & 0 \\ a_3/(2\sqrt{a_1}) & \sqrt{a_1} \end{bmatrix}$ gives a non-trivial solution to (2.1). ◻

We note that by Proposition 2.5, we can extend solutions to (2.1) for the 2×2 case to solutions for $(n \times n)$-matrices. Finally, we construct non-zero matrices $\mathbf{A}$ for which $\mathbf{X}^2\mathbf{A}\mathbf{X} = \mathbf{A}\mathbf{X}\mathbf{A}$ admits only the trivial solution.

THEOREM 3.3. *The equation* $\mathbf{X}^2\mathbf{A}\mathbf{X} = \mathbf{A}\mathbf{X}\mathbf{A}$ *admits only the trivial solution for any* $\mathbf{A} \in \mathrm{M}_2(\mathbb{R})$ *having two distinct negative eigenvalues or having a single negative eigenvalue of geometric multiplicity 1.*

Proof. For the first case, it is sufficient to assume $\mathbf{A} = \begin{bmatrix} -\lambda_1 & 0 \\ 0 & -\lambda_2 \end{bmatrix}$, where $\lambda_1 > \lambda_2 > 0$. Suppose $\mathbf{X} = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}$ is a solution. Then $|\mathbf{X}| = 0$ or $\pm\sqrt{\lambda_1\lambda_2}$ since

The Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 9, pp. 282-289, October 2002

ELA
http://math.technion.ac.il/iic/ela

$\mathbf{A}$ is nonsingular. By comparing the non-diagonal entries of $\mathbf{X}^2\mathbf{AX}$ and $\mathbf{AXA}$, we obtain the following two equations

$$\left\{\begin{array}{l} x_2(\lambda_1 x_1^2+\lambda_1 x_2x_3+\lambda_2 x_1x_4+\lambda_2 x_4^2+\lambda_1\lambda_2)=0 \\ x_3(\lambda_1 x_1^2+\lambda_1 x_1x_4+\lambda_2 x_2x_3+\lambda_2 x_4^2+\lambda_1\lambda_2)=0. \end{array}\right. \tag{3.1}$$

First we assume $0 \neq |\mathbf{X}| = \sqrt{\lambda_1\lambda_2}$. Then $x_2x_3 = x_1x_4 - \sqrt{\lambda_1\lambda_2}$. Thus (3.1) becomes

$$\left\{\begin{array}{l} x_2(\lambda_1 x_1^2+(\lambda_1+\lambda_2)x_1x_4+\lambda_2 x_4^2+\lambda_1\lambda_2-\lambda_1\sqrt{\lambda_1\lambda_2})=0 \\ x_3(\lambda_1 x_1^2+(\lambda_1+\lambda_2)x_1x_4+\lambda_2 x_4^2+\lambda_1\lambda_2-\lambda_2\sqrt{\lambda_1\lambda_2})=0. \end{array}\right. \tag{3.2}$$

If $x_2x_3 \neq 0$, then equations in (3.2) imply $\lambda_1\sqrt{\lambda_1\lambda_2} = \lambda_2\sqrt{\lambda_1\lambda_2} \Longrightarrow \lambda_1 = \lambda_2$, a contradiction. If $x_2x_3 = 0$, we compare the (1,1) entries of $\mathbf{X}^2\mathbf{AX}$ and $\mathbf{AXA}$ to obtain $-\lambda_1 x_1^3 = \lambda_1^2 x_1 \Longrightarrow x_1 = 0 \Longrightarrow |\mathbf{X}| = 0$, a contradiction again. Therefore $|\mathbf{X}| \neq \sqrt{\lambda_1\lambda_2}$. The same argument shows that $|\mathbf{X}| \neq -\sqrt{\lambda_1\lambda_2}$.

Now consider the case $|\mathbf{X}| = 0$, i.e., $x_1x_4 = x_2x_3$. By matrix multiplication, we have

$$\mathbf{X}^2\mathbf{AX} = -(x_1+x_4)(\lambda_1x_1+\lambda_2x_4)\begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} = \begin{bmatrix} \lambda_1^2 x_1 & \lambda_1\lambda_2 x_2 \\ \lambda_1\lambda_2 x_3 & \lambda_2^2 x_4 \end{bmatrix} = \mathbf{AXA}.$$

If $x_2 \neq 0$ or $x_3 \neq 0$, then $(x_1+x_4)(\lambda_1x_1+\lambda_2x_4) = -\lambda_1\lambda_2$ by comparing the non-diagonal entries. Apply this to the diagonal entries, we obtain $\lambda_1\lambda_2 x_1 = -\lambda_1^2 x_1$ and $\lambda_1\lambda_2 x_4 = -\lambda_2^2 x_4 \Longrightarrow x_1 = x_4 = 0$. Thus $\mathbf{X}^2\mathbf{AX} = \mathbf{0} \Longrightarrow \mathbf{AXA} = \mathbf{0} \Longrightarrow \mathbf{X} = \mathbf{0}$, since $\mathbf{A}$ is invertible. This gives only a trivial solution to (2.1). At last, consider the case of $x_2 = 0 = x_3$. Since $x_1x_4 = x_2x_3$, x_1 or $x_4 = 0$. If $x_1 = 0$, compare the (2,2)-entry of $\mathbf{X}^2\mathbf{AX}$ and $\mathbf{AXA}$, we have $\lambda_2 x_4^3 = -\lambda_2^2 x_4 \Longrightarrow x_4 = 0$. Similarly, $x_4 = 0 \Longrightarrow x_1 = 0$. Therefore $x_1 = x_2 = x_3 = x_4 = 0$ and $\mathbf{X}$ is a trivial solution.

Now assume $\mathbf{A}$ has a single negative eigenvalue of geometric multiplicity 1. Let $\mathbf{A} = \begin{bmatrix} a_1 & 0 \\ a_3 & a_1 \end{bmatrix}$ where $a_1 < 0$ and $a_3 \neq 0$. Assume $\mathbf{0} \neq \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}$ is a solution to (2.1). We first claim that $x_2 \neq 0$. If not, the diagonal entries of $\mathbf{X}^2\mathbf{AX} - \mathbf{AXA}$ are $a_1x_1(x_1^2 - a_1)$ and $a_1x_4(x_4^2 - a_1)$. Since a_1 is negative, it forces $x_1 = x_4 = 0$ and then $x_3 = 0$. Now assume $\mathbf{X}$ is a singular solution. Then the second row of $\mathbf{X}$ is k times the first row for some real number $k \neq 0$. By equating the second row minus k times the first row of both $\mathbf{X}^2\mathbf{AX}$ and $\mathbf{AXA}$, we obtain a contradiction. When $\mathbf{X}$ is a nonsingular solution, $|\mathbf{X}| = a_1$ or $-a_1$. Since $x_2 \neq 0$, $x_3 = \frac{x_1x_4 \pm a_1}{x_2}$. Then by equating the components of $\mathbf{X}^2\mathbf{AX}$ and $\mathbf{AXA}$, we obtain the following two equations:

$$\left\{\begin{array}{c} (x_1+x_4)x_2(a_1x_1 \pm a_3x_2 + a_1x_4) = 0 \\ (x_1+x_4)(a_1x_1x_4 \pm a_3x_2x_4 \pm a_1^2 + a_1x_4^2) = 0. \end{array}\right.$$

This implies $x_1 + x_4 = 0$. Then the $(1,1)$-component of $\mathbf{X}^2\mathbf{AX} - \mathbf{AXA}$ is $\pm a_1x_2a_3$ which can not be zero, a contradiction.

In conclusion, the equation (2.1) has no non-trivial solutions. □

Acknowledgements. We express our gratitude to Kee Y. Lam for his enlightening conversations and kind hospitality. We also express our sincere gratitude to one of the referees for the contribution of Theorem 2.6.

The Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 9, pp. 282-289, October 2002

ELA
http://math.technion.ac.il/iic/ela

REFERENCES

[1] Robert F. Brown. *The Lefschetz Fixed Point Theorem.* Scott, Foresman and Co., Glenview, Illinois, 1971.
[2] Kee Yuen Lam. Borsuk-Ulam type theorems and systems of bilinear equations. *Geometry from the Pacific Rim.* Walter de Gruyter & Co., Berlin, New York, 1997.
[3] Terry Lawson. *Linear Algebra.* John Wiley & Sons, Inc., New York, 1996.
[4] M. Mimura. *Homotopy Theory of Lie groups.* Handbook of Algebraic Topology, North-Holland, Amsterdam, 1995.
[5] E.H. Spanier. *Algebraic Topology.* McGraw-Hill, New York, 1966.