General Mathematics Vol. 11, No. 1–2 (2003), 47–52

A new differential inequality I

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Dedicated to Professor dr. Gheorghe Micula on his 60^{th} birthday

Abstract

We find conditions on the complex-valued functions A and B, in the unit disc U such that the differential inequality

$$Re\,[A(z)p^2(z)+B(z)p(z)+\alpha(zp'(z))^3-\beta(zp'(z))^2+\gamma(zp'(z))+\delta]>0$$

implies $Re\,p(z) > 0$, where $\alpha \geq 0$, $\beta \geq 0$, $\gamma \in \mathbb{R}$, $\delta < \dfrac{\alpha n^3}{8}+\dfrac{\beta n^2}{4}+\dfrac{\gamma n}{2}$ and $p \in \mathcal{H}[1,n]$.

2000 Mathematical Subject Classification: 30C80

Keywords: differential subordination, dominant.

1 Introduction and preliminaries

We let $\mathcal{H}[U]$ denote the class of holomorphic functions in the unit disc

$$U=\{z\in\mathbb{C}: \quad |z|<1\}.$$

For $a\in\mathbb{C}$ and $n\in\mathbb{N}^*$ we let

$$\mathcal{H}[a,n]=\{f\in\mathcal{H}[U], \quad f(z)=a+a_nz^n+a_{n+1}z^{n+1}+..., \ z\in U\}$$

and

$$\mathcal{A}_n=\{f\in\mathcal{H}[U], \quad f(z)=z+a_{n+1}z^{n+1}+a_{n+2}z^{n+2}+..., \ z\in U\}$$

with $\mathcal{A}_1=\mathcal{A}$.

In order to prove the new results we shall use the following lemma, which is a particular form of Theorem 2.3.i[1,p.35].

Lemma A. [1,p.35]. *Let* $\psi:\mathbb{C}^2\times U\to\mathbb{C}$ *a function which satisfies*

$$Re\ \psi(\rho i,\sigma;z)\le 0,$$

where $\rho,\sigma\in\mathbb{R}, \quad \sigma\le-\frac{n}{2}(1+\rho^2) \quad z\in U \quad and \quad n\ge 1.$

If $p\in\mathcal{H}[1,n]$ and

$$Re\ \psi\,(p(z),zp'(z);z)>0$$

then

$$Re\,p(z)\ >0.$$

2 Main results

Theorem 1. *Let* $\alpha \geq 0$, $\beta \geq 0$, $\gamma \in \mathbb{R}$, $\delta < \frac{\alpha n^3}{8} + \frac{\beta n^2}{4} + \frac{\gamma n}{2}$ *and let* n *be a positive integer. Suppose that the functions* $A, B : U \to \mathbb{C}$ *satisfy*

(1)
$$(i)\ Re\, A(z) > -\frac{3\alpha n^3}{8} - \frac{\beta n^2}{2} - \frac{\gamma n}{2}$$

$$(ii)\ (Im B(z))^2 \leq 4\left(\frac{3\alpha n^3}{8} + \frac{\beta n^2}{2} + \frac{\gamma n}{2} + Re\, A(z)\right) \cdot$$
$$\cdot \left(\frac{\alpha n^3}{8} + \frac{\beta n^4}{4} + \frac{\gamma n}{2} - \delta\right)$$

If $p \in \mathcal{H}[1, n]$ *and*

(2)
$$Re\, [A(z)p^2(z) + B(z)p(z) + \alpha(zp'(z))^3 -$$
$$-\beta(zp'(z))^2 + \gamma(zp'(z)) + \delta] > 0$$

then

$$Re\, p(z) > 0.$$

Proof. We let $\psi : \mathbb{C}^2 \times U \to \mathbb{C}$ be defined by

(3)
$$\psi(p(z), zp'(z); z) = A(z)p^2(z) + B(z)p(z) + \alpha(zp'(z))^3 -$$
$$-\beta(zp'(z))^2 + \gamma(zp'(z)) + \delta] > 0$$

From (2) we have

(4) $$Re\,\psi(p(z), zp'(z); z) > 0 \text{ for } z \in U.$$

For $\rho, \sigma \in \mathbb{R}$ satisfying $\sigma \leq -\frac{n}{2}(1+\rho^2)$, hence $-\sigma^2 \leq -\frac{n^2}{4}(1+\rho^2)^2$, $\sigma^3 \leq -\frac{n^3}{8}(1+\rho^2)^3$ and $z \in U$, by using (1) we obtain

$$Re\,\psi(\rho i, \sigma; z) = Re\,[A(z)(\rho i)^2 + B(z)\rho i +$$

$$+\alpha\sigma^3 - \beta\sigma^2 + \gamma\sigma + \delta] = -\rho^2 Re\,A(z) - \rho\, Im\, B(z) +$$

$$+\alpha\sigma^3 - \beta\sigma^2 + \gamma\sigma + \delta \leq$$

$$\leq -\rho^2 Re\,A(z) - \rho\, Im\, B(z) - \frac{\alpha n^3}{8}(1+\rho^2)^3 -$$

$$-\frac{\beta n^2}{4}(1+\rho^2)^2 - \frac{\gamma n}{2}(1+\rho^2) + \delta = -\frac{\alpha n^3}{8}\rho^6 -$$

$$-\left(\frac{3\alpha n^3}{8} + \frac{\beta n^2}{4}\right)\rho^4 - \left[\left(\frac{3\alpha n^3}{8} + \frac{\beta n^2}{2} + \frac{\gamma n}{2} + Re\,A(z)\right)\rho^2 +\right.$$

$$\left. +\rho\, Im\, B(z) + \frac{\alpha n^3}{8} + \frac{\beta n^2}{4} + \frac{\gamma n}{2} - \delta\right] \leq 0.$$

By using Lemma A we have that $Re\,p(z) > 0$.

If $\alpha = 0$, then Theorem 1 can be rewritten as follows:

Corollary 1. *Let $\beta \geq 0$, $\gamma \in \mathbb{R}$, $\delta < \frac{\beta n^2}{4} + \frac{\gamma n}{2}$ and let n be a positive integer. Suppose that the functions $A, B : U \to \mathbb{C}$ satisfy*

(5)

(i) $Re\,A(z) > -\frac{\beta n^2}{2} - \frac{\gamma n}{2}$

(ii) $(Im B(z))^2 \leq 4\left(\frac{\beta n^2}{2} + \frac{\gamma n}{2} + Re\,A(z)\right)\left(\frac{\beta n^4}{4} + \frac{\gamma n}{2} - \delta\right)$

If $p \in \mathcal{H}[1,n]$ *and*

$$(6) \qquad Re\,[A(z)p^2(z) + B(z)p(z) - \beta(zp'(z))^2 + \gamma(zp'(z)) + \delta] > 0$$

then

$$Re\, p(z) > 0.$$

If $\beta = 0$, then Theorem 1 can be rewritten as follows:

Corollary 2.

Let $\alpha \geq 0,\ \gamma \in \mathbb{R},\ \delta < \dfrac{\alpha n^3}{8} + \dfrac{\gamma n}{2}$ *and let* n *be a positive integer. Suppose that the functions* $A, B : U \to \mathbb{C}$ *satisfy*

$$(7) \qquad \begin{aligned} &(i)\ Re\, A(z) > -\frac{3\alpha n^3}{8} - \frac{\gamma n}{2} \\ &(ii)\ (Im B(z))^2 \leq 4\left(\frac{3\alpha n^3}{8} + \frac{\gamma n}{2} + Re\, A(z)\right)\left(\frac{\alpha n^3}{8} + \frac{\gamma n}{2} - \delta\right) \end{aligned}$$

If $p \in \mathcal{H}[1,n]$ *and*

$$(8) \qquad Re\,[A(z)p^2(z) + B(z)p(z) + \alpha(zp'(z))^3 + \gamma(zp'(z)) + \delta] > 0$$

then

$$Re\, p(z) > 0.$$

References

[1] S.S. Miller and P.T. Mocanu, *Differential Subordinations. Theory and Applications*, Marcel Dekker Inc., New York, Basel, 2000.

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