Singular Values of Trilinear Forms

Bo Bernhardsson and Jaak Peetre

CONTENTS

Introduction
1. Millennial Equation: The General Case
2. The Binary Case
3. Some General Nonsense
4. Singular Values of Trilinear Forms Via Minimax Principles
Acknowledgements
Electronic Availability
References

Let $T : \mathcal{H}_1 \times \mathcal{H}_2 \times \mathcal{H}_3 \to \mathbb{C}$ be a trilinear form, where $\mathcal{H}_1$, $\mathcal{H}_2$, $\mathcal{H}_3$ are separable Hilbert spaces. In the hypothesis that at least two of the three spaces are finite dimensional we show that the norm square $\lambda = \|T\|^2$ is a root of a certain algebraic equation, usually of very high degree, which we baptize the millennial equation, because it is an analogue of the secular equation in the bilinear case. More generally, as indicated in the title, we can consider singular values of a trilinear form and their squares too satisfy the same equation. We work out the binary case (all three spaces are two dimensional). Even in this case the situation is complex, so, in the absence of any genuine results, we content ourselves with advancing a number of conjectures suggested by computer experiments. Finally, we connect the singular values of a trilinear form with the critical values of an associated family of a one parameter family of bilinear forms. Also here we have to offer mainly only experimental evidence.

INTRODUCTION

Trilinear forms play an increasingly important rôle in many parts of analysis — for instance in Fourier analysis, where they appear in the guise of paracommutators and compensated quantities (for a survey, see [Peng and Wong 2000]). In this paper we address the following problem. Suppose

$$T : \mathcal{H}_1 \times \mathcal{H}_2 \times \mathcal{H}_3 \to \mathbb{C}$$

is a trilinear form where $\mathcal{H}_1$, $\mathcal{H}_2$, $\mathcal{H}_3$ are separable Hilbert spaces. What is the *norm* of T,

$$\|T\| = \sup_{\|x\|,\|y\|,\|z\|\leq 1} |T(x,y,z)|?$$

In the hypothesis that at least two of our three spaces are finite dimensional we show (Section 1) that the norm square $\lambda = \|T\|^2$ is a root of a certain algebraic equation, usually of very high degree, which we baptize the *millennial equation*, because it is an analogue of the secular equation in the bilinear case. More generally, as indicated in the title, we can consider *singular values* of a trilinear form and their squares too satisfy the same equation.

AMS Subject Classification: 47B10.

Keywords: trilinear form, eliminant, discriminant, singular value, minimax principle, millennial equation.

1058-6458/2001 \$0.50 per page

In Section 2 we work out the *binary* case (all three spaces are two dimensional). Even in this case the situation is rather complex so, fault of any genuine results, we content ourselves with advancing a number of conjectures which we were led to by computer experiments using either Mathematica and/or Matlab (see, in particular, the remark in Section 2).

In Section 3, returning to the general situation, we briefly indicate a very general set-up of which the problem treated in Section 1 is but a special case.

Finally, in Section 4 we connect the singular values of a trilinear form with the critical values of an associated family of a one parameter family of bilinear forms. Also here we have to offer mainly only experimental evidence.

For previous work on trilinear forms related to our present investigation we refer to [Cobos et al. 1992; 1999; 2000a; 2000b]. See also the survey [Cobos et al. 1997].

1. MILLENNIAL EQUATION: THE GENERAL CASE

The idea of the *millennial equation* is really hidden in [Cobos et al. 1992, Appendix]. The millennial equation does for trilinear forms what the secular equation does for bilinear forms. In general, there are three cases: the field of scalars $\mathbb{K} = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ (see Section 3). Here we shall consider the millennial equation for $\mathbb{K} = \mathbb{C}$ but only in the case when at most one of the underlying Hilbert spaces is infinite dimensional while the remaining ones are finite dimensional. Otherwise, we were led to infinite dimensional determinants and other complications.

Let $\mathcal{H}_1$, $\mathcal{H}_2$ and $\mathcal{H}_3$ be Hilbert spaces with generic elements x, y and z respectively. Our policy is to "eliminate" the variables one by one. Let the first one be the potentially infinite dimensional variable x, the remaining ones y and z being finite dimensional. The elimination of x is effected by observing that the study of trilinear forms

$$T : \mathcal{H}_1 \times \mathcal{H}_2 \times \mathcal{H}_3 \to \mathbb{C}$$

is equivalent to the study of bilinear forms $B : \mathcal{H}_2 \times \mathcal{H}_3 \to \mathcal{H}_1'$, where $\mathcal{H}_1'$ is the dual Hilbert space to $\mathcal{H}_1$; by the F. Riesz lemma there is a canonical antiholomorphic isometry from $\mathcal{H}_1$ to $\mathcal{H}_1'$ denoted $x \mapsto \bar{x}$. The formal connection is given by the formula $T(x, y, z) = [x, B(y, z)]$, where $[\cdot\,,\cdot]$ is the duality between $\mathcal{H}_1$ and $\mathcal{H}_1'$.

Consider the problem of critical points of the functional $\|B(y, z)\|^2$ under the subsidiary condition

$$\|y\|^2\|z\|^2 = 1.$$

It is the same as looking at the critical points of the functional $\|B(y, z)\|^2 - \lambda\|y\|^2\|z\|^2$ where λ is a Lagrange multiplier. The possible values of the square root $\lambda^{\frac{1}{2}}$ may be viewed as an analogue of the *singular*, *Schmidt*, or *approximation numbers* in the bilinear case; the largest of them coincides with the norm of B. The existence of a singular value is guaranteed by making the assumption that T be compact [Cobos et al. 1992].

Remark. Actually, the norms of T and B agree, which is a consequence of Cauchy's inequality. Similarly we get essentially the same Lagrange multipliers when considering the critical points of the functional $|T(x, y, z)|^2$ under the subsidiary condition $\|x\|^2\|y\|^2\|z\|^2 = 1$. So, in a way, what we have is a kind of generalization of this inequality.

We introduce the notation $B(y, z) = B(y)z = B(z)y$ where $B(y)$ and $B(z)$ are regarded as linear operators, in $\mathcal{H}_3$ and $\mathcal{H}_2$ respectively, with values in $\mathcal{H}_1'$. (One might perhaps equip the operators $B(z)$ and $B(y)$ with subscripts, putting $B(y, z) = B_3(z)y = B_2(y)z$. But if we make identify xyz with the indices 123, as is customary in classical invariant theory, these subscripts become superfluous.) Differentiating we find the two necessary and sufficient conditions

$$\begin{aligned} B(z)^*B(z)y &= \lambda\|z\|^2 y, \\ B(y)^*B(y)z &= \lambda\|y\|^2 z, \end{aligned} \tag{1–1}$$

for a pair $(y, z) \in \mathcal{H}_2 \times \mathcal{H}_3$ to be a critical point. Set

$$A(z) \stackrel{\text{def}}{=} \lambda\|z\|^2 E - B(z)^*B(z),$$

$$A(y) \stackrel{\text{def}}{=} \lambda\|y\|^2 E - B(y)^*B(y).$$

(Here and in what follows, E stands for the identity operator in the corresponding space.) We note that $A(z)$ and $A(y)$ are self-adjoint operators. Then we obtain the two equivalent conditions

$$A(z)y = 0 \text{ and } A(y)z = 0. \tag{1–2}$$

We now assume, for simplicity, that both $\mathcal{H}_2$ and $\mathcal{H}_3$ are finite dimensional, setting $\dim \mathcal{H}_2 = n_2$ and

$\dim \mathcal{H}_3 = n_3$ (and $\dim \mathcal{H}_1 = n_1$, if $\mathcal{H}_1$ is finite dimensional too). Let det be the *determinant function* on them. It follows from (1–2) that

$$\det A(z) = 0 \quad \text{and} \quad \det A(y) = 0. \tag{1–3}$$

Next we put into play the *adjoint operator* $A(z)^{\text{adj}}$; already Ivar Fredholm had an adjoint operator.

Remark. Recall that if C is a linear operator is a finite dimension vector space $\mathcal{W}$ then its adjoint operator C^{adj} is a generically defined operator satisfying the identity $CC^{\text{adj}} = E \det C$. The adjoint operator is implicit in Cramer's rule; in the infinite dimensional case Fredholm constructed it for certain integral operators. One has the formula $d \det C = \operatorname{tr}(C^{\text{adj}}\, dC)$.) Below we use the fact that if $\mathcal{W}$ is a finite dimensional Hilbert space and C is self-adjoint, then C^{adj} projects onto the nullspace or is zero. (Note the conflicting but traditional uses of "adjoint".) For instance, if C is a diagonal 3×3 matrix with entries $\lambda_1, \lambda_2, \lambda_3$ then C^{adj} is also diagonal with entries $\lambda_2\lambda_3$, $\lambda_1\lambda_3$, $\lambda_1\lambda_2$ and if, say, $\lambda_1 = 0$ then C^{adj} projects onto the first basis vector $(1, 0, 0)$.

Differentiation yields

$$\partial_z \det A(z,\zeta) = \operatorname{tr}(A(z)^{\text{adj}} \partial_z A(z,\zeta))$$

where we have

$$\partial_z A(z,\zeta) = \lambda\langle \zeta, z\rangle E - B(z)^* B(\zeta).$$

From (1–2), first formula, and using the result indicated in the previous remark, we infer that there exists a complex number ρ such that

$$A(z)^{\text{adj}}\cdot = \rho\langle\,\cdot\,, y\rangle y.$$

Continuing our calculation, we find

$$\begin{aligned}\partial_z \det A(z,\zeta) &= \rho \operatorname{tr}\big[\lambda\langle \zeta, z\rangle\langle\cdot\,, y\rangle y - \langle B(z)^* B(\zeta)\cdot\,, y\rangle y\big] \\ &= \rho\big(\lambda\langle\zeta, z\rangle \|y\|^2 - \langle B(z)^* B(\zeta) y, y\rangle\big).\end{aligned}$$

The second term in the last expression within brackets can be transformed as follows:

$$\begin{aligned}\langle B(z)^* B(\zeta) y,\, y\rangle &= \langle B(\zeta) y,\, B(z) y\rangle \\ &= \langle B(y)\zeta,\, B(y) z\rangle \\ &= \langle \zeta,\, B(y)^* B(y) z\rangle\,.\end{aligned}$$

Thus finally, in view of (1–2), second formula,

$$\begin{aligned}\partial_z \det A(z,\zeta) &= \rho\big\langle \zeta, \lambda\|y\|^2 z - B(y)^* B(y) z\big\rangle \\ &= \rho\langle \zeta, A(y) z\rangle = 0.\end{aligned}$$

This achieves the elimination of y. We summarize the situation as follows.

Proposition. *Let z be the second component of a critical point of the functional $\|B(y,z)\|^2 - \lambda\|y\|^2\|z\|^2$, with corresponding critical value λ. Then we have the equations*

$$\det A(z) = 0 \ \textit{and}\ \partial_z \det A(z) = 0$$

where $A(z) = \lambda\|z\|^2 E - B(z)^* B(z)$ and $B(z)$ is the linear operator defined by $B(z)y = B(y,z)$.

(A similar, symmetrical result can be obtained starting with (1–1), first formula, but this does not concern us at present.)

Because the operator $A(z)$ is Hermitean we have also

$$\bar{\partial}_z \det A(z) = 0.$$

In other words, we can also state our result as follows.

Corollary. *The point z is a critical point of the function $f(z) \stackrel{\text{def}}{=} \det A(z)$.*

The fact that $f(z)$ is only analytic, not holomorphic, is a nuisance. To emphasize this we may write, instead of $f(z)$, also $f(z, \bar z)$ and, indicating the λ-dependence, even $f(z, \bar z, \lambda)$; we call f the *eliminant.*

Remark. $f(z, \bar z)$ is of bidegree (n_2, n_2).

This forces us to pass to the *complexification* $\mathcal{H}_3^{\mathbb{C}}$ of $\mathcal{H}_3$ (at present considered as a *real* Hilbert space obtained from the complex one forgetting the complex structure). The elements of $\mathcal{H}_3^{\mathbb{C}}$ are thus pairs (z, w) where $z \in \mathcal{H}_3$, $w \in \mathcal{H}_3'$; a pair (z, w) comes from an element of $\mathcal{H}_3$ if and only if $w = \bar z$. So we have now the holomorphic polynomial $f(z, w) = f(z, w, \lambda)$ on $\mathcal{H}_3^{\mathbb{C}}$. The corollary implies that its *discriminant* $\Delta = \Delta(\lambda)$ must vanish. In other words, we can, at last, write down the millennial equation:

$$\boxed{\Delta(\lambda) = 0}\,. \tag{1–4}$$

It is an algebraic equation of a very high degree, its exact determination in terms of n_1, n_2 and n_3 being an open problem. It contains also *imaginary* roots, the meaning of which is not yet clear.

Remark. Recall (from [Gel'fand et al. 1994], for example) that if X is any algebraic variety then its discriminant Δ is a polynomial such that $\Delta = 0$ is the equation of the dual variety X^*, if the latter is a hypersurface.

Example. Consider the circle $X^2 + Y^2 = 1$ in the affine plane $\mathbb{C}^2$. Then a line $aX + Yb + c = 0$ is tangent to it precisely when $a^2 + b^2 = c^2$. That is, the discriminant is $\Delta = a^2 + b^2 - c^2$.

Example. Consider the quartic Veronese map $t \mapsto [1{:}t{:}t^2{:}t^3{:}t^4]$, that is, it is an algebraic curve in $\mathbb{P}^4$ with parametric equation

$$X_0 = 1, X_1 = t, X_2 = t^2, X_3 = t^3, X_4 = t^4.$$

A hyperplane

$$a_0X_0 + a_0X_1 + a_0X_2 + a_0X_3 + a_0X_4 = 0$$

cuts the curve at a point if and only if t satisfies the algebraic equation $a_0 + a_1t + a_2t^2 + a_3t^3 + a_4t^4 = 0$. It is tangent precisely when t is a double root (or a root of higher multiplicity). Thus Δ is the usual discriminant of the polynomial,

$$\Delta = a_4^6 \prod_{j \neq k} (t_j - t_k),$$

where t_j $(j = 1, 2, 3, 4)$ are the roots.

2. THE BINARY CASE

We turn to the case of binary trilinear forms, that is, all the three Hilbert spaces are two dimensional, $n_1 = n_2 = n_3 = 2$. In inhomogeneous notation, such a form T can be written

$$T(x, y, z) = \sum_{j,k,l=0,1} a_{jkl} x^j y^k z^l.$$

Here x, y, z are complex variables and we use the convention that $x^0 = 1$, $x^1 = x$ etc. (so that we have the point $[1{:}x]$ in the projective space corresponding to $\mathcal{H}_1$ etc.). The maximum norm of T is given by

$$\|T\| = \sup \frac{|T(x,y,z)|}{(1+|x|^2)^{1/2}(1+|y|^2)^{1/2}(1+|z|^2)^{1/2}}.$$

It is also often convenient to abandon tensor notation and resort to *British* (or *literal*) notation, like the one employed by Cayley, Sylvester, and Salmon in the nineteenth century. Instead of a_{jkl}, with j, k, $l = 1, 2$, we denote the coefficients by a_1, a_2, a_3, b_1, b_2, b_3, c, d according to the table below; observe the symmetry under reflection about the middle line, which in turn is a reflection of the invariance under the transformations $x \mapsto x^{-1}$ etc. (inversion).

a_{000}	d
a_{001}	a_3
a_{010}	a_2
a_{011}	b_1
a_{100}	a_1
a_{101}	b_2
a_{110}	b_3
a_{111}	c

Thus we may then write

$$T(x, y, z) \\ = d + a_1x + a_2y + a_3z + b_1yz + b_2xz + b_3xy + cxyz.$$

Eliminating x and y according to the general scheme indicated in Section 1 we obtain for the eliminant $f = f(z, \bar{z})$ an expression of the type

$$f = A|z|^4 + 2\,\mathrm{Re}\,Bz|z|^2 \\ +C|z|^2 + 2\,\mathrm{Re}\,Dz^2 + 2\,\mathrm{Re}\,Ez + F, \quad (2\text{–}1)$$

where the six coefficients $A, \dots, F$ are given by

$$A = \lambda^2 - (|a_3|^2+|b_1|^2+|b_2|^2+|c|^2)\lambda + |b_1b_2 - a_3c|^2;$$

$$B = -(b_1\bar{a}_2+b_2\bar{a}_1+a_3\bar{d}+c\bar{b}_3)\lambda \\ +(b_1b_2-a_3c)\overline{(a_1b_1+a_2b_2-a_3b_3-cd)};$$

$$C = 2\lambda^2 - \big(|a_1|^2+|a_2|^2+|a_3|^2+|b_1|^2+|b_2|^2+|b_3|^2 \\ +|c|^2+|d|^2\big)\lambda + |a_1b_1+a_2b_2-a_3b_3-cd|^2;$$

$$D = (b_1b_2-a_3c)\overline{(a_1a_2-b_3d)};$$

$$E = -(b_1\bar{a}_2+b_2\bar{a}_1+a_3\bar{d}+c\bar{b}_3)\lambda \\ +\overline{(a_1a_2-b_3d)}(a_1b_1+a_2b_2-a_3b_3-cd);$$

$$F = \lambda^2 - (|a_1|^2+|a_2|^2+|b_3|^2+|d|^2) + |a_1a_2-b_3d|^2.$$

Note that A, C, F are real, while B, D, E are complex, which is a reflection of the fact that f is a real polynomial in z and $\bar{z}$.

To find the discriminant Δ of f, which will give us then the millennial equation, it will be convenient to work for a while with a quite general function of the type (2–1), which we go on denoting by the same letter f. A necessary and sufficient condition for z to be a singular point is that we have the two equations $f(z, \bar{z}) = 0$ and $f'_z(z, \bar{z}) = 0$. As f is real, as we just observed, it is clear that the second equation is equivalent to $f'_{\bar{z}}(z, \bar{z}) = 0$. But the point is that it also be replaced by the equation $2f(z,\bar z)-zf'_z(z,\bar z)=0$. We "decouple" the variables z and $\bar z$, that is, we consider the system

$$\begin{aligned} f'_z(z,w)&=0,\\ 2f(z,w)-zf'_z(z,w)&=0, \end{aligned} \tag{2–2}$$

where, if we wish, $w=\bar z$ may be viewed as an auxiliary parameter. Eliminating z between these two equations we obtain the relation

$$\mathfrak{F}(w)=0,$$

where $\mathfrak{F}$ is the *resultant* of the two polynomials in (2–2). It turns out that it is a quartic polynomial:

$$\mathfrak{F}(w)=u_0w^4+u_1w^3+u_2w^2+u_3w+u_4. \tag{2–3}$$

where the coefficients $u_0,\dots,u_4$ are given by

$$\left.\begin{aligned} u_0&=\bar B^2-4A\bar D,\\ u_1&=2\bar BC-4B\bar D-4A\bar E,\\ u_2&=C^2-4D\bar D+2\bar BE-4B\bar E-4AF,\\ u_3&=2CE-4D\bar E-4BF,\\ u_4&=E^2-4DF. \end{aligned}\right\} \tag{2–4}$$

Remark. For certain reasons we call $\mathfrak{F}$ the *focal polynomial*; namely in a special case it gives the foci of a conic. Similarly, its roots may be called *generalized foci.*

Now we recall some salient facts connected with quartic polynomials and their invariants. Our main source here has been the excellent book [Schur 1968]. (However, as in [Sch] the *binomial* notation for polynomials is employed in that reference, our formulae below and the formulae there are not quite comparable. In addition, our invariants have a different normalization. In order to get back the formulae in [Schur 1968] we have to multiply the numbers u_0,u_1,u_2,u_3,u_4 by $1,4,6,4,1$ respectively, and renormalize.)

In particular, the discriminant of a polynomial like the one in (2–3) is given by

$$\begin{aligned}\Delta=u_1^2u_2^2u_3^2&-4u_0u_2^3u_3^2-4u_1^3u_3^3+18u_0u_1u_2u_3^3\\ &-27u_0^2u_3^4-4u_1^2u_2^3u_4+16u_0u_2^4u_4+18u_1^3u_2u_3u_4\\ &-80u_0u_1u_2^2u_3u_4-6u_0u_1^2u_3^2u_4+144u_0^2u_2u_3^2u_4\\ &-27u_1^4u_4^2+144u_0u_1^2u_2u_4^2-128u_0^2u_2^2u_4^2\\ &-192u_0^2u_1u_3u_4^2+256u_0^3u_4^3.\end{aligned}$$

If we plug in here the values given in (2–3), we obtain the 12-tic polynomial in the coefficients A, B, C, D, E, F and their complex conjugates, with 1010 coefficients; we still denote it by the same letter Δ. Because of this enormous size it does not make sense to write it down explicitly here; it would have taken quite many pages.

Luckily, we can instead make use of some well-known facts about invariants of quartic polynomials. Namely:

(1) The ring of all invariants is a polynomial ring with two generators P and Q given by the formulae

$$P=u_2^2-3u_1u_3+12u_0u_4 \tag{2–5}$$

and

$$Q=-2u_2^3+9u_1u_2u_3-27u_0u_3^2-27u_1^2u_4+72u_0u_2u_4. \tag{2–6}$$

[Schur 1968, p. 45, (59) and (60)].

(2) The discriminant Δ is given by the explicit formula

$$\Delta=4P^3-Q^2 \tag{2–7}$$

[Schur 1968, p. 52].

We are still going to use the letters P and Q after making the substitution in (2–2). Thus, using (2–7) we can determine Δ directly by first determining P and Q from (2–6) and (2–7).

In particular, we may apply the above result in the case of our binary trilinear form. Using for the coefficients $A,\dots,F$ the expressions given immediately after (2–1), we can determine the corresponding focal polynomial $\mathfrak{F}$ (see (2–3), (2–4)) and from it the values of the basic invariants P and Q and so the value of the discriminant Δ (see (2–5)–(2–7)). *Thus we have, in principle, determined the millennial equation in the case of a binary trilinear form.*

Remark. We say "in principle" because although it is possible to write down the millennial equation for any form with numerically given coefficients, it is beyond human – and machine – capacity to obtain a "closed" expression for it. However, experimenting with numerically given coefficients a,b,c,d using Mathematica we discovered various interesting properties which we have not been able to prove rigorously so far. First of all, the millennial equation is always of degree 12 but $\lambda=0$ seems to be a zero of multiplicity 4. So factoring with λ^4 we obtain a polynomial of degree 8; let it be denoted $\tilde{\Delta}$, so $\Delta = \lambda^4\tilde{\Delta}$. We call $\tilde{\Delta}$ the *reduced* millennial polynomial. It appears that the equation $\tilde{\Delta} = 0$ has *real* coefficients and, moreover, there appear to be (at least) 4 real roots and (at most) 2 pairs of pairwise conjugate roots. The real part of all roots seem to be positive. It is tempting to conjecture that the largest real root equals the square of the trilinear form.

Below we give some concrete illustrations of what was just said. First we observe that with no loss of generality we may assume that $\|T\| = 1$. This implies, in the first place, that the millennial equation has the root 1 but also other simplifications are possible. Namely, after some preliminary unitary transformations [Cobos et al. 1997; 2000b] we may then assume that $d = 1$, $a_1 = a_2 = a_3 = 0$, so there are left only the parameters b and c, these numbers being subject to the restriction:

$$|b_1|^2+|b_2|^2+|b_3|^2+\tfrac{1}{2}|c|^2+|2b_1b_2b_3+\tfrac{1}{2}c^2| \le 1. \quad (2\text{–}8)$$

We may also arrange (see again [Cobos et al. 1997; 2000b]) that the b's are non-negative real numbers.

Remark. In this situation the interpretation of (2–8) is that, given the values of b, the point c moves inside an ellipse $\mathfrak{E}$ in the complex plane, which is centered at the origin and has minor axis parallel to the abscissa axis and major axis parallel to the ordinate axis. It will be convenient to put

$$\boldsymbol{S} = b_1^2 + b_2^2 + b_3^2; \quad \boldsymbol{P} = b_1b_2b_3.$$

Then the equation of $\mathfrak{E}$ can be written

$$\frac{x^2}{1-\boldsymbol{S}-2\boldsymbol{P}} + \frac{y^2}{1-\boldsymbol{S}+2\boldsymbol{P}} = 1,$$

with $x = \operatorname{Re} c$, $y = \operatorname{Im} c$. The foci of $\mathfrak{E}$ are at the points $\pm 2\sqrt{\boldsymbol{P}}\, i$. In general, we have $\boldsymbol{S} + 2\boldsymbol{P} \le 1$. In the case of equality the ellipse $\mathfrak{E}$ degenerates into a segment and the foci become the extremities of this segment. Similarly, if $\boldsymbol{P} = 0$, that is, at least one of the b's vanishes, the ellipse becomes a circle and the foci shrink to the origin. Say that $b_3 = 0$ making no reality assumptions on b_1 and b_2. Then (2–8) reduces to $|b_1|^2+|b_2|^2+|c|^2 \le 1$ which equation defines a ball in $\mathbb{C}^3$.

Example 1. Consider the trilinear form

$$T = 1 + \tfrac{1}{2}(yz + xz + xy),$$

that is, $b_1 = b_2 = b_3 = \frac{1}{2}$, $c = 0$, which case in several respects is highly atypical. In particular, $\boldsymbol{S} + 2\boldsymbol{P} = 1$ so (2–8) holds with equality. The corresponding *reduced* millennial equation reads

$$1600\lambda^8 - 6640\lambda^7 + 11164\lambda^6 + 9845\lambda^5 + 4973\lambda^4 - 1490\lambda^3 + 262\lambda^2 - 25\lambda + 1 = 0,$$

up to a numerical multiplier. Its left hand side, the millennial polynomial, admits the factorization

$$(\lambda-1)^3(5\lambda-1)^2(4\lambda-1)^3.$$

Thus the roots are $1, \frac{1}{5}, \frac{1}{4}$ with multiplicities $3, 2, 3$ respectively.

Example 2. A perhaps more typical case is

$$b_1 = \tfrac{2}{5};\ b_2 = \tfrac{1}{5};\ b_3 = \tfrac{2}{3};\ c = -\tfrac{1}{10} + \tfrac{1}{5}i.$$

In this situation we find the roots

1
0.02171
0.20749
$0.43496 \pm 0.29173i$
0.49134
$1.06247 \pm 0.07727i$

Example 3. Again let

$$b_1 = \tfrac{1}{3};\ b_2 = \tfrac{5}{6};\ b_3 = \tfrac{1}{7};\ c = -\tfrac{6}{35} + \tfrac{8}{35}i.$$

This time the roots are

1
0.0127861
0.162849
0.753678
$0.875813 \pm 0.146734i$
$0.993718 \pm 0.00898095i$

Remark. In all our computations we have taken the b's and c to have rational values. If we instead take numbers in decimal notation, an amusing phenomenon takes place. Namely, apparently due to the fact that the machine no longer can make certain cancellations, the degree turns out to be 18, not 12!

3. SOME GENERAL NONSENSE

Leaving the binary case, we once more return to the general arena. We have seen (Section 1) that it is helpful, in the complex case $\mathbb{K} = \mathbb{C}$, to pass to the complexification of the underlying Hilbert spaces. That has led us to ask whether it might be possible to give a formulation which is valid directly in the complexified situation. Indeed, it is the case! This is in line with the bold general ideas of H. Weyl [1939] and R. Howe [1989] in invariant theory.

There are three cases corresponding to the series of simple Lie groups **A**, **BD** and **C** corresponding to the cases $\mathbb{K} = \mathbb{C}, \mathbb{R}, \mathbb{H}$ respectively.

Let $\mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_3$ be three finite dimensional vector spaces over the complex numbers $\mathbb{C}$, with generic elements denoted x, y, z. Let $\mathcal{V}_1', \mathcal{V}_2', \mathcal{V}_3'$ be the dual spaces, with generic elements denoted u, v, w. The corresponding dualities will be written $[xu]$ etc.

A (general linear group). Assume that $T(x, y, z)$ and $S(u, v, w)$ are trilinear forms on $\mathcal{V}_1 \times \mathcal{V}_2 \times \mathcal{V}_3$ and $\mathcal{V}_1' \times \mathcal{V}_2' \times \mathcal{V}_3'$ respectively. Consider the following expression:

$$\Phi = \Phi(\lambda) = T(x, y, z)S(u, v, w) - \lambda[xu][yv][zw].$$

For λ fixed the equation $\Phi = 0$ defines a cone on the product of the above two products. We are interested in the *singular values*, that is, those values of λ for which the cone has a singular ray.

BD (orthogonal group). Assume that our spaces each are equipped with a nondegenerate quadratic form, written (x, x) etc. If $T(x, y, z)$ is a trilinear form on $\mathcal{V}_1 \times \mathcal{V}_2 \times \mathcal{V}_3$ we may consider the expression

$$\Phi = \Phi(\lambda) = T(x, y, z)^2 - \lambda(xx)(yy)(zz)$$

and ask the same question, namely, for which values λ does the corresponding cone $\Phi = 0$ possess a singular ray?

C (symplectic group). Finally, we equip the same spaces each with a nondegenerate alternating form, written (x, x') etc. If $T(x, y, z)$ is a trilinear form on $\mathcal{V}_1 \times \mathcal{V}_2 \times \mathcal{V}_3$ we may consider the expression

$$\Phi = \Phi(\lambda) = T(x, y, z)T(x', y', z') - \lambda(xx')(yy')(zz')$$

and repeat the same question.

Our problem is to determine the singular values in each of the above cases. Eliminating two of the variables (and the corresponding dual variables) there remains only one variable, say, z (and the corresponding dual variable, w) there remains only an equation of the form $f(z, w) = 0$. As there exists a singular ray, we conclude that the discriminant must vanish,

$$\boxed{\Delta(\lambda) = 0}.$$

This is an algebraic equation in λ of usually very high degree.

Problem. Determine this degree in each concrete case.

Another interesting observation: the coefficients of $\Delta(\lambda)$ are algebraic invariants.

4. SINGULAR VALUES OF TRILINEAR FORMS VIA MINIMAX PRINCIPLES

We recall the notion of approximation numbers, s-numbers or *singular values* of a linear operator L (these words are used synonymously in the current literature; for a historical account, see the excellent book [Pietsch 1987]). The singular values $\sigma_1 \geq \sigma_2 \geq \dots$ of a linear operator L between two Hilbert spaces, which for simplicity are assumed finite dimensional here, are defined to be the eigenvalues of $(L^*L)^{1/2}$. It is well known that these numbers describe the distance of L to lower-rank operators

$$\sigma_i(L) = \min\{\|L - L_i\| : \dim \operatorname{Im}(L_i) < i\}.$$

Note that the norm of L equals $\sigma_1(L)$. Furthermore the minimax principle reads

$$\begin{aligned}\sigma_j(L) &= \min_{\operatorname{codim} W < j} \max_{0 \neq u \in W} \frac{\|Lu\|}{\|u\|} \\ &= \max_{\dim W \geq j} \min_{0 \neq u \in W} \frac{\|Lu\|}{\|u\|},\end{aligned}$$

which can be translated to results on bilinear forms using the standard equivalence. These facts motivated us to perform some numerical experiments with the trilinear forms $T(x, y, z)$ treated as one-parameter families of bilinear forms with x, y, and z, respectively, as parameter.

Introduce the two by two matrices A_i, B_i, C_i by

$$\begin{aligned}T(x, y, z) = \sum_i x_i y^T A_i z &= \sum_i y_i\, z^T B_i x \\ &= \sum_i z_i\, x^T C_i y.\end{aligned}$$

With the trilinear form in Section 2 in inhomogeneous notation, with $a_1 = a_2 = a_3 = 0$ and $d = 1$ we have

$$A_0 = \begin{pmatrix} 1 & 0 \\ 0 & b_1 \end{pmatrix}, \quad B_0 = \begin{pmatrix} 1 & 0 \\ 0 & b_2 \end{pmatrix}, \quad C_0 = \begin{pmatrix} 1 & 0 \\ 0 & b_3 \end{pmatrix},$$

$$A_1 = \begin{pmatrix} 0 & b_2 \\ b_3 & c \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0 & b_3 \\ b_1 & c \end{pmatrix}, \quad C_1 = \begin{pmatrix} 0 & b_1 \\ b_2 & c \end{pmatrix}.$$

We have now both maximized and minimized the first and second singular values of the matrices

$$\sum_i x_i A_i, \quad \sum_i y_i B_i, \quad \sum_i z_i C_i$$

over complex vectors x, y and z of unit length. The results we have are only numerical and for the binary case with $\dim \mathcal{H}_j = 2$ for $j = 1, 2, 3$. In the binary case this optimization can be reduced to vectors of the form

$$\begin{pmatrix} e^{i\theta_2} \cos\theta_1 \\ \sin\theta_1 \end{pmatrix}, \quad \text{with } \theta_1 \in [0, \pi), \ \theta_2 \in [0, 2\pi).$$

For Example 2 with parameters $(\frac{2}{5}, \frac{1}{5}, \frac{2}{3}, -\frac{1}{10} + \frac{i}{5})$ the maxima and minima of the square of the singular values are as follows:

	$\max\sigma_1^2$	$\min\sigma_1^2$	$\max\sigma_2^2$	$\min\sigma_2^2$
$\sum x_i A_i$	1.0000	0.20749	0.49134	0.0000
$\sum y_i B_i$	1.0000	0.20749	0.49134	0.0000
$\sum z_i C_i$	1.0000	0.49134	0.20749	0.0000

For Example 3 with parameters $(\frac{1}{3}, \frac{5}{6}, \frac{1}{7}, -\frac{6}{35} + \frac{8i}{35})$ we get the results

	$\max\sigma_1^2$	$\min\sigma_1^2$	$\max\sigma_2^2$	$\min\sigma_2^2$
$\sum x_i A_i$	1.0000	0.162849	0.753678	0.0000
$\sum y_i B_i$	1.0000	0.753678	0.162849	0.0000
$\sum z_i C_i$	1.0000	0.162849	0.753678	0.0000

Similar experiments with other cases indicate that the maxima and minima of the singular values correspond to solutions of the millennial equation. Not all of the real solutions of the millennial equation turn up this way, however.

Parallel to the bilinear form $B(y, z)$ introduced in Section 1 we can introduce the bilinear forms $A(x, y) : \mathcal{H}_1 \times \mathcal{H}_2 \to \mathcal{H}_3'$, $C(z, x) : \mathcal{H}_3 \times \mathcal{H}_1 \to \mathcal{H}_2'$. Then the numerical results above indicate that for Example 2

$$\max_{\|x\|=1} \min_{\|y\|=1} \|A(x,y)\| = \max_{\|y\|=1} \min_{\|z\|=1} \|B(y,z)\| = \min_{\|z\|=1} \max_{\|x\|=1} \|C(z,x)\|$$

and

$$\min_{\|x\|=1} \max_{\|y\|=1} \|A(x,y)\| = \min_{\|y\|=1} \max_{\|z\|=1} \|B(y,z)\| = \max_{\|z\|=1} \min_{\|x\|=1} \|C(z,x)\|.$$

Note also that when $\dim \mathcal{H}_1 = \dim \mathcal{H}_3$ one has

$$\max_{\|y\|=1} \min_{\|x\|=1} \|A(x,y)\| = \max_{\|y\|=1} \min_{\|z\|=1} \|B(y,z)\|$$

(and symmetrically in the other coordinates), so there are many equivalent ways of writing these optimizations.

Open Problem. Find a minimax principle for trilinear forms and relate the singular values to the solutions to the millennial equation.

ACKNOWLEDGEMENT

We are grateful to Jonathan Arazy who insisted upon that scalar trilinear forms be viewed as Hilbert space valued bilinear forms (compare [Arazy and Friedman 1978]). Likewise we thank the referee for several precious suggestions.

ELECTRONIC AVAILABILITY

The authors will send upon request a file with the code used, which is a straightforward translation into Mathematica of the formulae in this section.

REFERENCES

[Arazy and Friedman 1978] J. Arazy and Y. Friedman, *Contractive projections in C_1 and C_∞*, Mem. Amer. Math. Soc. **200**, Amer. Math. Soc., Providence, RI, 1978.

[Cobos et al. 1992] F. Cobos, T. Kühn, and J. Peetre, "Schatten-von Neumann classes of multilinear forms", *Duke Math. J.* **65**:1 (1992), 121–156.

[Cobos et al. 1997] F. Cobos, T. Kühn, and J. Peetre, "On the structure of bounded trilinear forms", 1997. See http://www.maths.lth.se/matematiklu/personal/jaak/structure.ps.

[Cobos et al. 1999] F. Cobos, T. Kühn, and J. Peetre, "On $\mathfrak{S}_p$-classes of trilinear forms", *J. London Math. Soc.* (2) **59**:3 (1999), 1003–1022.

[Cobos et al. 2000a] F. Cobos, T. Kühn, and J. Peetre, "Extreme points of the complex binary trilinear ball", *Studia Math.* **138**:1 (2000), 81–92.

[Cobos et al. 2000b] F. Cobos, T. Kühn, and J. Peetre, "Remarks on symmetries of trilinear forms", *Rev. Real Acad. Ciencias Fisicas, Exactas y Naturales de Madrid* **94** (2000), 441–449.

[Gel'fand et al. 1994] I. M. Gel'fand, M. M. Kapranov, and A. V. Zelevinsky, *Discriminants, resultants, and multidimensional determinants*, Birkhäuser, Boston, 1994.

[Howe 1989] R. Howe, "Remarks on classical invariant theory", *Trans. Amer. Math. Soc.* **313**:2 (1989), 539–570.

[Peng and Wong 2000] L. Peng and M. W. Wong, "Compensated compactness and paracommutators", *J. London Math. Soc.* (2) **62**:2 (2000), 505–520.

[Pietsch 1987] A. Pietsch, *Eigenvalues and s-numbers*, Cambridge Studies in Adv. Math. **13**, Cambridge University Press, Cambridge, 1987.

[Schur 1968] I. Schur, *Vorlesungen über Invariantentheorie*, Grundlehren der math. Wissenschaften **143**, Springer, Berlin, 1968.

[Weyl 1939] H. Weyl, *The classical groups: their invariants and representations*, Princeton Math. Series **1**, Princeton Univ. Press, Princeton, 1939.

Bo Bernhardsson, Bo Bernhardsson, Department of Automatic Control, Lund University, Box 118, S-221 00 Lund, Sweden (bob@control.lth.se)

Jaak Peetre, Centre for Mathematical Sciences, Division of Mathematics, Lund University, Box 118, S-221 00 Lund, Sweden (jaak@maths.lth.se)

Received March 20, 2000; accepted in revised form March 5, 2001