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# Probabilities and Lebesgue measure

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Dedicated to Professor dr. Gheorghe Micula on his $60^{th}$ birthday

**Abstract**

In this paper we give some applications of Lebesgue measure and we will also estimate an integral from probability theory.

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**1.** Lebesgue measure is frequent used in problems of the probability theory, in physics and other domains. It is sufficiently to recall that in the probability theory, a Borel measurable application is also a random variable defined on a selection space.

The Lebesgue measure (L) is of great importance in applications on $\mathbb{R}^n$ and we know that is invariant with respect to all the travels.

The next theorems justifies our terminology.

**Theorem 1.** *Let $\mu$ be a $\sigma$-finite measure on borelian $\sigma$-algebra of the $\mathbb{R}^n$ space. If in addition,*

*1.* $\mu\{x|0 < x_i \leq 1,\ i = \overline{1,n}\} = 1$

*2. $\mu(E) = \mu(E + a)$ for any borelian set $E$ and for any $a \in \mathbb{R}^n$, then $\mu$ is a Lebesgue measure $(L)$ on $\mathbb{R}^n$.*

**Theorem 2.** *Let $L$ be the Lebesgue measure on $\mathbb{R}^n$ and $S$ be a nondegenerate linear transform on $\mathbb{R}^n$. Then*

$$L(S(E)) = |det\ S| \cdot L(E)$$

*holds for any borelian set $E$.*

**Theorem 3. (The formula of integration by change of variable).**

*Let $\Omega \subset \mathbb{R}^n$ be an open set and let $L$ be a Lebesgue measure on $\Omega$. Let $T(x) = (y_1(x), ..., y_n(x))'$, $x = (x_1, ..., x_n)' \in \Omega$ be a given homeomorphism $T : \Omega \to \mathbb{R}^n$ with the continuous derivatives $\frac{\partial y_i}{\partial x_j}, i, j = \overline{1, n}$ on $\Omega$ and we note with $\tau(T, x) = \left(\left(\frac{\partial y_i}{\partial x_j}\right)\right), i, j = \overline{1, n}$, the nondegenerate Jacobian matrix for all $x \in \Omega$.*

Then for any non-negative borelian function $f$ defined on the open set $T(\Omega)$ we have

$$\int_{T\Omega} f(y)dy = \int_{\Omega} f(Tx) \cdot |\tau(T, x)|dx$$

where by $dx, dy$ we mean the integration with respect to the Lebesgue measure.

Using these theorems it's possible to establish many results with applications in the probability theory, in physics and in mechanics a.s.o.

**2.** Now, we will refer to an integral on $\mathbb{R}^n$ frequently meet in the probability theory and in mathematical statistics.

For any matrix $\sum$ of the type $(n \times n)$ positive defined with real elements and for any column-defined matrix m, we have

$$\text{(1)} \quad \int_{\mathbb{R}^n} \exp\left\{-\frac{1}{2}(x - m)' {\sum}^{-1} (x - m)\right\} dx = (\sqrt{2\pi})^n \cdot \left(\det \sum\right)^{\frac{1}{2}}$$

where $x'$ is the transpose of the $x - vector$.

Since the Lebesgue measure is invariant for any travel, the left hand said will be independent of m. For $m = 0$, because the $\sum$ is positive defined we have the representation $\sum = C'^C$ where C is a nondegenerate matrix.

Here we use the transformation $C\ x = y$ and the left-hand side of the (1) equality is to write

$$|det\ C| \cdot \int\limits_{\mathbb{R}^n} \exp\ \left\{-\frac{1}{2}(y_1^2 + ... + y_n^2)\right\} dy_1...dy_n =$$

$$= |det\ C| \cdot \left(\int\limits_{\mathbb{R}}^{\mathbb{R}^n} e^{-\frac{1}{2}y^2} \cdot dy\right)^n = |det\ C| \left(\sqrt{2\pi}\right)^n = |det\ \sum|^{\frac{1}{2}} \cdot (\sqrt{2\pi})^n.$$

**3.** Another application refers to finding of the Lebesgue measure of a set having the form

$$A = \{x | P_k(Tx) \leq 1\}$$

where $P_k(x) = \sum\limits_{i=1}^{n} x_i^{2k}$, and T is a nondegenerated linear transform of the $\mathbb{R}^n$ in itself.

Evidently if $x$ goes through on A, the point Tx goes thorough the set $B = \{x/P_k(x) \leq 1\}$.

From the Theorem (2) we have

$$L(B) = L(TA) = |det\ T| \cdot L(A). \tag{2}$$

We note $t_n = L(B)$. But the Lebesgue measure on $\mathbb{R}^n$ is the product of the measures of $\mathbb{R}^{n-1}$ and $\mathbb{R}$.

The section of set B in the point $x_n = x$ is the set

$$B_x = \{(x_1, ..., x_{n-1}) | (x_1^{2k} + ... + x_{n-1}^{2k})\}.$$

On the one hand we have $B_x = \phi$ for $|x| > 1$, and on the other hand for $|x| < 1$ we divide both members of the inequality from the expression $Bb_x$

by $1 - x^{2k}$. In view of relation (2) we have:

$$L(B_x) = t_{n-1}(1 - x^{2k})^{\dfrac{n-1}{2k}}$$

and find that:

$$t_n = L(B) = t_{n-1} \cdot \int\limits_{-1}^{1} (1 - x^{2k})^{\dfrac{n-1}{2k}} \cdot dx =$$

$$= 2t_{n-1} \cdot \int\limits_{0}^{1} (1-y)^{\dfrac{n-1}{2k}} \cdot y^{\dfrac{1}{2k}-1} \cdot \frac{dy}{2k} = \frac{t_{n-1}}{k} \cdot \frac{\Gamma\left(\dfrac{n-1}{2k}+1\right)\Gamma\left(\dfrac{1}{2k}\right)}{\Gamma\left(\dfrac{n}{2k}+1\right)}$$

From this recursion formula we obtain

$$t_n = \frac{\left[\Gamma\left(\dfrac{1}{2k}\right)\right]^n}{k^n \cdot \Gamma\left(\dfrac{n}{2k}+1\right)}$$

Therefore:

$$L\{x|P_k(T_x) \le 1\} = \frac{|det\ T|^{-1}}{k^n} \cdot \frac{\left[\Gamma\left(\dfrac{1}{2k}\right)\right]^n}{\Gamma\left(\dfrac{n}{2k}+1\right)}$$

In particular, the volume of the unite spherical in $\mathbb{R}^n$ is

$$L\left\{x|\sum_i\ x_i^2 \le 1\right\} = \frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}+1\right)}.$$

**4.** In the same way there, can be computed the Lebesgue measure of a standard simplex

$$W = \left\{x|x_i \ge 0, i = \overline{1,n}, \sum_{i=1}^{n} x_i \le 1\right\}$$

Then arque as above to obtain

$$L(W) = \frac{1}{n!}.$$

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