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On the Scalar Measure of Non-Normality of Matrices - Dimension vs. Structure

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Dedicated to Professor dr. Gheorghe Micula on his 60^{th} birthday

Abstract

The aim of this paper is to analyze the relative importance of the dimension and the structure of the square matrices in the quantification of their non-normality. We envisage non-normal matrices which come from numerical analysis of o. d. e. and p. d. e. as well as from various iterative processes where the parameter dimension varies. The main result consists in an upper bound for the departure from normality. This bound is a product of two factors, is based directly on the entries of the matrices, and is elementary computable. The first factor depends exclusively on the dimension of matrices and the second, called the aspect factor, is intimately related to the structure of the matrices. In some special situations, the aspect factor is independent of the dimension, and shows that, in these cases, only the dimension is responsible for the departure from normality.

An upper bound for the field of values is also obtained. Some numerical experiments are carried out. They underline the idea that the aspect factor and pseudospectrum are complementary aspects of the non-normality.

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1 Introduction

There exist two major concepts with respect to the measure of the non-normality of square matrices. The first one is due to P. Henrici [4] (see also Chaitin-Chatelin and Fraysee [2]) and it provides some scalar measures of non-normality. The second one, more recently introduced, is that of pseudospectrum of a matrix and is systematically treated by L. N. Trefethen in a large series of papers from which we quote [8]-[11]. We have to observe that an almost exhaustive bibliography concerning the pseudospectra of matrices is available in the book in preparation [11] or on the site http://www.comlab.ox.ac.uk/projects/pseudospectra.

As it is well known, the non-normality of matrices and consequently, of linear operators assumed, is responsible for a surprising and sometimes critical behavior of some numerical algorithms and procedures. In spite of this, the estimation of non-normality is not yet a routine matter among scientists and engineers who deal with such matrices. However, a realistic approach of some important numerical methods (such as collocation type methods) and problems (mainly those non-self-adjoint) requires the quantification of the non-normality of the matrices involved. In this respect, we try to refine

some of the results of P. Henrici on some scalar measures of non-normality. Consequently, in the second section, we display our main result which consists in an upper bound for the Frobenius (euclidian) norm departure from normality. This bound is a product of two factors. The first one has order $O\left(n^{\frac{5}{4}}\right)$ for a square matrix of dimension n and the second, called *aspect factor,* reflects the structure of the matrix and can be independent of n. For such matrices, whenever their dimensions are increased, the parameter n is itself the unique responsible for the departure from normality.
In the third section, using the above bound for the departure, an upper bound for the field of values is worked out. In the fourth section, we analyze some particular matrices which come from numerical analysis of differential and partial differential equations.

2 An upper bound for the Frobenius norm departure from normality

In the seminal paper [4], Henrici introduced the following departure from normality of an arbitrary $n \times n$ matrix A with complex entries

$$\Delta_{\nu}(A) := \inf_{\underbrace{A = U(\Lambda + M)U^*}_{(Schur\ \ decomposition)}} \nu(M), \tag{1}$$

where U is unitary and Λ is diagonal and is made up of the eigenvalues of A. The symbol * denotes as usually the conjugate transpose of a vector or a matrix while ν stands for a norm of A. The main result from the above quoted paper reads as follows:

$$\Delta_{\varepsilon}(A) \leq \left(\frac{n^3-n}{12}\right)^{1/4} (\varepsilon(A^*A - AA^*)^{1/2}, \tag{2}$$

where ε stands for the Frobenius (*euclidian*) norm of A. We also use the following three norms

$$\rho\,(A) := \max_{1\leq i\leq n} \sum_{j=1}^{n} |a_{ij}|\,, \quad \alpha\,(A) := \sum_{i,j=1}^{n} |a_{ij}|\,, \quad \gamma\,(A) := \max_{1\leq j\leq n} \sum_{i=1}^{n} |a_{ij}|\,.$$

Our main result is contained into the next theorem.

Theorem 1. *For an arbitrary $n\times n$ matrix A, the Frobenius norm departure from normality satisfies*

$$\Delta_{\varepsilon}(A) \leq \left(\frac{n^3\,(n^2-1)}{3}\right)^{\frac{1}{4}} \rho\,(A)\,\gamma\,(A)\,. \tag{3}$$

Proof. From [7] we use the following inequalities

$$n^{\frac{-1}{2}}\varepsilon\,(A) \;\;\leq\;\; \sigma\,(A) \leq \varepsilon\,(A)\,, \tag{4}$$

$$n^{\frac{-1}{2}}\sigma\,(A) \;\;\leq\;\; \rho\,(A) \leq \sqrt{n}\sigma\,(A)\,, \tag{5}$$

and we also observe that $\rho\,(A^*) = \gamma\,(A)$. The left inequalities (4) and (5) introduced in the right hand side of (2) prove the theorem.

Remark 1. *Whenever the norms $\rho\,(A)$ and $\gamma\,(A)$ of the matrix A, do not depend on n, we can rewrite the result of this theorem in the following asymptotic form*

$$\Delta_{\varepsilon}(A) = O\left(n^{\frac{5}{4}}\right),$$

for large values of the dimension n.

The quantity $\rho\,(A)\,\gamma\,(A)$ is called the ***aspect factor*** *of the matrix A.*

3 A bound for the field of values

It is well known that for normal matrices the field of values and the convex hull of their eigenvalues are two identical sets. The field of values of a non-normal matrix is still convex, although it may extend beyond the convex hull. The precise equation of the boundary of the field of values of a non-normal matrix, as well as its diameter and area are also known. Starting from another result of Henrici [4] we try to give a bound for the relative distance of the boundary of the field of values from the convex hull.

Theorem 2. *If ξ is a point of the field of values of a $n \times n$ matrix A, then there exists a point η in the convex hull of eigenvalues of A such that the distance $|\xi - \eta|$ satisfies*

$$|\xi - \eta| \leq \frac{1}{2} \left(\frac{n^7 \left(n^2 - 1\right)}{12} \right)^{1/4} \rho\left(A\right) \gamma\left(A\right). \tag{6}$$

Proof. From [7], we use the inequality

$$\alpha\left(A\right) \leq n\varepsilon(A),$$

for any matrix M from Schur decomposition of A, and from [4] we know that

$$|\xi - \eta| \leq \frac{1}{2} \Delta_{\alpha}(A),$$

where the constant $\frac{1}{2}$ cannot be replaced by any smaller constant. A simple manipulation of these two inequalities proves the theorem.

4 Numerical examples

Our first two examples illustrate the independence of the aspect factor with regard to n.

Example 1. A typical case where the aspect factor is independent of n was found in the book of Gottlieb and Orszag [3] on the page 53. The matrix of a difference approximation to a mixed initial-boundary value problem reads as follows

$$L_N = -\frac{1}{h}L,$$

where the norms of $N \times N$ matrix L are independent of N and has the shape

$$L = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -2 & 2 \end{pmatrix}.$$

In fact we have $\rho(L)\gamma(L) = 12$. The pseudospectrum is depicted in Fig.1 and it was computed using a slightly modified code from [9]. The numerical experiments were carried out for N ranging from 4 to 200.

Example 2. In the same spirit, in the second example we consider a tridiagonal matrix from [10], of the form $\begin{pmatrix} a & b & c \end{pmatrix}$ where the vectors a, b, and c contain $\left(\frac{1}{4}\right)' s$, $0's$ and respectively $1's$.. The vector b represents the main diagonal, a is the first subdiagonal, and c is the first upper diagonal. In this case the aspect factor equals $\left(\frac{5}{4}\right)^2$.

Example 3. The third example comes from the book of Canuto, Hussaini, Quarteroni and Zang [1], page 130 where a Chebyshev tau approximation is used to solve a two-point boundary value problem depending on a parameter λ. The structure of the corresponding quasi-tridiagonal matrix involved is the following:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 & 1 \\ x & x & x & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & x & x & x & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & x & x & x & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & x & x & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & x & 1 \end{pmatrix},$$

where $x's$ denotes some non-zero coefficients listed on the above quoted page. The pseudospectrum of this matrix is depicted in Fig.2. In this case the aspect factor equals

$$\left[2 + \frac{1}{4n\,(n-1)}\right] n,$$

which means that this factor goes to $+\infty$ for $n \to \infty$.

Example 4. The last example comes from the book of Schmid and Henningson [6]. On the page 522, the authors consider the matrix $A = \begin{pmatrix} -1/\,\mathrm{Re} & 0 \\ 1 & -2/\,\mathrm{Re} \end{pmatrix}$ where the Reynolds number Re approaches $+\infty$, this being the most interesting situation in fluid mechanics. The corresponding aspect factor equals

$$\left(1+\frac{1}{\mathrm{Re}}\right)\left(1+\frac{2}{\mathrm{Re}}\right),$$

which goes to 1 when $\mathrm{Re} \to +\infty$.

5 Concluding remarks

From these examples, and a lot of others which we have considered, it seems plausible to conceive the aspect factor as a companion quantity for the pseudospectra whenever one attempts to measure the non-normality of a given matrix. In the important cases mentioned in the Remark above the importance of dimension is set apart from that of structure in the characterization of non-normality. Moreover, the estimations (3) and (6) offer upper bounds for the Frobenius norm departure from normality and, respectively, for the field of values. They are based directly on the entries of A, and are comparable to that reported by Lee in [5], because they are of the same order of magnitude of computation($O\left(n^2\right)$).

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