General Mathematics Vol. 10, No. 3–4 (2002), 63–68

New inequalities obtained by means of quadrature formulae

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Dedicated to Professor D. D. Stancu on his 75th birthday.

Abstract

New inequalities are obtained by means of the quadrature formulae. The results of [5] are extended.

2000 Mathematical Subject Classification: 26D15, 65D32

In [1] we have studied two procedures of using the quadrature formulae in obtaining inequalities. In this paper we obtain new inequalities by these methods.

I. Let $w : [a, b] \to (0, \infty)$ be a weight function.

Proposition 1. *For each polynomial $p_{2m}(x) \geq 0,\ x \in [a,b]$ of the degree $2m$ and with dominant coefficient equal 1, the inequality*

$$\int_a^b w(x)p_{2m}(x)dx \geq \frac{1}{a_m^2}\int_a^b w(x)Q_m^2(x)dx \tag{1}$$

is valid, with equality only if

$$p_{2m}(x) = \frac{1}{a_m^2}Q_m^2,$$

where $Q_m(x)$ is the polynomial of degree m, with the dominant coefficient a_m, out of the system of orthogonal polynomials on the interval $[a,b]$ refering to the weight $w(x)$.

Proof. The validity of Proposition 1 is obtained from the Gauss quadrature formula (see [2], [3]):

$$\int_a^b w(x)f(x)dx = \sum_{i=1}^{m} A_i f(x_i) + R_{2m-1}(f),$$

in which the coefficients $A_i, i = \overline{1,m}$, are positive and the remainder is given by

$$R_{2m-1}(f) = \frac{1}{(2m)!}\cdot\frac{1}{a_m^2}f^{(2m)}(c)\int_a^b w(x)Q_m^2(x)dx, c \in (a,b).$$

Remark 1. *For $w(x) = (1-x)^\alpha(1+x)^\beta,\ x \in (-1,1),\ \alpha > -1,\ \beta > -1$, from (1) it results the inequality given by F. Locher in [5].*

Remark 2. *For $w(x) = x^\alpha e^{-x},\ x \in (0,+\infty),\ \alpha > -1$, we obtain the Proposition 5 from [1].*

Remark 3. *If* $w(x) = e^{-x^2}$*,* $x \in (-\infty, +\infty)$*, then for each polynomial* $p_{2m}(x) \geq 0$*,* $x \in (-\infty, +\infty)$*, of degree* $2m$ *and with the dominant coefficient equal to 1, the inequality*

$$\int_{-\infty}^{+\infty} e^{-x^2} p_{2m}(x)dx \geq \frac{m!\sqrt{\pi}}{2^m},$$

is valid, with equality only if

$$p_{2m}(x) = \frac{1}{2^{2m}} H_m^2(x),$$

where $H_m(x)$ *is the Hermite polynomial.*

II. The Gauss-Kronrad quadrature formula for the Legendre weight function, $w(x) = 1$, on $[-1, 1]$, has the form

$$\int_{-1}^{1} f(x)dx = \sum_{i=1}^{n+1} A_i f(x_i) + \sum_{j=1}^{n} B_j f(a_j) + R_n(f), \tag{2}$$

where $a_j, j = \overline{1,n}$, are the zeros of the n-th degree Legendre polynomial, $P_n(x)$, and the x_i, A_i, $i = \overline{1, n+1}$, B_j, $j = \overline{1,n}$, are chosen such (2) has maximum degree of exactness ($3n + 1$ for n or $3n + 2$ if n is odd). It is known that the x_i are simple, all contained in the interval (-1, 1) and they interlace with a_j, that is

$$x_{n+1} < a_n < x_n < ... < x_3 < a_2 < x_2 < a_1 < x_1 \tag{3}$$

(see [7] - [9]). Moreover, all coefficients of (2) are positive (the positivity of the A_i is equivalent to the interlacing property (3); see[5]).

Let us $f \in C^{(3n+2)}[-1, 1]$ and n even, then

$$R_n(t) = \frac{(n!)^2}{2^n(3n+2)!(2n)!} \int_{-1}^{1} P_n(x) w_{n+1}^2(x) f_{(c_x)}^{(3n+2)} dx, \ c_x \in (-1, 1),$$

where $w_{n+1}(x) = \prod_{i=1}^{n+1}(x - x_i)$ and it satisfies the following orthogonality relation

$$\int_{-1}^{1} p_n(x)w_{n+1}(x)x^k dx = 0,\ k = \overline{0,n}.$$

When n is odd, if we assume $f \in C^{3n+3}[-1,1]$, then

$$R_n(t) = \frac{(n!)^2}{2^n(3n+3)!(2n)!}\int_{-1}^{1} P_n(x)w_{n+1}^2(x)f^{(3n+3)}(c_x)dx,\ c_x \in (-1,1), (see\ [7]).$$

Now we obtain:

Proposition 2. *If n is even, then for each polynomial $p_{3n+2}(x) \geq 0$, $x \in [-1,1]$, of degree $3n+2$ and with the dominant coefficient equal 1, the inequality*

$$\int_{-1}^{1} p_{3n+2}(x)dx \geq \frac{(n!)^2}{2^n(2n)!}\int_{-1}^{1} P_n(x)w_{n+1}^2(x)dx$$

is valid.

III. Let's consider the Euler's quadrature formula (see [2], [4])

(4)
$$\int_a^b f(x)dx = \frac{b-a}{2}[f(a)+f(b)]+$$

$$+\sum_{i=1}^{n-1}\frac{(b-a)^{2i}}{(2i)!}B_{2i}[f^{(2i-1)}(a) - f^{(2i-1)}(b)] + R(f),$$

with

(5)
$$R(f) = -\frac{(b-a)^{2n+1}B_{2n}}{(2n)!}f^{(2n)}(c),\ c \in (a,b)$$

where $B_{2j}, j = \overline{1,n}$, are the Bernoulli numbers. If $f \in c^{(2n)}[a,b]$, with $f^{(2n)}(x) \geq 0$ for any $x \in [a,b]$, and $B_{2n} > 0$, then from (4) and (5) we obtain the inequality

$$(6) \int_a^b f(x)dx \leq \frac{b-a}{2}[f(a)+f(b)]+\sum_{i=1}^{n-1}\frac{(b-a)^{2i}}{(2i)!}B_{2i}[f^{(2i-1)}(a)-f^{(2i-1)}(b)].$$

If $B_{2n} < 0$, then we have the inequality

$$(7) \int_a^b f(x)dx \geq \frac{b-a}{2}[f(a)+f(b)]+\sum_{i=1}^{n}\frac{(b-a)^{2i}}{(2i)!}B_{2i}[f^{(2i-1)}(a)-f^{(2i-1)}(b)].$$

For $f^{(2n)}(n) \leq 0$ on $[a,b]$, the inequality (6) and (7) reverse the order.

The inequalities (6) and (7) generalize the results from [1].

If in (6) we insert $f(x) = 1/x$, $x \in [a,b]$, $0 < a < b$, then we find the inequality

$$\ln\frac{b}{a} < \frac{b^2-a^2}{2ab}.$$

From here, for $a = 1, b = 1+x, x > 0$, it results

$$\ln(1+x) < \frac{x(x+2)}{2(x+1)}.$$

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