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THE CONVERGENCE RATE OF THE CHEBYSHEV SEMIITERATIVE METHOD UNDER A PERTURBATION OF THE FOCI OF AN ELLIPTIC DOMAIN*

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Abstract. The Chebyshev semiiterative method (CHSIM) is a powerful method for finding the iterative solution of a nonsymmetric real linear system $A\mathbf{x} = \mathbf{b}$ if an ellipse excluding the origin well fits the spectrum of A. The asymptotic rate of convergence of the CHSIM for solving the above system under a perturbation of the foci of the optimal ellipse is studied. Several formulae to approximate the asymptotic rates of convergence, up to the first order of a perturbation, are derived. These generalize the results about the sensitivity of the asymptotic rate of convergence to a perturbation of a real-line segment spectrum by Hageman and Young, and by the first author. A numerical example is given to illustrate the theoretical results.

Key words. Chebyshev semiiterative method, Asymptotic rate of convergence.

AMS subject classifications. 65F10

1. Introduction. Let $A\mathbf{x} = \mathbf{b}$ be a nonsingular real linear system. With a regular splitting $A = M - N$ where M is nonsingular, the above system is written as an equivalent fixed-point form

$$\mathbf{x} = T\mathbf{x} + \mathbf{c}, \tag{1.1}$$

where $T = M^{-1}N$ and $\mathbf{c} = M^{-1}\mathbf{b}$. Then 1 is not in the spectrum of T, $\sigma(T)$. Let $\{p_m\}$ be a sequence of polynomials with deg $p_m \leq m$ and $p_m(1) = 1$. Assume that Ω is a compact set excluding the point $z = 1$ but containing $\sigma(T)$ and that the complement of Ω in the extended complex plane is simply connected. The *asymptotic rate of convergence* of the semiiterative method (cf. Varga [11]) induced by $\{p_m\}$ for Ω is defined by

$$\kappa(\Omega, \{p_m\}) := \overline{\lim}_{m\to\infty} \|p_m\|_{\Omega}^{1/m},$$

and

$$\kappa(\Omega) := \inf_{\{p_m\}} \kappa(\Omega, \{p_m\})$$

is called the *asymptotic convergence factor* (ACF) for Ω; cf. Eiermann, Niethammer and Varga [3]. If $\kappa(\Omega, \{p_m\}) = \kappa(\Omega)$ for some $\{p_m\}$, then the semiiterative method induced by $\{p_m\}$ is called *asymptotically optimal.*

If an ellipse excluding 1 well fits $\sigma(T)$, then a Chebyshev semiiterative method (CHSIM) for solving (1.1) is determined by its foci. An adaptive procedure for estimating the foci of the optimal ellipse whose major axis either lies on the real axis or

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parallel to the imaginary axis based on the power method was developed by Manteuffel [8]. This adaptive dynamic scheme was modified based on the GMRES Algorithm by Elman at al. [4] and further developed by Golub et al. [2], [5] as application of the modified moments. The hybrid iterative method relying on an approximation of the field of values of the coefficient matrix and of its inverse was proposed by Manteuffel and Starke [9]. All available software for iterative methods based on Chebyshev polynomials, such as Chebycode [1] adaptively seek to determine a good ellipse during the iterations by computing its foci.

For simplicity, we only discuss the case where the major axis lies along the x-axis since the other case can be handled analogously. Assume that $\partial\Omega$ is an optimal ellipse for $\sigma(T)$ in the sense that the parameters of CHSIM are chosen on the basis of the foci of $\partial\Omega$ for solving (1.1) is asymptotically optimal; cf. Niethammer and Varga [10]. In practice, the exact values of α and β are often not available. It is more realistic to assume that we are only given estimates, α_e and β_e, of α and β. The purpose of this paper is to consider how the convergence behavior changes if a CHSIM is corresponding to α_e and β_e.

If the length of the minor axis, denoted by b, is zero, the ellipse reduces to a line segment $[\alpha, \beta]$. This case has been thoroughly studied by Hageman and Young [6] and Li [7]. Their results are generalized here to the case of an elliptic domain. The same notation as in [7] is used here.

It is well-known that a unique conformal mapping Φ

$$\Phi(z) = \frac{(\sqrt{z-\beta} + \sqrt{z-\alpha})^2}{\beta - \alpha}$$

maps the exterior of $[\alpha, \beta]$ on the extended z-plane one-one onto the exterior of the unit circle with ∞ corresponding to ∞ and $\Psi'(\infty) > 0$. If $(d, 0)$ is the center of $\partial\Omega$ then the asymptotic convergence factor for Ω is given by

$$\kappa(\Omega) = \frac{|\Phi(d+ib)|}{|\Phi(1)|} = \frac{2(a+b)}{\left(\sqrt{1-\beta} + \sqrt{1-\alpha}\right)^2}\ , \tag{1.2}$$

where

$$a = \sqrt{b^2 + (\beta - \alpha)/2)^2} \tag{1.3}$$

is the length of the major semiaxis.

This paper is organized as follows. A general formula for the sensitivity of the ACF for a closed ellipse under a perturbation of its foci is introduced at the end of Section 1. The asymptotic rate of convergence of the CHSIM under a perturbation of each focus and both of foci are studied in Sections 2 and 3, respectively. These rates of convergence are compared in Section 4. A numerical example is given in Section 5 to illustrate the theoretical results.

It follows from (1.2) that

$$\frac{\partial\kappa}{\partial a} = \frac{\partial\kappa}{\partial b} = \frac{2}{\left(\sqrt{1-\beta} + \sqrt{1-\alpha}\right)^2}\ ,$$

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$$\frac{\partial\kappa}{\partial\alpha} = \frac{2(a+b)}{\left(\sqrt{1-\beta}+\sqrt{1-\alpha}\right)^3\sqrt{1-\alpha}} ,$$
$$\frac{\partial\kappa}{\partial\beta} = \frac{2(a+b)}{\left(\sqrt{1-\beta}+\sqrt{1-\alpha}\right)^3\sqrt{1-\beta}} .$$

Suppose that α, β and b have increments $\Delta\alpha$, $\Delta\beta$ and Δb, respectively. We denote by Ω_1 the closed ellipse with foci $\alpha+\Delta\alpha$, $\beta+\Delta\beta$ and a minor semiaxis of length $b+\Delta b$. Then

$$\Delta\kappa = \kappa(\Omega_1)-\kappa(\Omega) = \frac{\partial\kappa}{\partial\alpha}\Delta\alpha + \frac{\partial\kappa}{\partial\beta}\Delta\beta + \frac{\partial\kappa}{\partial a}\Delta a + \frac{\partial\kappa}{\partial b}\Delta b$$
$$+o\left(\sqrt{\Delta\alpha^2+\Delta\beta^2+\Delta a^2+\Delta b^2}\right).$$

With (1.3) and denoting

$$s = \frac{\sqrt{1-\beta}}{\sqrt{1-\alpha}} , \tag{1.4}$$

we have the ACF for the perturbed closed ellipse $\kappa(\Omega_1)$ represented by

$$\kappa(\Omega_1) = \kappa(\Omega)\left\{1+\frac{(\beta-\alpha)(\Delta\beta-\Delta\alpha)}{4a(a+b)}+\frac{\Delta\beta+s\Delta\alpha}{s(1+s)(1-\alpha)}+\frac{\Delta b}{a}\right\}$$
$$+o\left(\sqrt{\Delta\alpha^2+\Delta\beta^2+\Delta b^2}\right). \tag{1.5}$$

The expression (1.5) will be used to estimate the sensitivity of the asymptotic rate of convergence of a CHSIM to a perturbation of the foci in the following two sections.

2. Perturbation of either α or β. The asymptotic rate of convergence of a Chebyshev semiiterative method under a perturbation of either α or β is studied in this section. The size of a perturbation is denoted by a positive parameter ϵ. There are four different possible perturbations:

(a) an overestimate for α, $\alpha_o = \alpha-\epsilon$.
(b) an underestimate for α, $\alpha_u = \alpha+\epsilon$.
(c) an underestimate for β, $\beta_u = \beta-\epsilon$.
(d) an overestimate for β, $\beta_o = \beta+\epsilon$.

Let κ_{α_e} or κ_{β_e} be the asymptotic rate of convergence of the CHSIM whose parameters are selected on the basis of a perturbation of one focus α or β.

THEOREM 2.1. *The asymptotic rate of convergence of the CHSIM whose parameters are selected on the basis of a perturbation of one focus α or β for solving* (1.1) *is given by*

$$\kappa_{\alpha_o} = \kappa\left\{1+\left[\frac{-a+b+c}{2bc}-\frac{1}{(1+s)(1-\alpha)}\right]\epsilon+o(\epsilon)\right\}, \tag{2.1}$$

$$\kappa_{\alpha_u} = \kappa\left\{1+\left[\frac{a-b+c}{2bc}+\frac{1}{(1+s)\,(1-\alpha)}\right]\epsilon+o\,(\epsilon)\right\}, \tag{2.2}$$

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$$\kappa_{\beta_u} = \kappa \left\{ 1 + \left[\frac{a-b+c}{2bc} - \frac{1}{s\,(1+s)\,(1-\alpha)} \right] \epsilon + o\,(\epsilon) \right\}, \tag{2.3}$$

$$\kappa_{\beta_o} = \kappa \left\{ 1 + \left[\frac{-a+b+c}{2bc} + \frac{1}{s(1+s)(1-\alpha)} \right] \epsilon + o(\epsilon) \right\}, \tag{2.4}$$

where s is defined in (1.4).

Proof. We only consider the case (d) in detail. The other cases can be shown in an analogous way. The equation of the ellipse $\partial\Omega$ is given by

$$\frac{(x-d)^2}{a^2} + \frac{y^2}{b^2} = 1, \tag{2.5}$$

where $d = (\beta+\alpha)/2$. An ellipse $\partial\Omega_1$ with foci α and β_o is in the following form

$$\frac{\left(x - d - \frac{\epsilon}{2}\right)^2}{a_1^2} + \frac{y^2}{b_1^2} = 1, \tag{2.6}$$

where

$$a_1^2 = \left(c + \frac{\epsilon}{2}\right)^2 + b_1^2 \quad \text{and} \quad c = \frac{\beta-\alpha}{2}. \tag{2.7}$$

It follows from (2.5) and (2.6) that Ω is contained in Ω_1, the closed interior of $\partial\Omega_1$, if and only if the following inequality holds

$$b_1^2 \left(1 - \frac{\left(x-d-\frac{\epsilon}{2}\right)^2}{a_1^2} \right) \geq b^2 \left(1 - \frac{(x-d)^2}{a^2} \right) \text{ for } x \in [d-a, d+a],$$

or equivalently,

$$b_1^2 a^2 \left[a_1^2 - \left(x - d - \frac{\epsilon}{2} \right)^2 \right] \geq b^2 a_1^2 \left[a^2 - (x-d)^2 \right] \text{ for } x \in [d-a, d+a]. \tag{2.8}$$

We are going to find the minimum value of b_1 such that (2.8) holds. Let

$$b_1 = b + \eta\epsilon. \tag{2.9}$$

It suffices to find the smallest $\eta \geq 0$ such that (2.8) holds up to the first order of ϵ. Substituting a_1^2 and b_1^2 from (2.7) and (2.9) into (2.8) and then dropping the $o(\epsilon)$ term yields

$$2b\eta\epsilon a^4 - (b^2 + 2b\eta\epsilon)a^2[(x-d)^2 - (x-d)\epsilon] \geq -b^2(a^2 + 2b\eta\epsilon + c\epsilon)(x-d)^2,$$

or equivalently,

$$\epsilon\eta[2a^4 - 2a^2(x-d)^2 + 2b^2(x-d)^2] + \epsilon b(x-d)[a^2 + c(x-d)] \geq 0.$$

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This inequality is equivalent to the following:

$$\eta \geq \frac{b(x-d)[a^2+c(x-d)]}{2[c^2(x-d)^2-a^4]} = \frac{b(x-d)}{2[c(x-d)-a^2]}, \quad x \in [d-a, d+a]. \tag{2.10}$$

The right hand side of (2.10) achieves its maximum value when $x = d - a$. In other words, the smallest positive value of η, denoted by η again, is given by

$$\eta = \frac{b}{2(a+c)} + O(\epsilon)$$

and the minimum value of b_1, denoted by b_1 again, is given by

$$b_1 = b + \frac{b\epsilon}{2(a+c)} + o(\epsilon).$$

Comparing $\partial\Omega_1$ with $\partial\Omega$, we observe that

$$\Delta\alpha = 0, \quad \Delta\beta = \epsilon \quad \text{and} \quad \Delta b = \frac{b\epsilon}{2(a+c)} + o(\epsilon).$$

Thus, it follows from (1.5) that

$$\kappa(\Omega_1) = \kappa\left\{1 + \left[\frac{-a+b+c}{2bc} + \frac{1}{s(1+s)(1-\alpha)}\right]\epsilon + o(\epsilon)\right\},$$

where s is given by (1.4).

On the other hand, it follows from [10] that the asymptotic rate of convergence of the CHSIM whose parameters are selected on the basis of foci α and β_o, denoted by κ_{β_o}, is the same as the ACF for Ω_1, i.e.

$$\kappa_{\beta_o} = \kappa(\Omega_1).$$

This completes the proof. □

As $b \to 0$, Ω reduces to the line segment $[\alpha, \beta]$ and the limit of $(-a+b+c)/(2bc)$ is $1/(\beta-\alpha)$. Then, the two equations in (2.1) and (2.4) reduce to

$$\kappa_{\alpha_o} = \kappa\left\{1 + \frac{s\epsilon}{(\beta-\alpha)} + o(\epsilon)\right\} \quad \text{and} \quad \kappa_{\beta_o} = \kappa\left\{1 + \frac{\epsilon}{s(\beta-\alpha)} + o(\epsilon)\right\},$$

respectively. These estimates, which extends the results in [6], appeared in [7].

3. Perturbations of both α and β. Perturbations of both foci α and β of $\partial\Omega$ will be studied in this section. There are four different possible perturbations of both α and β

(a) overestimates for both α and β.
(b) underestimates for both α and β.
(c) an overestimate for α and an underestimate for β.
(d) an underestimate for α and an overestimate for β.

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We only discuss case (a) since the rest of the cases can be treated in a similar, but much simpler, fashion.

Let $\alpha_o = \alpha - \epsilon_1$ and $\beta_o = \beta + \epsilon_2$, where $\epsilon_1 > 0$ and $\epsilon_2 > 0$. The asymptotic rate of convergence of the CHSIM for solving (1.1), determined by the ellipse with foci α_o and β_o, is denoted by $\kappa_{\alpha_o,\beta_o}$. An ellipse $\partial\Omega_1$ with foci α_o and β_o is given by

$$\frac{\left(x - d - \frac{\epsilon_2-\epsilon_1}{2}\right)^2}{a_1^2} + \frac{y^2}{b_1^2} = 1,$$

where

$$a_1^2 - b_1^2 = \left(c + \frac{\epsilon_1 + \epsilon_2}{2}\right)^2 \quad \text{and} \quad c = \frac{\beta - \alpha}{2} \ . \tag{3.1}$$

The requirement that $\partial\Omega$ be contained in $\partial\Omega_1$ is equivalent to the following inequality

$$b_1^2 a^2 \left[a_1^2 - \left(x - d - \frac{\epsilon_2 - \epsilon_1}{2}\right)^2\right] \geq b^2 a_1^2 \left[a^2 - (x-d)^2\right], \ x \in [d-a, d+a]. \tag{3.2}$$

Let

$$b_1 = b + \eta(\epsilon_1 + \epsilon_2). \tag{3.3}$$

We are going to find the minimum value of b_1 or the smallest $\eta \geq 0$ such that (3.2) holds up to the first order of ϵ_1 and ϵ_2. Substituting a_1 and b_1 from (3.1) and (3.3) into (3.2) and dropping the $o(\epsilon_1)$ and $o(\epsilon_2)$ terms yields

$$\eta \geq \frac{-bc(x-d)^2(\epsilon_1+\epsilon_2) - a^2 b(x-d)(\epsilon_2 - \epsilon_1)}{2[a^4 - c^2(x-d)^2](\epsilon_1 + \epsilon_2)}, \qquad x \in [d-a, d+a].$$

It is clear that if $\epsilon_1 = \epsilon_2$ the minimum nonnegative value of η is 0. Hence $b_1 = b$ and $\Delta b = 0$.

Assume that $\epsilon_1 \neq \epsilon_2$ and let

$$g(x) = \frac{-bcx^2(\epsilon_1+\epsilon_2) - a^2 bx(\epsilon_2 - \epsilon_1)}{2[a^4 - c^2x^2]}, \quad x \in [-a, a].$$

Then

$$g'(x) = \frac{a^2 b[-c^2(\varepsilon_2 - \varepsilon_1)x^2 - 2a^2c(\varepsilon_1 + \varepsilon_2)x - a^4(\varepsilon_2 - \varepsilon_1)]}{2[a^4 - c^2x^2]^2}, \quad x \in [-a, a].$$

If we solve $g'(x) = 0$ for x, we get two solutions:

$$x_1 = -\frac{a^2}{c}\left(\frac{\epsilon_1 + \epsilon_2 - 2\sqrt{\epsilon_1\epsilon_2}}{\epsilon_2 - \epsilon_1}\right) \quad \text{and} \quad x_2 = -\frac{a^2}{c}\left(\frac{\epsilon_1 + \epsilon_2 + 2\sqrt{\epsilon_1\epsilon_2}}{\epsilon_2 - \epsilon_1}\right).$$

It is clear that $x_2 \notin [-a, a]$. But $x_1 \in [-a, a]$ if and only if

$$\frac{a\left|\sqrt{\epsilon_2} - \sqrt{\epsilon_1}\right|}{c\left(\sqrt{\epsilon_2} + \sqrt{\epsilon_1}\right)} \leq 1. \tag{3.4}$$

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Condition (3.4) is equivalent to

$$\max\left\{\frac{\epsilon_1}{\epsilon_2},\frac{\epsilon_2}{\epsilon_1}\right\} \le \left(\frac{1+e}{1-e}\right)^2, \tag{3.5}$$

where $e = c/a$ is the eccentricity of the ellipse $\partial\Omega$.

If we assume that (3.5) holds, then x_1 is the only critical point of g in the interval $[-a, a]$. It easily follows from $g(0) = 0$ and $g(x_1) = b(\epsilon_1 + \epsilon_2 - 2\sqrt{\epsilon_1\epsilon_2})/(4c) > 0$ that $g(x_1)$ is the maximum value of g on $[-a, a]$. Thus the minimum value of $\eta(\epsilon_1 + \epsilon_2)$ is $b(\epsilon_1 + \epsilon_2 - 2\sqrt{\epsilon_1\epsilon_2})/(4c) + o(\epsilon_1) + o(\epsilon_2)$. It follows from (3.3) that

$$\Delta b = \frac{b}{4c}(\epsilon_1 + \epsilon_2 - 2\sqrt{\epsilon_1\epsilon_2}) + o(\epsilon_1) + o(\epsilon_2).$$

If we assume that (3.5) does not hold, then $x_1 \notin [-a, a]$ and $g'(x)$ keeps the same sign as $\epsilon_1 - \epsilon_2$ on $[-a, a]$. The g achieves its maximum at $x = -a$ if $\epsilon_1 < \epsilon_2$ and at $x = a$ if $\epsilon_1 > \epsilon_2$. Consequently, the minimum positive value of η is

$$\frac{-c(\epsilon_1 + \epsilon_2) + a\,|\epsilon_1 - \epsilon_2|}{2b(\epsilon_1 + \epsilon_2)} + O(\epsilon_1) + O(\epsilon_2).$$

Thus by (3.3) we have

$$\Delta b = \frac{-c(\epsilon_1 + \epsilon_2) + a\,|\epsilon_1 - \epsilon_2|}{2b} + o(\epsilon_1) + o(\epsilon_2).$$

Once again, we apply (1.5) and the observations from [10]. This completes the proof of a part of Theorem 3.1 below. The rest of equations can be shown in an analogous way.

THEOREM 3.1. *The asymptotic rate of convergence of the CHSIM whose parameters are selected on the basis of a perturbation of both foci α and β for solving* (1.1) *is given by the following formulas.*

If $\epsilon_1 \ne \epsilon_2$ *and* $\max\left\{\frac{\epsilon_1}{\epsilon_2},\frac{\epsilon_2}{\epsilon_1}\right\} \le \left(\frac{1+e}{1-e}\right)^2$, *then*

$$\begin{aligned}\kappa_{\alpha_o,\beta_o} = \kappa\Bigg\{1 &+ \left[\frac{2a-b}{4ac} - \frac{1}{(1+s)\,(1-\alpha)}\right]\epsilon_1 - \frac{b}{2ac}\sqrt{\epsilon_1\epsilon_2}\\ &+ \left[\frac{2a-b}{4ac} + \frac{1}{s\,(1+s)\,(1-\alpha)}\right]\epsilon_2 + o\,(\epsilon_1) + o\,(\epsilon_2)\Bigg\}.\end{aligned}$$

If $\max\left\{\frac{\epsilon_1}{\epsilon_2},\frac{\epsilon_2}{\epsilon_1}\right\} > \left(\frac{1+e}{1-e}\right)^2$, *then*

$$\begin{aligned}\kappa_{\alpha_o,\beta_o} = \kappa\Bigg\{1 &+ \left[\frac{-a+b}{2bc} - \frac{1}{(1+s)\,(1-\alpha)}\right]\epsilon_1 + \frac{1}{2b}\,|\epsilon_1 - \epsilon_2|\\ &+ \left[\frac{-a+b}{2bc} + \frac{1}{s\,(1+s)\,(1-\alpha)}\right]\epsilon_2 + o\,(\epsilon_1) + o\,(\epsilon_2)\Bigg\},\end{aligned}$$

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and

$$\begin{aligned}\kappa_{\alpha_u,\beta_u} &= \kappa\left\{1+\left[\frac{a-b}{2bc}+\frac{1}{(1+s)\,(1-\alpha)}\right]\epsilon_1+\frac{1}{2b}\left|\epsilon_1-\epsilon_2\right|\right.\\ &\quad\left.+\left[\frac{a-b}{2bc}-\frac{1}{s\,(1+s)\,(1-\alpha)}\right]\epsilon_2+o\left(\epsilon_1\right)+o\left(\epsilon_2\right)\right\},\\ \kappa_{\alpha_o,\beta_u} &= \kappa\left\{1+\left[\frac{-a+b+c}{2bc}-\frac{1}{(1+s)\,(1-\alpha)}\right]\epsilon_1\right.\\ &\quad\left.+\left[\frac{a-b+c}{2bc}-\frac{1}{s\,(1+s)\,(1-\alpha)}\right]\epsilon_2+o\left(\epsilon_1\right)+o\left(\epsilon_2\right)\right\},\\ \kappa_{\alpha_u,\beta_o} &= \kappa\left\{1+\left[\frac{a-b+c}{2bc}+\frac{1}{(1+s)\,(1-\alpha)}\right]\epsilon_1\right.\\ &\quad\left.+\left[\frac{-a+b+c}{2bc}+\frac{1}{s\,(1+s)\,(1-\alpha)}\right]\epsilon_2+o\left(\epsilon_1\right)+o\left(\epsilon_2\right)\right\}.\end{aligned}$$

In particular, when $\epsilon_1=\epsilon_2=\epsilon$,

$$\kappa_{\alpha_o,\beta_o}=\kappa\left\{1+\left[\frac{a-b}{ac}+\frac{1-s}{s\,(1+s)\,(1-\alpha)}\right]\epsilon+o\left(\epsilon\right)\right\}, \tag{3.6}$$

$$\kappa_{\alpha_u,\beta_u}=\kappa\left\{1+\left[\frac{a-b}{bc}-\frac{1-s}{s\,(1+s)\,(1-\alpha)}\right]\epsilon+o\left(\epsilon\right)\right\}, \tag{3.7}$$

$$\kappa_{\alpha_o,\beta_u}=\kappa\left\{1+\left[\frac{1}{b}-\frac{1}{s(1-\alpha)}\right]\epsilon+o\left(\epsilon\right)\right\}, \tag{3.8}$$

$$\kappa_{\alpha_u,\beta_o}=\kappa\left\{1+\left[\frac{1}{b}+\frac{1}{s(1-\alpha)}\right]\epsilon+o\left(\epsilon\right)\right\}. \tag{3.9}$$

As $b\to 0$, Ω reduces to the line segment $[\alpha,\beta]$. We then have

$$\frac{a-b}{ac}\to\frac{2}{\beta-\alpha}\quad\text{and}\quad\frac{1-s}{s(1+s)(1-\alpha)}\to\frac{(1+s)}{s\,(\beta-\alpha)}.$$

Then it follows from (3.6) that

$$\kappa_{\alpha_o,\beta_o}=\kappa\left\{1+\frac{(1+s)\epsilon}{s(\beta-\alpha)}+o(\epsilon)\right\},$$

which appeared in [7].

As an application of Theorem 3.1, we consider how a perturbation of a point on $\partial\Omega$ affects the asymptotic rate of convergence of the Chebyshev method. For simplicity, assume that the two vertices of $\partial\Omega$ on the real axis and $z_1=(x_1,y_1)$ on the up right quarter of $\partial\Omega$ are three extreme eigenvalues of T. Let $z_e=z_1+\epsilon e^{it}$, where $\epsilon>0$ and $0\le t\le\pi/2$, be a perturbation of z_1 which lies on the outer normal vector at z_1.

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The ellipse $\partial\Omega_e$ containing the two vertices of $\partial\Omega$ and z_e is given by

$$\frac{(x-d)^2}{a^2} + \frac{y^2}{b_e^2} = 1, \tag{3.10}$$

where $b_e = b + \eta\epsilon$ for some $\eta \geq 0$. Substituting z_e into (3.10), we then have

$$\eta = \frac{b\left(b^2\left(x_1 - d\right)\cos t + a^2 y_1 \sin t\right)}{a^2 y_1^2}.$$

Since the slope of the normal vector at z_1 is $\tan t = a^2 y_1/(b^2(x_1 - d))$, then $\eta = b/(y_1 \sin t)$. Therefore $b_e = b + b\epsilon/(y_1 \sin t) + o(\epsilon)$. Thus we have

$$\Delta b = \frac{b}{y_1 \sin t}\epsilon + o(\epsilon), \qquad \Delta\alpha = \frac{b}{c}\Delta b, \qquad \Delta\beta = -\frac{b}{c}\Delta b.$$

It follows from (3.7) that the asymptotic rate of convergence of the CHSIM for solving (1.1) under the perturbation of z_1 is given by

$$\kappa_{\alpha_u,\beta_u} = \kappa\left\{1 + \frac{b}{y_1 \sin t}\left[\frac{1}{a+b} - \frac{b}{c}\left(\frac{1-s}{s\left(1+s\right)\left(1-\alpha\right)}\right)\right]\epsilon + o\left(\epsilon\right)\right\}.$$

4. Comparison of the asymptotic rates of convergence. Eight asymptotic rates of convergence derived above are compared in this section. We have shown the following inequalities in the case of a line segment spectrum (i.e., $b = 0$) in [7].

$$\kappa_{\alpha_o} \leq \kappa_{\beta_o} < \kappa_{\alpha_o,\beta_o} < \kappa_{\alpha_u,\beta_u} < \kappa_{\beta_u} \leq \kappa_{\alpha_u}(\text{or } \kappa_{\alpha_o,\beta_u}) \leq \kappa_{\alpha_u,\beta_o}. \tag{4.1}$$

However, the relationship among those rates of convergence in the case of an elliptic spectrum domain is more complicated. Notice that all formulas of the asymptotic rate of convergence derived can be unified by introducing the following notation

$$\kappa_* = \kappa c_*, \tag{4.2}$$

where the subscript $*$ denotes the type of perturbation, e.g., $\kappa_{\alpha_o} = \kappa c_{\alpha_o}$. It is trivial that $c_* > 1$. Then we extend Proposition 4 in [7] as follows.

$$c_{\alpha_o,\beta_u} = c_{\alpha_o} c_{\beta_u} + o(\epsilon), \tag{4.3}$$

$$c_{\alpha_u,\beta_o} = c_{\alpha_u} c_{\beta_o} + o(\epsilon), \tag{4.4}$$

$$c_{\alpha_u,\beta_u} = \frac{c_{\alpha_u}}{c_{\beta_o}} + o(\epsilon) = \frac{c_{\beta_u}}{c_{\alpha_o}} + o(\epsilon), \tag{4.5}$$

$$c_{\alpha_o,\beta_o} \geq c_{\alpha_o} c_{\beta_o} + o(\epsilon),$$

where the equality in the last expression holds if and only if $b = 0$.

The relations (4.3)–(4.5) can be interpreted as the fact that the effect of one perturbation is the same as the composition of the corresponding two perturbations. From (2.1)–(2.4), (4.2) and (4.5), we obtain the following theorem.

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Theorem 4.1. *The following inequalities hold*

$$\kappa_{\alpha_o} < \kappa_{\beta_u} (or\ \kappa_{\beta_o}) < \kappa_{\alpha_u} \quad and \quad \kappa_{\beta_u} < \kappa_{\beta_o} \Leftrightarrow e \leq \frac{2a}{1-\alpha},$$

where e is the eccentricity of $\partial\Omega$.

Proof. It follows from (2.1)–(2.4) that

$$\kappa_{\alpha_o} < \kappa_{\alpha_u}, \quad \kappa_{\alpha_o} < \kappa_{\beta_o} \quad \text{and} \quad \kappa_{\beta_u} < \kappa_{\alpha_u}.$$

The equalities (4.2) and (4.5) imply that

$$\kappa_{\alpha_o} < \kappa_{\beta_u} \quad \text{and} \quad \kappa_{\beta_o} \leq \kappa_{\alpha_u}.$$

The first part of theorem is proved. With the identity $2c = (1-\alpha)(1-s^2)$, one can easily verify that

$$\kappa_{\beta_u} \leq \kappa_{\beta_o} \Leftrightarrow s \leq \frac{b}{a} \Leftrightarrow e \leq \frac{2a}{1-\alpha}.$$

This completes the proof of theorem. □

In practice, we are only interested in the case of the optimal ellipse close to the point $z = 1$. The condition $e > 2a/(1-\alpha)$ means that the ellipse is flat enough. In this case, the relationship among four rates is consistent with (4.1). We conclude that an underestimate of α is more sensitive than the either underestimate of overestimate of β and that an overestimate of α is less sensitive than either underestimate or overestimate of β. Assume that the ellipse is not so flat (in the sense of $e \leq 2a/(1-\alpha)$). If only β needs to be estimated, then an underestimate of β is better than the overestimate by an equivalent amount. The example in the following section illustrates this point.

We can show the following theorem in an analogous way.

Theorem 4.2. *The following inequalities hold:*

$$\kappa_{\beta_o} < \kappa_{\alpha_o,\beta_o}, \qquad \kappa_{\alpha_u} < \kappa_{\alpha_u,\beta_o},$$
$$\kappa_{\alpha_u,\beta_u} < \kappa_{\beta_u} < \kappa_{\alpha_o,\beta_u} (or\ \kappa_{\alpha_o,\beta_o}) < \kappa_{\alpha_u,\beta_o}, \quad if\ e < \frac{2a}{1-\alpha},$$
$$\kappa_{\alpha_o,\beta_o} < \kappa_{\alpha_u,\beta_u} < \kappa_{\beta_u} < \kappa_{\alpha_o,\beta_u} < \kappa_{\alpha_u,\beta_o}, \quad if\ e > \frac{2a}{1-\alpha}.$$

We remark that divergence will never happen if only α is overestimated, while a big overestimate of β may cause divergence. An underestimate of α together with an overestimate of β is the worst case. We suggest in practice that α should never be underestimated. If several cycles of estimates of foci are needed, one may choose a fair overestimate of α and an underestimate of β. Then one should make a careful dynamic estimate of β.

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5. Example. Consider an elliptic partial differential equation of the following form,

$$-\Delta u + 2p_1 x u_x + 2p_2 y u_y - p_3 u = f,$$

with constants p_1, p_2 and $p_3 \geq 0$ on the unit square $[0,1] \times [0,1]$ and a boundary condition $u(x,y) = 0$, where f is a continuous function of x and y. Using the standard five-point discretization scheme, a linear system (1.1) with N unknowns is obtained. Values of $p_1 = 34$, $p_2 = 26$, $p_3 = 130$ and $N = 400$ are chosen in this example.

For the optimal ellipse fitting $\sigma(T)$, the two real foci $\alpha = -5.39131$, $\beta = -0.01912$ and the length of the minor semiaxis, $b = 2.47412$, are calculated. The asymptotic rate of convergence of the optimal Chebyshev method is given by $\kappa = 0.97903$. The exact solution $\mathbf{x}$ is generated with random numbers chosen from the interval $[-1,1]$ and the right hand side f is computed accordingly. The starting vector $\mathbf{x}_0 = 0$.

Let the size of the perturbation ϵ be 0.05. The values of the asymptotic rates of convergence κ_* from (2.1)–(2.4) and (3.6)–(3.9), and the values of c_* corresponding to different perturbations are shown in Table 1. The ratio of numbers of iterations, RNI $= \log\kappa / \log\kappa_*$, indicates how many extra iterations are proportionately required if the parameters of a CHSIM are selected on the basis of the perturbed foci α_e and/or β_e.

Observe also that both c_{α_u,β_u} and c_{α_o,β_u} are very close to $c_{\beta u}$. This means that if β is underestimated then the effect of either an underestimate or an overestimate for α on the asymptotic rate of convergence of the CHSIM is very small and consequently can be ignored. It is remarked that an overestimate of α is the best case while an underestimate of α with an overestimate of β is the worst case. Observe that $e < 2a/(1-\alpha)$ holds and therefore, the κ_* in column 2 satisfy the inequalities of Theorems 4.1 and 4.2.

All the computations were performed with MATLAB 5.3. The experimental asymptotic rate of convergence of the optimal Chebyshev iterative method is calculated as $\kappa_1 = 0.97904$, which is used to get data in the sixth column of Table 1.

	Approximations			Experimental Results		
Pertur.	κ_*	c_*	RNI	κ_*	c_*	RNI
α_o	0.97911	1.00008	1.004	0.97912	1.00008	1.004
α_u, β_u	0.97947	1.00045	1.022	0.97945	1.00042	1.020
β_u	0.97955	1.00053	1.026	0.97957	1.00054	1.026
α_o, β_u	0.97963	1.00061	1.030	0.97962	1.00059	1.029
α_o, β_o	0.99314	1.01441	3.079	0.99345	1.01472	3.223
β_o	0.99829	1.01967	12.39	0.99511	1.01641	4.321
α_u	0.99873	1.02012	16.68	0.99872	1.02010	16.54
α_u, β_o	1.01799	1.03979	div.	1.01877	1.04058	div.

Table 1. κ_*, c_* and RNIs under perturbations of α and β

As can be observed from Table 1, the experimental data matches the approximation data very well.

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