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THE SYMMETRIC LINEAR MATRIX EQUATION*

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Abstract. In this paper sufficient conditions are derived for the existence of unique and positive definite solutions of the matrix equations $X - A_1^* X A_1 - \ldots - A_m^* X A_m = Q$ and $X + A_1^* X A_1 + \ldots + A_m^* X A_m = Q$. In the case there is a unique solution which is positive definite an explicit expression for this solution is given.

Key words. Linear matrix equation, Positive definite solutions, Uniqueness of solutions.

AMS subject classifications. 15A24

1. Introduction. In this paper we will study the existence of solutions of the linear matrix equations

$$X - A_1^* X A_1 - \ldots - A_m^* X A_m = Q \tag{1.1}$$

and

$$X + A_1^* X A_1 + \ldots + A_m^* X A_m = Q, \tag{1.2}$$

where $Q, A_1, \ldots, A_m$ are arbitrary $n \times n$ matrices. We are particularly interested in (unique) positive definite solutions. The study of these equations is motivated by nonlinear matrix equations, which are of the same form as the algebraic Riccati equation. See for example the nonlinear matrix equation which appears in Chapter 7 of [10]. Solutions of these type of nonlinear matrix equations can be found by using Newton's method. In every step of Newton's method, an equation of the form (1.1) or (1.2) has to be solved.

We shall use the following notations: $\mathcal{M}(n)$ denotes the set of all $n \times n$ matrices, $\mathcal{H}(n) \subset \mathcal{M}(n)$ the set of all $n \times n$ Hermitian matrices and $\mathcal{P}(n) \subset \mathcal{H}(n)$ is the set of all $n \times n$ positive definite matrices. Instead of $X \in \mathcal{P}(n)$ we will also write $X > 0$. Further, $X \geq 0$ means that X is positive semidefinite. As a different notation for $X - Y \geq 0$ $(X - Y > 0)$ we will use $X \geq Y$ $(X > Y)$. The norm we use in this paper is the spectral norm, i.e., $\|A\| = \sqrt{\lambda^+(A^*A)}$ where $\lambda^+(A^*A)$ is the largest eigenvalue of A^*A. The $n \times n$ identity matrix will be written as I_n. Finally, in this paper stable always means stable with respect to the unit circle.

2. The equation $\mathbf{X - A_1^* X A_1 - \ldots - A_m^* X A_m = Q}$. We shall start this section by recalling some results concerning the Stein equation, which is a special case of (1.1). Observe that if $m = 1$ equation (1.1) becomes

$$X - A_1^* X A_1 = Q,$$

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which is the Stein equation, provided that $Q > 0$. It is well-known that this equation has a unique solution if and only if A_1 is stable. Moreover, this unique solution is positive definite. In addition, it is known that the stability of A_1 (and hence the unique solvability of the Stein equation) follows if there is a $\tilde{Q} \in \mathcal{P}(n)$ for which

$$\tilde{Q} - A_1^* \tilde{Q} A_1 > 0. \tag{2.1}$$

This and more about the Stein equation can be found in Section 13.2 of [7]. Another well-known fact is that in the case A_1 is stable, the unique solution of the Stein equation is given by

$$X = \sum_{i=0}^{\infty} A_1^{*i} Q A_1^i \, ; \tag{2.2}$$

see for example formula (5.3.6) in [6].

In the general case we can prove that a condition on $A_1, \ldots, A_m$, similar to the stability condition (2.1), is sufficient for the existence of a unique solution, which is positive definite. Our proof is based on the notion of the Kronecker product of two matrices.

Recall that the Kronecker product of two matrices $A, B \in \mathcal{M}(n)$, denoted as $A \otimes B$, is defined to be the matrix

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}B & a_{n2}B & \cdots & a_{nn}B \end{bmatrix} \in \mathcal{M}(n^2),$$

where a_{ij} is the $(i, j)-$entry of A. Further, for a matrix $A = [a_1 \ldots a_n], a_i \in \mathbb{C}^n, i = 1, \ldots, n$, $\mathrm{vec}(A)$ is defined as the vector

$$\mathrm{vec}(A) = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \in \mathbb{C}^{n^2}.$$

¿From Theorem 1 and Corollary 2 on page 414 in [7] we know that the following theorem holds true.

THEOREM 2.1. *A matrix $X \in \mathcal{M}(n)$ is a solution of equation* (1.1) *if and only if $x = \mathrm{vec}(X)$ is a solution of $Kx = q$, with $K = I_{n^2} - \sum_{j=1}^m A_j^T \otimes A_j^*$ and $q = \mathrm{vec}(Q)$. Consequently, equation* (1.1) *has a unique solution for any $Q \in \mathcal{M}(n)$ if and only if the matrix K is nonsingular.*

Kronecker products were also used in [5], [9] and [11] to solve (not necessarily symmetric) linear matrix equations. The authors of these three papers remark that, in general, nothing is known about the invertibility of the matrix K. In [5], the equation $AX + XB = C$ is discussed in detail. In this special case it is possible to

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express the eigenvalues of K in terms of the eigenvalues of A and B, and this is one method that is considered to derive a condition for the invertibility of K. In [11] the equation $Kx = q$ is converted to a dimension-reduced vector form, which is convenient for machine computations.

In case $m = 1$ the invertibility of K is guaranteed if A_1 is stable, i.e., if there is a $\tilde{Q} \in \mathcal{P}(n)$ such that (2.1) holds. We can generalize this result.

THEOREM 2.2. *Assume there exists a positive definite $n \times n$ matrix $\tilde{Q}$ such that $\tilde{Q} - \sum_{j=1}^{m} A_j^* \tilde{Q} A_j > 0$. Then the matrix*

$$K = I_{n^2} - \sum_{j=1}^{m} A_j^T \otimes A_j^*$$

is invertible. So in this case equation (1.1) *has a unique solution $\bar{X}$ for any $Q \in \mathcal{M}(n)$. Moreover, this unique solution is given by*

$$\bar{X} = Q + \sum_{i=1}^{\infty} \sum_{j_1,\ldots,j_i=1}^{m} A_{j_1}^* \cdots A_{j_i}^* Q A_{j_i} \cdots A_{j_1}\,; \tag{2.3}$$

hence $\bar{X} \in \mathcal{P}(n)$ if $Q \in \mathcal{P}(n)$.

Proof. First assume that $I_n - \sum_{j=1}^{m} A_j^* A_j > 0$, i.e., the assumption holds with $\tilde{Q} = I_n$. We will show that this implies that $\sum_{j=1}^{m} A_j^T \otimes A_j^*$ is stable. To do so, let λ be an eigenvalue of $\sum_{j=1}^{m} A_j^T \otimes A_j^*$ and x a corresponding eigenvector, i.e.,

$$(\sum_{j=1}^{m} A_j^T \otimes A_j^*)x = \lambda x.$$

This implies that X with $x = \mathrm{vec}(X)$ is a solution of

$$A_1^* X A_1 + \cdots + A_m^* X A_m = \lambda X.$$

This equation can be rewritten as

$$\lambda X = A^* \hat{X} A, \tag{2.4}$$

where A and $\hat{X}$ are the block matrices

$$A = \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_m \end{bmatrix}, \quad \hat{X} = \begin{bmatrix} X & 0 & \cdots & 0 \\ 0 & X & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \ldots & 0 & X \end{bmatrix} \in \mathcal{M}(n^2). \tag{2.5}$$

Note that X and $\hat{X}$ have the same eigenvalues, up to multiplicity, so $\|X\| = \|\hat{X}\|$. Further, $\|A\|^2 = \lambda^+(A^*A) = \|A^*A\|$. Hence, taking norms at both sides of equation (2.4) gives that

$$|\lambda|\|X\| = \|A^*\hat{X}A\| \leq \|A\|^2\|\hat{X}\| = \|A^*A\|\|X\|,$$

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so $|\lambda| \leq \|A^*A\| < 1$, because of the assumption $A^*A = \sum_{j=1}^m A_j^* A_j < I_n$. This proves that $\sum_{j=1}^m A_j^T \otimes A_j^*$ is indeed stable.

Next let $\tilde{Q} > 0$ be arbitrary and note that

$$\tilde{Q} - \sum_{j=1}^m A_j^* \tilde{Q} A_j > 0 \Longleftrightarrow I_n - \sum_{j=1}^m (\tilde{Q}^{-\frac{1}{2}} A_j^* \tilde{Q}^{\frac{1}{2}})(\tilde{Q}^{\frac{1}{2}} A_j \tilde{Q}^{-\frac{1}{2}}) > 0.$$

In the first part of the proof we have shown that it follows from the inequality on the right that the matrix $\sum_{j=1}^m (\tilde{Q}^{-\frac{1}{2}T} A_j^T \tilde{Q}^{\frac{1}{2}T}) \otimes (\tilde{Q}^{-\frac{1}{2}} A_j^* \tilde{Q}^{\frac{1}{2}})$ is stable. Using Corollary 1(a) and Corollary 2 on page 408 in [7] we see that

$$\begin{aligned}\sum_{j=1}^m (\tilde{Q}^{-\frac{1}{2}T} A_j^T \tilde{Q}^{\frac{1}{2}T}) \otimes (\tilde{Q}^{-\frac{1}{2}} A_j^* \tilde{Q}^{\frac{1}{2}}) &= \sum_{j=1}^m (\tilde{Q}^{-\frac{1}{2}T} \otimes \tilde{Q}^{-\frac{1}{2}})(A_j^T \otimes A_j^*)(\tilde{Q}^{\frac{1}{2}T} \otimes \tilde{Q}^{\frac{1}{2}}) \\ &= (\tilde{Q}^{\frac{1}{2}T} \otimes \tilde{Q}^{\frac{1}{2}})^{-1} (\sum_{j=1}^m A_j^T \otimes A_j^*)(\tilde{Q}^{\frac{1}{2}T} \otimes \tilde{Q}^{\frac{1}{2}}),\end{aligned}$$

which implies that $\sum_{j=1}^m A_j^T \otimes A_j^*$ is a stable matrix. So the existence of a $\tilde{Q} > 0$ such that $\tilde{Q} - \sum_{j=1}^m A_j^* \tilde{Q} A_j > 0$ is indeed sufficient for the stability of $\sum_{j=1}^m A_j^T \otimes A_j^*$.

¿From the stability of $\sum_{j=1}^m A_j^T \otimes A_j^*$ it follows that there exists an $H \in \mathcal{P}(n^2)$ such that

$$H - (\sum_{j=1}^m A_j^{T*} \otimes A_j) H (\sum_{j=1}^m A_j^T \otimes A_j^*) > 0,$$

which is equivalent to

$$(H^{-\frac{1}{2}} (\sum_{j=1}^m A_j^{T*} \otimes A_j) H^{\frac{1}{2}})(H^{\frac{1}{2}} (\sum_{j=1}^m A_j^T \otimes A_j^*) H^{-\frac{1}{2}}) < I_{n^2}. \tag{2.6}$$

Now let $\|\cdot\|_H$ be the norm defined by $\|X\|_H = \|H^{\frac{1}{2}} X H^{-\frac{1}{2}}\|$, then

$$\begin{aligned}\|\sum_{j=1}^m A_j^T \otimes A_j^*\|_H^2 &= \|H^{\frac{1}{2}} (\sum_{j=1}^m A_j^T \otimes A_j^*) H^{-\frac{1}{2}}\|^2 \\ &= \|(H^{-\frac{1}{2}} (\sum_{j=1}^m A_j^{T*} \otimes A_j) H^{\frac{1}{2}})(H^{\frac{1}{2}} (\sum_{j=1}^m A_j^T \otimes A_j^*) H^{-\frac{1}{2}})\|\end{aligned}$$

and this is smaller than 1 because of (2.6). With Theorem 8.1 in [2] it then follows that K is invertible. Recall from Theorem 2.1 that this implies that (1.1) has a unique solution for any Q. Moreover, Theorem 8.1 in [2] gives us that the inverse of K is given by

$$K^{-1} = \sum_{i=0}^\infty (\sum_{j=1}^m A_j^T \otimes A_j^*)^i = I_{n^2} + \sum_{i=1}^\infty (\sum_{j=1}^m A_j^T \otimes A_j^*)^i.$$

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With induction it can easily be proven that

$$\begin{aligned}(\sum_{j=1}^{m} A_j^T \otimes A_j^*)^i &= \sum_{j_1,\ldots,j_i=1}^{m} (A_{j_1}^T \otimes A_{j_1}^*)\cdots(A_{j_i}^T \otimes A_{j_i}^*)\\ &= \sum_{j_1,\ldots,j_i=1}^{m} (A_{j_1}^T \cdots A_{j_i}^T) \otimes (A_{j_1}^* \cdots A_{j_i}^*),\end{aligned}$$

so

$$K^{-1} = I_{n^2} + \sum_{i=1}^{\infty} \sum_{j_1,\ldots,j_i=1}^{m} (A_{j_1}^T \cdots A_{j_i}^T) \otimes (A_{j_1}^* \cdots A_{j_i}^*).$$

This implies that the unique solution $\bar{X}$ of (1.1) satisfies

$$\mathrm{vec}(\bar{X}) = K^{-1}\mathrm{vec}(Q) = \mathrm{vec}(Q) + (\sum_{i=1}^{\infty} \sum_{j_1,\ldots,j_i=1}^{m} (A_{j_1}^T \cdots A_{j_i}^T) \otimes (A_{j_1}^* \cdots A_{j_i}^*))\mathrm{vec}(Q)$$

and hence

$$\bar{X} = Q + \sum_{i=1}^{\infty} \sum_{j_1,\ldots,j_i=1}^{m} A_{j_1}^* \cdots A_{j_i}^* Q A_{j_i} \cdots A_{j_1}.$$

Thus $\bar{X}$ is indeed positive definite if Q is positive definite. □

REMARK 2.3. Note that in case $m = 1$ the condition in this theorem becomes (2.1), and (2.3) is exactly the expression for the unique solution of the Stein equation given in (2.2).

In [1] and [8] solutions of a matrix equation are considered as fixed points of some map $\mathcal{G}$. Also, in the case of equation (1.1) we are interested in fixed points of the map

$$\mathcal{G}_+(X) = Q + \sum_{j=1}^{m} A_j^* X A_j.$$

Although in [1] m is equal to 1, the results in that paper can be easily generalized to the case that $m \in \mathbb{N}$. In particular, Theorem 5.1 in [1] also holds for the map $\mathcal{G}_+$. Combining this theorem and Theorem 2.2 gives us the following result.

COROLLARY 2.4. *Let $Q \in \mathcal{P}(n)$ and assume there is a positive definite solution $\tilde{X}_0$ of the inequality*

$$X - A_1^* X A_1 - \cdots - A_m^* X A_m \geq Q. \tag{2.7}$$

Then (1.1) *has a unique solution. Moreover, this unique solution is positive definite and the sequence $\{\mathcal{G}_+^k(Q)\}_{k=0}^{\infty}$ increases to this unique solution, while the sequence $\{\mathcal{G}_+^k(\tilde{X}_0)\}_{k=0}^{\infty}$ decreases to this unique solution.*

Note that the condition in Theorem 2.2 is weaker than condition (2.7). We will now show that this corollary also gives us expression (2.3) for the unique solution.

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Well then, let $\bar{X}$ be the unique positive definite solution of (1.1). Then it follows from Corollary 2.4 that

$$\bar{X} = \lim_{k\to\infty} \mathcal{G}_+^k(Q),$$

so it is very useful to derive an expression for $\mathcal{G}_+^k(Q)$ first.

LEMMA 2.5. *For $k = 0, 1, 2, \ldots$ the following holds true:*

$$\mathcal{G}_+^k(Q) = Q + \sum_{i=1}^{k} \sum_{j_1,\ldots,j_i=1}^{m} A_{j_1}^* \cdots A_{j_i}^* Q A_{j_i} \cdots A_{j_1}. \tag{2.8}$$

Proof. We will prove this by induction. For $k = 0$ and $k = 1$ it is evident. Now assume that it is true for $k = l$, then we have to prove that it also holds for $k = l+1$. Well then,

$$\begin{aligned}
\mathcal{G}_+^{l+1}(Q) &= \mathcal{G}_+(\mathcal{G}_+^l(Q)) = \mathcal{G}_+(Q + \sum_{i=1}^{l} \sum_{j_1,\ldots,j_i=1}^{m} A_{j_1}^* \cdots A_{j_i}^* Q A_{j_i} \cdots A_{j_1}) \\
&= Q + \sum_{j=1}^{m} A_j^* Q A_j + \sum_{j=1}^{m} \sum_{i=1}^{l} \sum_{j_1,\ldots,j_i=1}^{m} A_j^* A_{j_1}^* \cdots A_{j_i}^* Q A_{j_i} \cdots A_{j_1} A_j \\
&= Q + \sum_{j=1}^{m} A_j^* Q A_j + \sum_{i=1}^{l} \sum_{j_1,\ldots,j_{i+1}=1}^{m} A_{j_1}^* \cdots A_{j_{i+1}}^* Q A_{j_{i+1}} \cdots A_{j_1} \\
&= Q + \sum_{j=1}^{m} A_j^* Q A_j + \sum_{i=2}^{l+1} \sum_{j_1,\ldots,j_i=1}^{m} A_{j_1}^* \cdots A_{j_i}^* Q A_{j_i} \cdots A_{j_1} \\
&= Q + \sum_{i=1}^{l+1} \sum_{j_1,\ldots,j_i=1}^{m} A_{j_1}^* \cdots A_{j_i}^* Q A_{j_i} \cdots A_{j_1},
\end{aligned}$$

so (2.8) also holds for $k = l+1$, which we had to show. □

It follows directly from this lemma that indeed

$$\lim_{k\to\infty} \mathcal{G}_+^k(Q) = Q + \sum_{i=1}^{\infty} \sum_{j_1,\ldots,j_i=1}^{m} A_{j_1}^* \cdots A_{j_i}^* Q A_{j_i} \cdots A_{j_1}\,; \tag{2.9}$$

hence $\bar{X}$ satisfies (2.3). Note that (2.9) can be used to find an approximation of the unique positive definite solution of (1.1) in numerical examples.

3. The equation $\mathbf{X + A_1^* X A_1 + \ldots + A_m^* X A_m = Q}$. In this section we will study the existence of (unique) positive definite solutions of equation (1.2). We can prove the following proposition analogously to Theorem 2.2.

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PROPOSITION 3.1. *Assume there exists a positive definite $n \times n$ matrix $\tilde{Q}$ such that $\tilde{Q} - \sum_{j=1}^{m} A_j^* \tilde{Q} A_j > 0$. Then the matrix*

$$L = I_{n^2} + \sum_{j=1}^{m} A_j^T \otimes A_j^*$$

is invertible. So in this case equation (1.2) has a unique solution $\bar{X}$ for any $Q \in \mathcal{M}(n)$. Moreover, this unique solution is given by

$$\bar{X} = Q + \sum_{i=1}^{\infty} \sum_{j_1,\ldots,j_i=1}^{m} (-1)^i A_{j_1}^* \cdots A_{j_i}^* Q A_{j_i} \cdots A_{j_1}. \tag{3.1}$$

In this case it doesn't follow that $\bar{X}$ is positive definite if Q is positive definite. Hence we need an additional condition, if we want $\bar{X}$ to be positive definite.

Let the map $\mathcal{G}_-$ be defined by

$$\mathcal{G}_-(X) = Q - \sum_{j=1}^{m} A_j^* X A_j.$$

This map will play the same role as $\mathcal{G}_+$ did in the previous section, i.e., the solutions of (1.2) are exactly the fixed points of $\mathcal{G}_-$. The following lemma can be proven analogously to Corollary 2.1 in [8].

LEMMA 3.2. *Let $Q \in \mathcal{P}(n)$ and assume that $Q - \sum_{j=1}^{m} A_j^* Q A_j > 0$. Then equation (1.2) has a solution in the set $[Q - \sum_{j=1}^{m} A_j^* Q A_j, Q]$ and all its positive semidefinite solutions are contained in this set.*

Combining Proposition 2.2 and Lemma 3.2 gives the first part of the main result of this section.

THEOREM 3.3. *Let $Q \in \mathcal{P}(n)$ and assume that $Q - \sum_{j=1}^{m} A_j^* Q A_j > 0$. Then (1.2) has a unique solution which is positive definite. Moreover, the sequence $\{\mathcal{G}_-^k(Q)\}_{k=0}^{\infty}$ converges to this unique solution.*

Proof. We only have to prove the convergence of $\{\mathcal{G}_-^k(Q)\}_{k=0}^{\infty}$ to the unique solution. With induction it can be proven that

$$\mathcal{G}_-^k(Q) = Q + \sum_{i=1}^{\infty} \sum_{j_1,\ldots,j_i=1}^{m} (-1)^i A_{j_1}^* \cdots A_{j_i}^* Q A_{j_i} \cdots A_{j_1}.$$

So the limit of $\mathcal{G}_-^k(Q)$ for k to infinity is equal to (3.1) and according to Proposition 3.1 this is the unique positive definite solution of (1.2). ▯

The following example shows that the condition that there exists a $\tilde{Q} > 0$ such that $\tilde{Q} - \sum_{j=1}^{m} A_j^* \tilde{Q} A_j > 0$ cannot be omitted in Theorem 2.2 and Proposition 3.1.

EXAMPLE 3.4. Let $m = 2$ and

$$Q = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \quad A_1 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}, \quad A_2 = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

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Then $Q - \sum_{j=1}^m A_j^* Q A_j = 0$. It can be easily checked that for every $a \in (-1,1)$ the matrix

$$\bar{X} = \begin{bmatrix} 1 & ai \\ -ai & 1 \end{bmatrix}$$

is a positive definite solution of (1.2). So it is a consequence of Theorem 3.3 that there does not exist positive matrices $\tilde{Q}$ such that $\tilde{Q} - \sum_{j=1}^2 A_j^* \tilde{Q} A_j > 0$. This can also be seen directly. Indeed, let $\tilde{Q} > 0$ be given by

$$\tilde{Q} = \begin{bmatrix} q_{11} & q_{12} \\ \bar{q}_{12} & q_{22} \end{bmatrix}$$

so that

$$\tilde{Q} - \sum_{j=1}^2 A_j^* \tilde{Q} A_j = \begin{bmatrix} \frac{1}{2}q_{11} - \frac{1}{2}q_{22} & \frac{3}{2}q_{12} - \frac{1}{2}\bar{q}_{12} \\ \frac{3}{2}\bar{q}_{12} - \frac{1}{2}q_{12} & \frac{1}{2}q_{22} - \frac{1}{2}q_{11} \end{bmatrix}.$$

The eigenvalues of this matrix are $\pm\sqrt{(\frac{1}{2}q_{11} - \frac{1}{2}q_{22})^2 + |\frac{3}{2}\bar{q}_{12} - \frac{1}{2}q_{12}|^2}$, so this matrix cannot be positive definite.

Because there does not exist a $\tilde{Q} > 0$ such that $\tilde{Q} - \sum_{j=1}^2 A_j^* \tilde{Q} A_j > 0$, there cannot exist an $\bar{X} > 0$ such that $\bar{X} - \sum_{j=1}^2 A_j^* \bar{X} A_j = Q > 0$, so in this case (1.1) does not have any positive definite solution.

4. Two Related Equations. Recall that the matrices K and L are invertible if there exists a positive definite $\tilde{Q}$ such that $\tilde{Q} - \sum_{j=1}^m A_j^* \tilde{Q} A_j > 0$. In practice it can be hard to find such $\tilde{Q}$. A first attempt may be to try $\tilde{Q} = Q$ or $\tilde{Q} = I_n$. There are matrices $A_1, \ldots, A_m$ such that $\tilde{Q} - \sum_{j=1}^m A_j^* \tilde{Q} A_j > 0$ is not satisfied for one of these choices of $\tilde{Q}$, but such that $\tilde{Q} - \sum_{j=1}^m A_j \tilde{Q} A_j^* > 0$ is satisfied for one of these $\tilde{Q}$.

Example 4.1. Let the matrices A_1 and A_2 be given by

$$A_1 = \begin{bmatrix} -0.1652 & -0.4786 \\ -0.2499 & 0.6463 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -0.0180 & 0.2205 \\ -0.0874 & 0.6405 \end{bmatrix}.$$

Then the eigenvalues of $A_1^* A_1 + A_2^* A_2$ are 0.0780 and 1.1253 and the eigenvalues of $A_1 A_1^* + A_2 A_2^*$ are 0.2799 and 0.9234. So we have $A_1^* A_1 + A_2^* A_2 \not< I_2$ and $A_1 A_1^* + A_2 A_2^* < I_2$.

Note that if there exists a $\tilde{Q} > 0$ such that $\tilde{Q} - \sum_{j=1}^m A_j \tilde{Q} A_j^* > 0$, then the matrices

$$K^* = I_{n^2} - \sum_{j=1}^m A_j^{*T} \otimes A_j,$$

$$L^* = I_{n^2} + \sum_{j=1}^m A_j^{*T} \otimes A_j$$

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are invertible. Hence the equations

$$X - A_1XA_1^* - \cdots - A_mXA_m^* = Q, \tag{4.1}$$

$$X + A_1XA_1^* + \cdots + A_mXA_m^* = Q$$

both have a unique solution. Moreover, the unique solution of (4.1) is positive definite. Because the invertibility of K^* and L^* is equivalent to the invertibility of K and L, it follows that also (1.1) and (1.2) have a unique solution. The unique solution of (1.1) is even positive definite. Indeed, recall from the proof of Theorem 2.2 that the existence of a $\tilde{Q} \in \mathcal{P}(n)$ such that $\tilde{Q} - \sum_{j=1}^m A_j\tilde{Q}A_j^* > 0$ implies that $\sum_{j=1}^m A_j^{*T} \otimes A_j$ is stable, which is equivalent to stability of $\sum_{j=1}^m A_j^T \otimes A_j^*$. Hence, following the proof of Theorem 2.2, the unique solution of (1.1) is positive definite. Interchanging the roles of A_j and A_j^* completes the proof of the following theorem.

THEOREM 4.2. *Let Q be positive definite. Equation* (1.1) *has a unique solution which is positive definite if and only if equation* (4.1) *has a unique solution which is positive definite.*

Using this theorem we can prove that not only the invertibility of K and L is equivalent to the invertibility of K^* and L^*, but also that the sufficient condition for the invertibility of K and L (see Theorem 2.2 and Proposition 3.1) is equivalent to this sufficient condition applied to K^* and L^*.

COROLLARY 4.3. *There exists a $\tilde{Q} \in \mathcal{P}(n)$ such that $\tilde{Q} - \sum_{j=1}^m A_j^*\tilde{Q}A_j > 0$ if and only if there exists a $\bar{Q} \in \mathcal{P}(n)$ such that $\bar{Q} - \sum_{j=1}^m A_j\bar{Q}A_j^* > 0$.*

Proof. Assume that $\tilde{Q} - \sum_{j=1}^m A_j^*\tilde{Q}A_j > 0$ for a $\tilde{Q} > 0$. With Theorem 2.2 we know that in this case (1.1) has a unique solution, which is positive definite, for any $Q \in \mathcal{P}(n)$. Then it follows from the previous theorem that also equation (4.1) has a unique solution which is positive definite, say $\bar{X}$. But then $\bar{X} - \sum_{j=1}^m A_j\bar{X}A_j^* = Q > 0$, which proves one implication of the theorem. The other implication is proven by reversing the roles of A_j and A_j^*, $j = 1, \ldots, m$. □

5. Positive Cones and Positive Operators. In this section we will prove a slightly weaker result than Theorem 2.2. The proof is based on the theory of positive cones and linear operators mapping a cone into itself; see for example Section I.4 in [3] and Chapter 1 and 2 in [4]. Before giving the proof, we will give an overview of the definitions and results in [3] and [4], which we need in this section.

DEFINITION 5.1. *Given a real Banach space $\mathcal{B}$, a positive cone in $\mathcal{B}$ is a nonempty subset $\mathcal{C}$ of $\mathcal{B}$ that satisfies the following properties:*

(i) *$x + y \in \mathcal{C}$, whenever $x, y \in \mathcal{C}$,*

(ii) *$\lambda x \in \mathcal{C}$, whenever $\lambda \in [0, \infty)$ and $x \in \mathcal{C}$,*

(iii) *the zero vector is the only element $x \in \mathcal{C}$ for which x and $-x$ belong to $\mathcal{C}$.*

DEFINITION 5.2. *A cone $\mathcal{C}$ is called solid if it has a nonempty interior. It is called reproducing if every element of $\mathcal{B}$ is the difference of two elements of $\mathcal{C}$.*

It is easy to see that every solid cone is reproducing.

DEFINITION 5.3. *A linear operator $\mathcal{K}$ on $\mathcal{B}$ is called a positive operator in the lattice sense if it maps the positive cone $\mathcal{C}$ into itself. Given $u_0 \in \mathcal{C}\backslash\{0\}$, we call $\mathcal{K}$*

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u_0-positive, if for every $x \in \mathcal{C}\backslash\{0\}$ there exist $m \in \mathbb{N}$ and $\alpha, \beta \in (0,\infty)$ such that $\alpha u_0 \leq \mathcal{K}^m(x) \leq \beta u_0$.

The following result about the spectral radius of any positive operator is Theorem I.4.1 in [3].

THEOREM 5.4. *Let $\mathcal{K}$ be a power compact positive operator (i.e., $\mathcal{K}^m$ compact for some $m \in \mathbb{N}$) on a Banach space $\mathcal{B}$ with a reproducing cone $\mathcal{C}$. Then either the spectral radius of $\mathcal{K}$ vanishes or the spectral radius of $\mathcal{K}$ is a positive eigenvalue with at least one corresponding eigenvector in $\mathcal{C}$.*

The proof of Theorem 2.2 which will be given in this section involves Theorem I.4.4 in [3], for the special case $c = 1$. For completeness we state this theorem below

THEOREM 5.5. *Let $\mathcal{B}$ be a (real or complex) Banach space with reproducing cone $\mathcal{C}$ and let $\mathcal{K}$ be a u_0-positive operator on $\mathcal{B}$ with spectral radius $\rho(\mathcal{K})$. Consider the equation $x - \mathcal{K}(x) = y$, where $y \in \mathcal{C}$. Then the following statements hold true:*

(*i*) *For $\rho(\mathcal{K}) < 1$, there is a unique solution $x \in \mathcal{C}$ for every $y \in \mathcal{C}$, which is given by the absolutely convergent series*

$$x = \sum_{k=0}^{\infty} \mathcal{K}^k(y).$$

(*ii*) *For $\rho(\mathcal{K}) = 1$ and $y \in \mathcal{C}$ there is no solution $x \in \mathcal{C}$ unless $y = 0$. In that case all the solutions in $\mathcal{C}$ are positive multiples of the positive eigenvector corresponding to the eigenvalue $\rho(\mathcal{K})$.*

(*iii*) *For $\rho(\mathcal{K}) > 1$ and $y \in \mathcal{C}$ there do not exist any solutions $x \in \mathcal{C}$ unless $y = 0$. In this case $x = 0$ is the only solution in $\mathcal{C}$.*

Now let $\mathcal{B} = \mathcal{H}(n), \mathcal{C} = \mathcal{P}(n) \cup \{0\}$ and

$$\mathcal{K}(X) = \sum_{j=1}^{m} A_j^* X A_j.$$

It is obvious that $\mathcal{H}(n)$ is indeed a real Banach space and that $\mathcal{P}(n) \cup \{0\}$ is indeed a positive cone. Moreover, its interior is equal to $\mathcal{P}(n)$, so it is not empty. Hence $\mathcal{P}(n) \cup \{0\}$ is even a reproducing cone. Now let A and $\hat{X}$ be as in (2.5). Then $\mathcal{K}$ can be written as

$$\mathcal{K}(X) = A\hat{X}A.$$

If $\ker A = \{0\}$, then $\mathcal{K}$ maps $\mathcal{C}$ into itself. Indeed, if $X \in \mathcal{P}(n)$, then $\hat{X} \in \mathcal{P}(mn)$, which implies, together with $\ker A = \{0\}$, that $\mathcal{K}(X) \in \mathcal{P}(mn)$. The condition $\ker A = \{0\}$ is also sufficient for the I_n-positivity of $\mathcal{K}$, as the following lemma shows.

LEMMA 5.6. *Let A be an $mn \times n$ matrix. If $\ker A = \{0\}$, then $\mathcal{K}$ is I_n-positive.*

Proof. Recall that $\ker A = \{0\}$ implies that $A^*\hat{X}A > 0$ for $X > 0$. Now let $\alpha = \lambda^+(A^*\hat{X}A)$ and $\beta = \lambda^-(A^*\hat{X}A)$. Then $\alpha, \beta > 0$ and $\alpha I_n \leq \mathcal{K}(X) \leq \beta I_n$. This proves the lemma. □

So our choices of $\mathcal{B}, \mathcal{C}$ and $\mathcal{K}$ satisfy the conditions of Theorem 5.5, if we assume that $\ker A = \{0\}$. Hence we can use this theorem to derive conditions for the existence

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of a unique solution in $\mathcal{C}$ of the equation $X - \mathcal{K}(X) = Q, Q \in \mathcal{C}$, which is exactly equation (1.1).

THEOREM 5.7. *Let* $Q \in \mathcal{C}$ *and* A *be an* $mn \times n$ *matrix such that* $\ker A = \{0\}$ *and such that there exists* $\tilde{Q} \in \mathcal{P}(n)$ *satisfying* $\tilde{Q} - \sum_{j=1}^{m} A_j^* \tilde{Q} A_j > 0$. *Then* (1.1) *has a unique solution* $\bar{X} \in \mathcal{C}$ *for every* $Q \in \mathcal{C}$. *Moreover,*

$$\bar{X} = \sum_{k=0}^{\infty} \mathcal{K}^k(Q) = Q + \sum_{i=1}^{\infty} \sum_{j_1,\ldots,j_i=1}^{m} A_{j_1}^* \cdots A_{j_i}^* Q A_{j_i} \cdots A_{j_1}. \tag{5.1}$$

Proof. Note that the matrix $\tilde{Q}$ is a positive definite solution of $X - \mathcal{K}(X) = P$ for some $P \in \mathcal{P}(n)$. So it follows from Theorem 5.5(i) and (ii) that $\rho(\mathcal{K}) < 1$. Hence, with part (ii) of that theorem, the theorem follows immediately. □

REMARK 5.8. Note that $\bar{X} = 0$ is the unique solution of (1.1) if and only if $Q = 0$. Hence it follows that equation (1.1) has a unique solution in $\mathcal{P}(n)$ for all $Q \in \mathcal{P}(n)$, under the conditions of the theorem.

Recall that in Section 2 the matrix A did not need to satisfy $\ker A = \{0\}$. We will now show that Theorem 5.7 also holds for arbitrary $mn \times n$ matrices A. First we will reduce (1.1) to the special case that $Q = I_n$.

LEMMA 5.9. *Let* $Q \in \mathcal{P}(n)$ *and* A *be an* $mn \times n$ *matrix. Then* $\bar{X}$ *is a solution of* (1.1) *if and only if* $\bar{Y} = Q^{-\frac{1}{2}} \bar{X} Q^{-\frac{1}{2}}$ *is a solution of*

$$Y - \tilde{A}_1^* Y \tilde{A}_1 - \cdots - \tilde{A}_m^* Y \tilde{A}_m = I_n, \tag{5.2}$$

where $\tilde{A}_j = Q^{\frac{1}{2}} A_j Q^{-\frac{1}{2}},\ j = 1, \ldots, m$.

That this lemma is true, can be easily seen, so we will not give the proof.

REMARK 5.10. It is obvious that (1.1) has a unique positive definite solution if and only if (5.2) has a unique positive definite solution. Further, if $\tilde{Q} \in \mathcal{P}(n)$ satisfies $\tilde{Q} - \sum_{j=1}^{\infty} A_j^* \tilde{Q} A_j > 0$, then $\bar{Q} = Q^{-\frac{1}{2}} \tilde{Q} Q^{-\frac{1}{2}}$ satisfies $\bar{Q} - \sum_{j=1}^{\infty} \tilde{A}_j^* \bar{Q} \tilde{A}_j > 0$.

Now assume that $A \neq 0$ and $\ker A \neq \{0\}$. A decomposition of $\mathbb{C}^n$ is given by

$$\mathbb{C}^n = (\ker A)^{\perp} \oplus \ker A.$$

Let d be the dimension of $\ker A$. Because $x \in \ker A$ if and only if $x \in \ker A_j,\ j = 1, \ldots, m$, we can write the matrices $A_j,\ j = 1, \ldots, m$, with respect to this decomposition as follows:

$$A_j = \begin{bmatrix} A_1^{(j)} & 0 \\ A_2^{(j)} & 0 \end{bmatrix},$$

where $A_1^{(j)}$ is an $(n-d) \times (n-d)$ matrix. Further, a positive definite solution of (1.1) with $Q = I_n$ is necessarily of the form

$$X = \begin{bmatrix} X_1 & 0 \\ 0 & I_d \end{bmatrix}, \quad X_1 \in \mathcal{P}(n-d),$$

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So equation (1.1) with $Q = I_n$ reduces to

$$X_1 - \sum_{j=1}^{m} A_1^{(j)*} X_1 A_1^{(j)} = I_d + \sum_{j=1}^{m} A_2^{(j)*} A_2^{(j)}. \tag{5.3}$$

Note that X is a solution of (1.1) if and only if X_1 is a solution of (5.3).

LEMMA 5.11. *Let $Q = I_n$ and A be an $mn \times n$ matrix such that* $\ker A \neq \{0\}$. *Then* (1.1) *has a unique positive definite solution if and only if* (5.3) *has a unique positive definite solution. Moreover, if there exists a $\tilde{Q} \in \mathcal{P}(n)$ such that $\tilde{Q} - \sum_{j=1}^m A_j^* \tilde{Q} A_j > 0$, then there exists a $\tilde{Q}_1 \in \mathcal{P}(n-d)$ such that $\tilde{Q}_1 - \sum_{j=1}^m A_1^{(j)*} \tilde{Q}_1 A_1^{(j)} > 0$.*

Proof. We only have to prove the last part. Well then, assume that $\tilde{Q} > 0$ satisfies $\tilde{Q} - \sum_{j=1}^m A_j^* \tilde{Q} A_j > 0$. Write $\tilde{Q}$ with respect to the decomposition of $\mathbb{C}^n$:

$$\tilde{Q} = \begin{bmatrix} \tilde{Q}_{11} & \tilde{Q}_{12} \\ \tilde{Q}_{12}^* & \tilde{Q}_{22} \end{bmatrix}, \text{ where } \tilde{Q}_{11} \text{ is an } (n-d) \times (n-d) \text{ matrix.}$$

Note that

$$\begin{bmatrix} \tilde{Q}_{11} & \tilde{Q}_{12} \\ \tilde{Q}_{12}^* & \tilde{Q}_{22} \end{bmatrix} = S^* \begin{bmatrix} \tilde{Q}_{11} - \tilde{Q}_{12}\tilde{Q}_{22}^{-1}\tilde{Q}_{12}^* & 0 \\ 0 & \tilde{Q}_{22} \end{bmatrix} S,$$

where

$$S = \begin{bmatrix} I_{n-d} & 0 \\ \tilde{Q}_{22}^{-1}\tilde{Q}_{12}^* & I_d \end{bmatrix}.$$

Because S is invertible, it follows that $\tilde{Q} - \sum_{j=1}^m A_j^* \tilde{Q} A_j > 0$ if and only if

$$S^{-*}\tilde{Q}S^{-1} - \sum_{j=1}^{m} (S^{-*}A_j^* S^*)(S^{-*}\tilde{Q}S^{-1})(SA_jS^{-1}) > 0. \tag{5.4}$$

¿From the equality

$$SA_jS^{-1} = \begin{bmatrix} A_1^{(j)} & 0 \\ \tilde{Q}_{22}^{-1}\tilde{Q}_{12}^* A_1^{(j)} + A_2^{(j)} & 0 \end{bmatrix}$$

it follows that (5.4) can be written as

$$\begin{aligned} &\begin{bmatrix} \tilde{Q}_1 & 0 \\ 0 & \tilde{Q}_{22} \end{bmatrix} - \sum_{j=1}^m \begin{bmatrix} A_1^{(j)*} & \tilde{A}_2^{(j)*} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{Q}_1 & 0 \\ 0 & \tilde{Q}_{22} \end{bmatrix} \begin{bmatrix} A_j^{(j)} & 0 \\ \tilde{A}_j^{(j)} & 0 \end{bmatrix} = \\ &= \begin{bmatrix} \tilde{Q}_1 & 0 \\ 0 & \tilde{Q}_{22} \end{bmatrix} - \sum_{j=1}^m \begin{bmatrix} A_1^{(j)^*}\tilde{Q}_1 A_1^{(j)} + \tilde{A}_2^{(j)*}\tilde{Q}_{22}\tilde{A}_2^{(j)} & 0 \\ 0 & 0 \end{bmatrix} > 0, \end{aligned} \tag{5.5}$$

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where $\tilde{Q}_1 = \tilde{Q}_{11} - \tilde{Q}_{12}\tilde{Q}_{22}^{-1}$ and $\tilde{A}_2^{(j)} = \tilde{Q}_{22}^{-1}\tilde{Q}_{12}^{*}A_1^{(j)} + A_2^{(j)}$. It is well-known that (5.5) is satisfied if and only if

$$\begin{cases} \tilde{Q}_1 - \sum_{j=1}^{m}(A_1^{(j)^*}\tilde{Q}_1 A_1^{(j)} + \tilde{A}_2^{(j)*}\tilde{Q}_{22}\tilde{A}_2^{(j)}) > 0, \\ \tilde{Q}_{22} > 0. \end{cases}$$

This implies that

$$\tilde{Q}_1 - \sum_{j=1}^{m} A_1^{(j)^*}\tilde{Q}_1 A_1^{(j)} > \sum_{j=1}^{m} \tilde{A}_2^{(j)*}\tilde{Q}_{22}\tilde{A}_2^{(j)} \geq 0,$$

which proves the lemma. □

Let A_1 be the matrix given by

$$A_1 = \begin{bmatrix} A_1^{(1)} \\ A_1^{(2)} \\ \vdots \\ A_1^{(m)} \end{bmatrix}.$$

If $A_1 = 0$, then it is obvious that (5.3) has a unique solution. If $\ker A_1 = \{0\}$, then the existence of a unique positive definite solution follows from Theorem 5.7. Otherwise, we reduce (5.3) first to an equation of the form (5.2), but of lower dimensions, and then to one of the form (5.3), also of lower dimensions. The reduction process will end in a finite number of steps and it will result in the equation

$$X^{(red)} - \sum_{j=1}^{m} A_j^{(red)} X^{(red)} A_j^{(red)} = Q^{(red)}, \tag{5.6}$$

with $Q^{(red)} > 0$ and $A^{(red)} = 0$ or $\ker A^{(red)} = \{0\}$, where

$$A^{(red)} = \begin{bmatrix} A_1^{(red)} \\ A_2^{(red)} \\ \vdots \\ A_m^{(red)} \end{bmatrix}.$$

Subsequently applying Lemma 5.9, Remark 5.10 and Lemma 5.11 proves the following corollary.

Corollary 5.12. *Let $Q \in \mathcal{P}(n)$ and $A \neq 0$ be an $mn \times n$ matrix such that $\ker A \neq \{0\}$. Then* (1.1) *has a unique positive definite solution if and only if* (5.6) *has a unique positive definite solution. Moreover, $Q^{(red)} > 0$ and if there exists a $\tilde{Q} > 0$ such that $\tilde{Q} - \sum_{j=1}^{m} A_j^{*}\tilde{Q}A_j > 0$, then there exists a $\tilde{Q}^{(red)} > 0$ such that $\tilde{Q}^{(red)} - \sum_{j=1}^{m} A_j^{(red)*}\tilde{Q}^{(red)}A_j^{(red)} > 0$.*

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Now we are able to prove the main result of this section.

THEOREM 5.13. *Let $Q \in \mathcal{P}(n)$ and A an $mn \times n$ matrix such that there exists a $\tilde{Q} \in \mathcal{P}(n)$ such that $\tilde{Q} - \sum_{j=1}^{m} A_j^* \tilde{Q} A_j > 0$. Then* (1.1) *has a unique solution in* $\mathcal{P}(n)$.

Proof. If $\ker A = \{0\}$, then this is just the first part of Theorem 5.7. If $\ker A \neq \{0\}$, then we can reduce equation (1.1) to equation (5.6) with $A^{(red)} = 0$ or $\ker A^{(red)} = \{0\}$. In case $A^{(red)} = 0$, it immediately follows that $X^{(red)} = Q^{(red)}$ is the unique positive definite solution of (5.6). With Corollary 5.12 it then follows that also (1.1) has a unique positive definite solution. In case $\ker A^{(red)} = \{0\}$ we can apply Theorem 5.7, because we know from Corollary 5.12 that the conditions of this theorem are satisfied. So (5.6) has a unique positive definite solution and thus also equation (1.1). $\square$

REMARK 5.14. In this case we can also show that the unique solution is given by (5.1). Because the unique positive definite solution of (1.1) satisfies the inequality in Corollary 2.4, we can apply this theorem. This gives us that the unique solution is indeed given by (5.1).

REMARK 5.15. Note the subtle difference between Theorem 2.2 and Theorem 5.13. Although the hypotheses in both theorems are the same, Theorem 2.2 gives us the existence of a unique solution which turns out to be positive definite, whereas Theorem 5.13 gives us the existence of a unique positive definite solution. Hence Theorem 2.2 is a stronger result.

At first sight, the method based on Kronecker products and the method used in this section might be very different. However, the existence of a $\tilde{Q} > 0$ such that $\tilde{Q} - \sum_{j=1}^{m} A_j^* \tilde{Q} A_j > 0$ plays an important role in both cases . It is sufficient for the stability of $\sum_{j=1}^{m} A_j^T \otimes A_j^*$ and for $\rho(\mathcal{K}) < 1$. It is easy to prove that these properties are equivalent.

LEMMA 5.16. *The spectrum of $\mathcal{K}$ is equal to the spectrum of $\sum_{j=1}^{m} A_j^T \otimes A_j^*$. So $\rho(\mathcal{K}) < 1$ if and only if $\sum_{j=1}^{m} A_j^T \otimes A_j^*$ is stable.*

REMARK 5.17. From this lemma it follows that in this case we can also use Theorem 5.5 to prove Proposition 3.1. Indeed, the existence of a $\tilde{Q} > 0$ such that $\tilde{Q} - \sum_{j=1}^{m} A_j^* \tilde{Q} A_j > 0$ implies that $\rho(\mathcal{K}) < 1$. Hence the matrix $\sum_{j=1}^{m} A_j^T \otimes A_j^*$ is stable, which implies that L is invertible.

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