General Mathematics Vol. 11, No. 3–4 (2003), 5–26

On higher order Cauchy-Pompeiu formula in Clifford analysis and its applications

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Abstract

In this paper, we firstly construct the kernel functions which are necessary for us to study universal Clifford analysis. Then we obtain the higher order Cauchy-Pompeiu formulas for functions with values in a universal Clifford algebra, which are different from those in [2]. As applications we give the mean value theorem and as special case the higher order Cauchy's integral formula.

2000 Mathematical Subject Classification: 35C10, 31B05, 30G35.

Keywords: Universal Clifford algebra, Cauchy-Pompeiu formula.

1 Introduction

As is well-known the Cauchy integral formula plays a very important role in the classical theory of functions of one complex variable. R. Delanghe,

F. Brackx, F. Sommen, V. Iftimie and many other authors have studied the theory of functions with values in a Clifford algebra. In Clifford analysis, the Cauchy integral formula has been set up and it leads to many important theorems, which are similar to classical results in classical complex analysis. Some examples are the residue theorem, the maximum modulus theorem, the Morera theorem and so on (see, e.g., [5–9, 11–14, 16–17]). In [9], R.Delanghe, F.Brackx have studied the k-regular functions and have given the corresponding Cauchy integral formula. In [10], Du and Zhang have obtained the Cauchy integral formula with respect to the distinguished boundary for functions with values in a universal Clifford algebra and some of its applications.

Let D be a bounded domain with the smooth boundary ∂D in the complex plane $\mathcal{C}$, and $\omega \in C^1(D,\mathcal{C})\bigcap C(\overline{D},\mathcal{C})$. The following generalized form of the Cauchy integral formula for functions of one complex variable is known as the Cauchy-Pompeiu formula [15].

$$\begin{cases} w(z) = \dfrac{1}{2\pi i}\displaystyle\int_{\partial D}\dfrac{w(\zeta)}{\zeta - z}\mathrm{d}\zeta - \dfrac{1}{\pi}\displaystyle\iint_D \dfrac{w_{\overline{\zeta}}(\zeta)}{\zeta - z}\mathrm{d}\xi\mathrm{d}\eta, \\ w(z) = -\dfrac{1}{2\pi i}\displaystyle\int_{\partial D}\dfrac{w(\zeta)}{\overline{\zeta} - \overline{z}}\mathrm{d}\overline{\zeta} - \dfrac{1}{\pi}\displaystyle\iint_D \dfrac{w_{\zeta}(\zeta)}{\overline{\zeta} - \overline{z}}\mathrm{d}\xi\mathrm{d}\eta, \end{cases} \qquad \zeta = \xi + i\eta, \ \ z \in D,$$

with the Kolossov-Wirtinger operators

$$w_\zeta = \frac{\partial w}{\partial \zeta} = \frac{1}{2}\left(\frac{\partial w}{\partial \xi} - i\frac{\partial w}{\partial \eta}\right), \quad w_{\overline{\zeta}} = \frac{\partial w}{\partial \overline{\zeta}} = \frac{1}{2}\left(\frac{\partial w}{\partial \xi} + i\frac{\partial w}{\partial \eta}\right).$$

The Cauchy-Pompeiu formulae and the Pompeiu operators were recently extended to the situation of Clifford analysis in many papers (see, e.g., [1–3]). In [4], the Cauchy-Pompeiu formulae for functions with values in a universal Clifford algebra were obtained. In order to study higher order

Cauchy-Pompeiu formulae for functions with values in a universal Clifford algebra, in this paper, the Cauchy-Pompeiu formulae for functions with values in a universal Clifford algebra will be considered only in the case of $s = n$. The other cases will be discussed in a forthcoming paper.

2 Preliminaries and notations

Let $V_{n,s}(0 \le s \le n)$ be an n–dimensional $(n \ge 1)$ real linear space with basis $\{e_1, e_2, \cdots, e_n\}$, $C(V_{n,s})$ be the 2^n–dimensional real linear space with basis

$$\{e_A, A = (h_1, \cdots, h_r) \in \mathcal{P}N, 1 \le h_1 < \cdots < h_r \le n\},$$

where N stands for the set $\{1, \cdots, n\}$ and $\mathcal{P}N$ denotes for the family of all order-preserving subsets of N in the above way. Sometimes, $e_\emptyset$ is written as e_0 and e_A as $e_{h_1 \cdots h_r}$ for $A = \{h_1, \cdots, h_r\} \in \mathcal{P}N$. The product on $C(V_{n,s})$ is defined by

$$(2.1) \quad \begin{cases} e_A e_B = (-1)^{\#((A\cap B)\setminus S)}(-1)^{P(A,B)} e_{A\triangle B}, \quad \text{if } A, B \in \mathcal{P}N, \\ \\ \lambda\mu = \sum\limits_{A\in\mathcal{P}N} \sum\limits_{B\in\mathcal{P}N} \lambda_A \mu_B e_A e_B, \quad \text{if } \lambda = \sum\limits_{A\in\mathcal{P}N} \lambda_A e_A,\ \mu = \sum\limits_{A\in\mathcal{P}N} \mu_A e_A. \end{cases}$$

where S stands for the set $\{1, \cdots, s\}$, $\#(A)$ is the cardinal number of the set A, the number $P(A, B) = \sum\limits_{j\in B} P(A, j)$, $P(A, j) = \#\{i, i \in A, i > j\}$, the symmetric difference set $A\triangle B$ is also order-preserving in the above way, and $\lambda_A \in \mathcal{R}$ is the coefficient of the e_A–component of the Clifford number λ. It follows at once from the multiplication rule (2.1) that e_0 is the identity element written now as 1 and in particular,

$$
(2.2)\qquad \begin{cases} e_i^2 = 1, & \text{if } \ i = 1, \cdots, s, \\ e_j^2 = -1, & \text{if } \ j = s+1, \cdots, n, \\ e_i e_j = -e_j e_i, & \text{if } \ 1 \le i < j \le n, \\ e_{h_1} e_{h_2} \cdots e_{h_r} = e_{h_1 h_2 \cdots h_r}, & \text{if } \ 1 \le h_1 < h_2 \cdots, < h_r \le n. \end{cases}
$$

Thus $C(V_{n,s})$ is a real linear, associative, but non-commutative algebra and it is called the universal Clifford algebra over $V_{n,s}$.

In the sequel, we constantly use the following conjugate:

$$
(2.3)\qquad \begin{cases} \overline{e_A} = (-1)^{\sigma(A)+\#(A\cap S)} e_A, & \text{if } \ A \in \mathcal{P}N, \\ \overline{\lambda} = \sum\limits_{A\in\mathcal{P}N} \lambda_A \overline{e_A}, & \text{if } \ \lambda = \sum\limits_{A\in\mathcal{P}N} \lambda_A e_A, \end{cases}
$$

where $\sigma(A) = \#(A)(\#(A)+1)/2$. Sometimes λ_A is also written as $[\lambda]_A$, in particular, the coefficient $\lambda_\emptyset$ is denoted by λ_0 or $[\lambda]_0$, which is called the scalar part of the Clifford number λ.

From (2.3), it is easy to check:

$$
(2.4)\qquad \begin{cases} \overline{e_i} = e_i, & \text{if } \ i = 0, 1, \cdots, s, \\ \overline{e_j} = -e_j, & \text{if } \ j = s+1, \cdots, n, \\ \overline{\lambda\mu} = \overline{\mu}\,\overline{\lambda}, & \text{for any } \ \lambda, \mu \in C(V_{n,s}). \end{cases}
$$

We introduce the norm on $C\left(V_{n,s}\right)$

$$
(2.5)\qquad |\lambda| = \sqrt{(\lambda,\lambda)} = \left(\sum_{A\in\mathcal{P}N} \lambda_A^2 \right)^{\frac{1}{2}}.
$$

Let Ω be an open non empty subset of $\mathcal{R}^n$. Functions f defined in Ω and with values in $C\left(V_{n,s}\right)$ will be considered, i.e.,

$$f : \Omega \longrightarrow C(V_{n,s}) \,.$$

They are of the form

$$f(x) = \sum_A f_A(x) e_A, \;\; x = (x_1, x_2, \cdots, x_n) \in \Omega,$$

where the symbol $\sum\limits_A$ is abbreviated from $\sum\limits_{A \in \mathcal{P}N}$ and $f_A(x)$ is the e_A–component of $f(x)$. Obviously, f_A are real–valued functions in Ω, which are called the e_A–component functions of f. Whenever a property such as continuity, differentiability, etc. is ascribed to f, it is clear that in fact all the component functions f_A possess the cited property. So $f \in C^{(r)}(\Omega, C\left(V_{n,s}\right))$ is very clear.

The conjugate of the function f is the function $\overline{f}$ given by

$$\overline{f}(x) = \sum_A f_A(x) \overline{e_A}, \;\; x \in \Omega.$$

The following is an obvious fact.

Remark 2.1. $f \in C^{(r)}(\Omega, C\left(V_{n,s}\right))$ if and only if $\overline{f} \in C^{(r)}(\Omega, C\left(V_{n,s}\right))$.

Introduce the following operators

$$D_1 = \sum_{k=1}^{s} e_k \frac{\partial}{\partial x_k} : \;\; C^{(r)}(\Omega, C\left(V_{n,s}\right)) \longrightarrow C^{(r-1)}(\Omega, C\left(V_{n,s}\right)) \,,$$

$$D_2 = \sum_{k=s+1}^{n} e_k \frac{\partial}{\partial x_k} : \;\; C^{(r)}(\Omega, C\left(V_{n,s}\right)) \longrightarrow C^{(r-1)}(\Omega, C\left(V_{n,s}\right)) \,.$$

Let f be a function with value in $C(V_{n,s})$ defined in Ω, the operators D_1 and D_2 act on function f from the left and right being governed by the rules

$$D_1[f] = \sum_{k=1}^{s}\sum_{A} e_k e_A \frac{\partial f_A}{\partial x_k}, \qquad [f]D_1 = \sum_{k=1}^{s}\sum_{A} e_A e_k \frac{\partial f_A}{\partial x_k},$$
$$D_2[f] = \sum_{k=s+1}^{n}\sum_{A} e_k e_A \frac{\partial f_A}{\partial x_k}, \qquad [f]D_2 = \sum_{k=s+1}^{n}\sum_{A} e_A e_k \frac{\partial f_A}{\partial x_k}.$$

Definition 2.1. *A function $f \in C^{(r)}(\Omega, C(V_{n,s}))$ $(r \geq 1)$ is called (D_1) left $(right)$ regular in Ω if $D_1[f] = 0\,([f]D_1 = 0)$.*

A function $f \in C^{(r)}(\Omega, C(V_{n,s}))\,(r \geq 1)$ is called (D_2) left $(right)$ regular in Ω if $D_2[f] = 0\,([f]D_2 = 0)$. f is said to be (D_1) $((D_2))$ biregular if and only if it is both (D_1) $((D_2))$ left and (D_1) $((D_2))$ right regular.

Definition 2.2. *A function $f \in C^{(r)}(\Omega, C(V_{n,s}))$ $(r \geq 1)$ is said to be LR regular in Ω if and only if it is both (D_1) left regular and (D_2) right regular, i.e., $D_1[f] = 0$ and $[f]D_2 = 0$ in Ω.*

Frequent use will be made of the notation $\mathcal{R}^n_z$ where $z \in \mathcal{R}^n$, which means to remove z from $\mathcal{R}^n$. In particular $\mathcal{R}^n_0 = \mathcal{R}^n \setminus \{0\}$.

Example 2.1. *Suppose*

$$H(x) = \frac{1}{\rho_1^s(x)} \sum_{k=1}^{s} x_k e_k, \;\; x = (x_1, x_2, \cdots, x_n) \in \mathcal{R}^s_0 \times \mathcal{R}^{n-s}$$

where

$$\rho_1(x) = \left(\sum_{k=1}^{s} x_k^2\right)^{\frac{1}{2}},$$

and

$$E(x) = \frac{1}{\rho_2^{n-s}(x)} \sum_{k=s+1}^{n} x_k e_k, \;\; x = (x_1, x_2, \cdots, x_n) \in \mathcal{R}^s \times \mathcal{R}^{n-s}_0$$

where

$$\rho_2(x) = \left(\sum_{k=s+1}^{n} x_k^2 \right)^{\frac{1}{2}},$$

then H, E, HE, and EH are both (D_1) and (D_2) biregular, respectively, in $\mathcal{R}_0^s \times \mathcal{R}^{n-s}$, $\mathcal{R}^s \times \mathcal{R}_0^{n-s}$ and $\mathcal{R}_0^s \times \mathcal{R}_0^{n-s}$ (see [10]*).*

Example 2.2.*Suppose that $H(x)$ and $E(x)$ are as above, then by* (2.3) *and* (2.4)*, $\overline{H} = H$, $\overline{E} = -E$, $\overline{HE} = -EH$ and $\overline{EH} = -HE$, so $\overline{H}, \overline{E}, \overline{HE}, \overline{EH}$ are both (D_1) biregular and (D_2) biregular, respectively, in $\mathcal{R}_0^s \times \mathcal{R}^{n-s}$, $\mathcal{R}^s \times \mathcal{R}_0^{n-s}$ and $\mathcal{R}_0^s \times \mathcal{R}_0^{n-s}$.*

As can be seen from the above Example 2.1–2.2, we often need to consider the especial case $\Omega = \Omega_1 \times \Omega_2$ where Ω_1 is an open non empty set in $\mathcal{R}^s$ and Ω_2 is an open non empty set in $\mathcal{R}^{n-s}$. In this case, the points in $\Omega_1 \times \Omega_2$ are denoted alternatively by

$$x = (x_1, x_2, \cdots, x_n) = (x^S, x^{N\setminus S})$$

where $x^S = (x_1, x_2, \cdots, x_s) \in \Omega_1$ and $x^{N\setminus S} = (x_{s+1}, x_{s+2}, \cdots, x_n) \in \Omega_2$. Correspondingly, the functions defined in Ω are denoted alternatively by

$$f(x) = f(x^S, x^{N\setminus S}).$$

It is also seen that H in Example 2.1 may be really treated as the function from $\Omega_1 \subset \mathcal{R}^s$ to $C(V_{n,s})$. In this manner, thereinafter we would rather write $f \in C^{(r)}(\Omega_1, C(V_{n,s}))$ than $f \in C^{(r)}(\Omega, C(V_{n,s}))$. The meaning of the symbol $C^{(r)}(\Omega_2, C(V_{n,s}))$ is similar and obvious.

Example 2.3. *For fixed $z = (z^S, z^{N\setminus S}) \in \mathcal{R}^n$, $H(x-z)$, $\overline{H}(x-z)$, $E(x-z)$, $\overline{E}(x-z)$, $(HE)(x-z)$, $(EH)(x-z)$, $\overline{HE}(x-z)$, $\overline{EH}(x-z)$ are both (D_1)*

biregular and (D_2) *biregular, respectively, in* $\mathcal{R}^s_{z^S} \times \mathcal{R}^{n-s}$, $\mathcal{R}^s \times \mathcal{R}^{n-s}_{z^{N\setminus S}}$ *and* $\mathcal{R}^s_{z^S} \times \mathcal{R}^{n-s}_{z^{N\setminus S}}$ *(see* [10]*).*

Since we shall only consider the case of $s = n$ in this paper, we shall denote the operator D_1 as D.

Definition 2.3. *A function* $f \in C^{(r)}(\Omega, C(V_{n,n}))$ $(r \geq 1)$ *is called left (right) regular in* Ω *if* $D[f] = 0\,([f]D = 0)$ *in* Ω*;*

A function $f \in C^{(r)}(\Omega, C(V_{n,n}))$ $(r \geq k)$ *is called left (right) k-regular in* Ω *if* $D^k[f] = 0\,([f]D^k = 0)$ *in* Ω.

Let M be an n–dimensional differentiable oriented manifold with boundary contained in some open non empty set $\Omega \subset \mathcal{R}^n$. The differential space with basis $\{\mathrm{d}x^1, \mathrm{d}x^2, \cdots, \mathrm{d}x^n\}$ is denoted by V_n. Let $G(V_n)$ be the Grassmann algebra over V_n with basis $\left\{\mathrm{d}x^A, A \in \mathcal{P}N\right\}$. The exterior product on $G(V_n)$ also may be defined by

$$
(2.6)\quad \begin{cases}
\mathrm{d}x^A \wedge \mathrm{d}x^B = (-1)^{P(A,B)} \mathrm{d}x^{A\cup B}, & \text{if } \ A, B \in \mathcal{P}N, A \bigcap B = \emptyset, \\[2ex]
\mathrm{d}x^A \wedge \mathrm{d}x^B = 0, & \text{if } \ A, B \in \mathcal{P}N, A \bigcap B \neq \emptyset, \\[2ex]
\eta \wedge \upsilon = \sum\limits_A \sum\limits_B \eta^A \upsilon^B \mathrm{d}x^A \wedge \mathrm{d}x^B, & \text{if } \ \eta = \sum\limits_A \eta^A \mathrm{d}x^A, \ \upsilon = \sum\limits_A \upsilon^A \mathrm{d}x^A,
\end{cases}
$$

where η^A and υ^A are real and $\sum\limits_A$ is the same as before. Obviously, as a rule,

$$
(2.7)\quad \begin{cases}
\mathrm{d}x^\emptyset = \mathrm{d}x^0 = 1, \\[2ex]
\mathrm{d}x^{h_1} \wedge \mathrm{d}x^{h_2} \cdots \wedge \mathrm{d}x^{h_r} = \mathrm{d}x^{h_1 h_2 \cdots h_r}, \ \text{if } 1 \leq h_1 < h_2 \cdots, < h_r \leq n, \\[2ex]
\mathrm{d}x^A \wedge \mathrm{d}x^B = (-1)^{\#(A)\#(B)} \mathrm{d}x^B \wedge \mathrm{d}x^A, \quad \text{if } \ A, B \in \mathcal{P}N.
\end{cases}
$$

If moreover we construct the direct product algebra $\mathcal{W} = (C(V_{n,s}), G(V_n))$, then we may consider a function $\Upsilon : M \longrightarrow \mathcal{W}$ of the form

$$\Upsilon(x) = \sum_{A} \sum_{\#(B)=p} \Upsilon_{A,B}(x) e_A \mathrm{d}x^B,$$

where all $\Upsilon_{A,B}$ are of the class $C^{(r)}$ $(r \geq 1)$ in Ω and p is fixed, $0 \leq p \leq n$. Υ is called a $C(V_{n.s})$–valued p–differential form.

Let furthermore C be a p–chain on M, then we define

$$\int_C \Upsilon(x) = \sum_{A} \sum_{\#(B)=p} e_A \int_C \Upsilon_{A,B}(x) \mathrm{d}x^B.$$

In the sequel, since we shall only consider the case of $s = n$, we shall use the following $C(V_{n.n})$–valued $(n-1)$–differential form, which is exact and written as

$$\mathrm{d}\theta = \sum_{k=1}^{n} (-1)^{k-1} e_k \mathrm{d}\widehat{x}_k^N,$$

where

$$\mathrm{d}\widehat{x}_k^N = \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^{k-1} \wedge \mathrm{d}x^{k+1} \cdots \wedge \mathrm{d}x^n.$$

3 Kernel functions

In this section, we shall construct the kernel functions which play a crucial role to obtain the Cauchy-Pompeiu formula in universal Clifford analysis, and we will give some of its properties. Suppose

$$
(3.1) \qquad H_j^*(x) = \begin{cases} \dfrac{1}{2^{i-1}(i-1)! \prod\limits_{r=1}^{i}(2r-n)} \dfrac{1}{\omega_n} \dfrac{\mathbf{x}^{2i}}{\rho^n(x)}, \\ \quad j = 2i, j < n, i = 1, 2, \cdots, \\ \dfrac{1}{2^i i! \prod\limits_{r=1}^{i}(2r-n)} \dfrac{1}{\omega_n} \dfrac{\mathbf{x}^{2i+1}}{\rho^n(x)}, \\ \quad j = 2i+1, j < n, i = 0, 1, \cdots, \end{cases}
$$

where $\mathbf{x} = \sum\limits_{k=1}^{n} x_k e_k$, $\rho(x) = \left(\sum\limits_{k=1}^{n} x_k^2\right)^{\frac{1}{2}}$, and ω_n denotes the area of the unit sphere in $\mathcal{R}^n$. We denote

$$
(3.2) \quad A_j = \begin{cases} \dfrac{1}{2^{i-1}(i-1)! \prod\limits_{r=1}^{i}(2r-n)} & j = 2i, j < n, i = 1, 2, \cdots, \\ \dfrac{1}{2^i i! \prod\limits_{r=1}^{i}(2r-n)} & j = 2i+1, j < n, i = 0, 1, \cdots, \end{cases}
$$

then

$$
H_j^*(x) = \frac{A_j}{\omega_n} \frac{\mathbf{x}^j}{\rho^n(x)}, \qquad j < n.
$$

From (3.1), it is easy to check that,

$$
(3.3) \qquad \begin{cases} H_1^*(x) & = \dfrac{1}{\omega_n} \dfrac{\mathbf{x}}{\rho^n(x)}, \\ H_{2i+1}^*(x) & = \dfrac{1}{2i} H_{2i}^*(x) \mathbf{x}, \\ H_{2i}^*(x) & = \dfrac{1}{2i-n} H_{2i-1}^*(x) \mathbf{x}. \end{cases}
$$

Lemma 3.1. *Let $H_j^*(x)$ be as above, then we have,*

$$\left\{\begin{array}{l} D\left[H_1^*(x)\right]=\left[H_1^*(x)\right]D=0, x\in\mathcal{R}_0^n, \\ D\left[H_{j+1}^*(x)\right]=\left[H_{j+1}^*(x)\right]D=H_j^*(x), x\in\mathcal{R}_0^n, \\ \quad \textit{for any } 1\leq j<n-1. \end{array}\right. \tag{3.4}$$

Proof. First we know that the following equality is just the special case of $s=n$ of example 2.1:

$$D\left[H_1^*(x)\right]=\left[H_1^*(x)\right]D=0, x\in\mathcal{R}_0^n.$$

In the following, we will prove that the second equality in (3.4) holds by induction, and in the sequel, we suppose $x\in\mathcal{R}_0^n$.

Step 1. For $j=1$, we rewrite $H_1^*(x)$ as $H_1^*(x)=\sum\limits_{j=1}^{n}H_{1j}^*(x)e_j$, then from (3.3) we have

$$\begin{aligned} D\left[H_2^*(x)\right] &= \frac{1}{2-n}D\left[H_1^*(x)\mathbf{x}\right] \\ &= \frac{1}{2-n}\sum_{i=1}^{n}D\left[H_1^*(x)x_ie_i\right] \\ &= \frac{1}{2-n}\sum_{i=1}^{n}\sum_{j=1}^{n}e_jH_1^*(x)\delta_{ij}e_i \left(\text{since} D\left[H_1^*(x)\right]=0\right) \\ &= \frac{1}{2-n}\sum_{i=1}^{n}\sum_{j=1}^{n}\left(-H_1^*(x)e_j+2H_{1j}^*(x)e_je_j\right)\delta_{ij}e_i \\ &= \frac{1}{2-n}\sum_{i=1}^{n}\left(-H_1^*(x)e_i+2H_{1i}^*(x)\right)e_i \\ &= \frac{1}{2-n}\sum_{i=1}^{n}\left(-H_1^*(x)+2H_{1i}^*(x)e_i\right) \\ &= \frac{1}{2-n}\left(-nH_1^*(x)+2H_1^*(x)\right) \\ &= H_1^*(x) \end{aligned}$$

In view of $H_2^*(x)$ being a scalar function, and so

$$[H_2^*(x)]\,D = D\,[H_2^*(x)] = H_1^*(x).$$

For $j=2$, from (3.3), and in view of $H_2^*(x)$ being a scalar function, we have,

$$\begin{aligned}
D\,[H_3^*(x)] &= \sum_{k=1}^{n} D\left[\frac{1}{2}H_2^*(x)x_k e_k\right]\\
&= \frac{1}{2}\sum_{k=1}^{n}\left(D\,[H_2^*(x)]\,x_k e_k + H_2^*(x)\right)\\
&= \frac{1}{2}\sum_{k=1}^{n}\left(H_1^*(x)x_k e_k + H_2^*(x)\right)\\
&= \frac{1}{2}\left(H_1^*(x)\mathbf{x} + nH_2^*(x)\right)\\
&= \frac{1}{2}\left((2-n)H_2^*(x) + nH_2^*(x)\right)\\
&= H_2^*(x).
\end{aligned}$$

Similarly, by (3.3) again, and in view of $\mathbf{x}H_1^*(x) = H_1^*(x)\mathbf{x} = (2-n)H_2^*(x)$, we have,

$$\begin{aligned}
[H_3^*(x)]\,D &= \left[\frac{1}{2}H_2^*(x)\sum_{k=1}^{n} x_k e_k\right] D\\
&= \frac{1}{2}\sum_{k=1}^{n}\left[H_2^*(x)x_k e_k\right] D\\
&= \frac{1}{2}\sum_{k=1}^{n}\left(H_2^*(x) + x_k e_k\,[H_2^*(x)]\,D\right)\\
&= \frac{1}{2}\left(nH_2^*(x) + \mathbf{x}H_1^*(x)\right)\\
&= H_2^*(x).
\end{aligned}$$

Step 2. Suppose (3.4) holds for $j \le 2k-1$, or clearly,

$$D\left[H_{j+1}^*(x)\right] = \left[H_{j+1}^*(x)\right] D = H_j^*(x), j \le 2k-1.$$

Now we will prove that the following equality holds for $j = 2k$:

$$D\left[H^*_{2k+1}(x)\right] = \left[H^*_{2k+1}(x)\right]D = H^*_{2k}(x).$$

From (3.3) and the induction hypothesis, in view of $H^*_{2k}(x)$ being a scalar function, we have,

$$\begin{aligned}
D\left[H^*_{2k+1}(x)\right] &= \sum_{i=1}^{n} D\left[\frac{1}{2k}H^*_{2k}(x)x_i e_i\right] \\
&= \frac{1}{2k}\sum_{i=1}^{n}\left(D\left[H^*_{2k}(x)\right]x_i e_i + H^*_{2k}(x)\right) \\
&= \frac{1}{2k}\sum_{i=1}^{n}\left(H^*_{2k-1}(x)x_i e_i + H^*_{2k}(x)\right) \\
&= \frac{1}{2k}\left(H^*_{2k-1}(x)\mathbf{x} + nH^*_{2k}(x)\right) \\
&= \frac{1}{2k}\left((2k-n)H^*_{2k}(x) + nH^*_{2k}(x)\right) \\
&= H^*_{2k}(x).
\end{aligned}$$

Meanwhile, by (3.3) and the induction hypothesis again, in view of $\mathbf{x}H^*_{2k-1}(x) = H^*_{2k-1}(x)\mathbf{x} = (2k-n)H^*_{2k}(x)$, in a similar way one can check

$$\left[H^*_{2k+1}(x)\right]D = H^*_{2k}(x).$$

Step 3. Suppose (3.4) holds for $j \leq 2k$, or clearly,

$$D\left[H^*_{j+1}(x)\right] = \left[H^*_{j+1}(x)\right]D = H^*_{j}(x), j \leq 2k.$$

Now we will prove that the following equality holds for $j = 2k+1$:

$$D\left[H^*_{2k+2}(x)\right] = \left[H^*_{2k+2}(x)\right]D = H^*_{2k+1}(x).$$

From (3.3) and the induction hypothesis, we have,

$$
\begin{aligned}
D\left[H_{2k+2}^*(x)\right] &= \frac{1}{2k+2-n} D\left[H_{2k+1}^*(x)\mathbf{x}\right] \\
&= \frac{1}{2k+2-n}\sum_{i=1}^{n} D\left[H_{2k+1}^*(x)x_i e_i\right] \\
&= \frac{1}{2k+2-n}\sum_{i=1}^{n} \left(D\left[H_{2k+1}^*(x)\right]x_i e_i + e_i H_{2k+1}^*(x)e_i\right) \\
&= \frac{1}{2k+2-n}\left((2k-n)H_{2k+1}^*(x) + 2H_{2k+1}^*(x)\right) \\
&= H_{2k+1}^*(x).
\end{aligned}
$$

In view of $H_{2k+2}^*(x)$ being a scalar function, and so

$$
\left[H_{2k+2}^*(x)\right] D = D\left[H_{2k+2}^*(x)\right] = H_{2k+1}^*(x).
$$

So, from the above three steps, the result follows.

Lemma 3.2. *Let $H_k^*(x)(k<n)$ be as above, then we have,*

$$
(3.5) \qquad \begin{cases} D^k\left[H_k^*(x)\right] &= \left[H_k^*(x)\right]D^k = 0, x \in \mathcal{R}_0^n, \\ D^j\left[H_k^*(x)\right] &= \left[H_k^*(x)\right]D^j = H_{k-j}^*(x), x \in \mathcal{R}_0^n, \ \ j<k. \end{cases}
$$

Proof. It may be directly proved by Lemma 3.1.

Similarly, we have:

Lemma 3.3. *Let $H_j^*(x)$ be as above, then we have,*

$$
(3.6) \quad \begin{cases} D\left[H_1^*(x-z)\right] &= \left[H_1^*(x-z)\right]D = 0, x \in \mathcal{R}_z^n, \\ D\left[H_{j+1}^*(x-z)\right] &= \left[H_{j+1}^*(x-z)\right]D = H_j^*(x-z), x \in \mathcal{R}_z^n, \\ & \qquad \textit{for any } 1 \le j < n-1. \end{cases}
$$

Lemma 3.4. *Let $H_k^*(x)(k < n)$ be as above, then we have,*

$$(3.7)\quad \begin{cases} D^k\left[H_k^*(x-z)\right] = \left[H_k^*(x-z)\right]D^k = 0, x \in \mathcal{R}_z^n, \\ \\ D^j\left[H_k^*(x-z)\right] = \left[H_k^*(x-z)\right]D^j = H_{k-j}^*(x-z), x \in \mathcal{R}_z^n,\ \ j < k. \end{cases}$$

Remark 3.1. From Lemma 3.2 and Lemma 3.4, $H_k^*(x)(k < n)$ are left (right) k-regular functions with values in the universal Clifford algebra $C(V_{n,n})$ in $\mathcal{R}_0^n$; $H_k^*(x-z)(k < n)$ are left (right) k-regular functions with values in the universal Clifford algebra $C(V_{n,n})$ in $\mathcal{R}_z^n$.

4 Higher order Cauchy-Pompeiu formula

Lemma 4.1. *Let M be an n–dimensional differentiable compact oriented manifold contained in some open non empty subset $\Omega \subset \mathcal{R}^n$, $f \in C^{(r)}\left(\Omega, C\left(V_{n,n}\right)\right)$, $g \in C^{(r)}\left(\Omega, C\left(V_{n,n}\right)\right)$, $r \geq 1$, and moreover ∂M is given the induced orientation. Then*

$$\int\limits_{\partial M} f(x)\mathrm{d}\theta g(x) = \int\limits_{M} \left(\left(\left[f(x)\right]D\right)g(x) + f(x)\left(D\left[g(x)\right]\right)\right)\mathrm{d}x^N.$$

Proof. It has been proved in [5,7].

Theorem 4.1.(Higher order Cauchy-Pompeiu formula) *Suppose that M is an n–dimensional differentiable compact oriented manifold contained in some open non empty subset $\Omega \subset \mathcal{R}^n$, $f \in C^{(r)}\left(\Omega, C\left(V_{n,n}\right)\right)$, $r \geq k$, $k < n$, moreover ∂M is given the induced orientation, $H_j^*(x)$ is as above. Then, for $z \in \overset{\circ}{M}$*

$$f(z) = \sum_{j=0}^{k-1} (-1)^j \int_{\partial M} H^*_{j+1}(x-z) \mathrm{d}\theta D^j f(x) + \tag{4.1}$$
$$+(-1)^k \int_M H^*_k(x-z) D^k f(x) \mathrm{d}x^N.$$

Proof. Assume $z \in \overset{\circ}{M}$. Take $\delta > 0$ such that $B(z,\delta) \subset \overset{\circ}{M}$, denoting

$$\Theta(\delta) = \sum_{j=0}^{k-1} (-1)^j \int_{\partial(M \setminus B(z,\delta))} H^*_{j+1}(x-z) \mathrm{d}\theta D^j f(x),$$

$$\Delta(\delta) = (-1)^{k-1} \int_{M \setminus B(z,\delta)} H^*_k(x-z) D^k f(x) \mathrm{d}x^N.$$

by Lemma 3.1, Lemma 3.2 and Lemma 4.1, we have

$$\Theta(\delta) = \Delta(\delta). \tag{4.2}$$

In view of the weak singularity of the kernel H^*_k, the existence of the integral over the manifold M in (4.1) follows, and so we have,

$$\lim_{\delta \to 0} \Delta(\delta) = (-1)^{k-1} \int_M H^*_k(x-z) D^k f(x) \mathrm{d}x^N. \tag{4.3}$$

Thus we have,

$$\Theta(\delta) = \sum_{j=0}^{k-1} (-1)^j \int_{\partial M} H^*_{j+1}(x-z) \mathrm{d}\theta D^j f(x) - \Theta_1(\delta), \tag{4.4}$$

where

$$\Theta_1(\delta) = \sum_{j=0}^{k-1} (-1)^j \int_{\partial B(z,\delta)} H^*_{j+1}(x-z) \mathrm{d}\theta D^j f(x), \tag{4.5}$$

where $\partial B(z,\delta)$ is given the induced orientation.

By Stoke's formula, it is easy to check that

$$\lim_{\delta \to 0} \Theta_1(\delta) = f(z). \tag{4.6}$$

Taking limit $\delta \to 0$ in (4.2) and combining (4.3), (4.4), (4.5) with (4.6), (4.1) follows.

Remark 4.1. Introducing the operators

$$\left(T_k f\right)(z) = (-1)^k \int\limits_M H_k^*(x-z) f(x) \mathrm{d}x^N, 1 \leq k < n, \tag{4.7}$$

then (4.1) may be rewritten as

$$f(z) = \sum_{j=0}^{k-1} (-1)^j \int\limits_{\partial M} H_{j+1}^*(x-z) \mathrm{d}\theta D^j f(x) + \left(T_k D^k f\right)(z). \tag{4.8}$$

For $k = 1$, the operator T_1 is just the Pompeiu operator T, and we call (4.1) the higher order Cauchy-Pompeiu formula in the universal Clifford algebra $C(V_{n,n})$.

5 Some applications

In this section, we will give some applications of the Cauchy-Pompeiu formula and for example, the Cauchy integral formula and the mean value theorem.

Theorem 5.1.(Cauchy integral formula) *Suppose that M is an n–dimensional differentiable compact oriented manifold contained in some open non empty subset $\Omega \subset \mathcal{R}^n$, and let f be a left k-regular function in M,* *moreover ∂M is given the induced orientation, $H_j^*(x)$ is as above. Then*

$$\sum_{j=0}^{k-1}(-1)^j \int\limits_{\partial M} H_{j+1}^*(x-z)\mathrm{d}\theta D^j f(x) = \begin{cases} 0, & \text{if } z \notin M, \\ f(z), & \text{if } z \in \overset{\circ}{M}\,. \end{cases} \tag{5.1}$$

Proof. By Theorem 4.1 and Stoke's formula, in view of function f being a left k-regular function in M, the result follows.

Theorem 5.2.(Mean Value Theorem) *Let Ω be an open non empty set in $\mathcal{R}^n$, and let f be a left k-regular function in Ω, t is chosen in such a way that $\overline{B}(a,t) \subset \Omega$, then*

$$f(a) = \frac{1}{t^n \omega_n} \sum_{j=0}^{\left[\frac{k-1}{2}\right]} t^{2j} A_{2j+1} \int\limits_{B(a,t)} \big((\mathbf{x}-\mathbf{a})D^{2j+1}f(x) + (n-2j)D^{2j}f(x)\big)\,\mathrm{d}x^N, \tag{5.2}$$

where $\omega_n = \dfrac{2\pi^{n/2}}{\Gamma(n/2)}$ *denotes the area of the unit sphere in* $\mathcal{R}^n$, $A_{2j+1} = \dfrac{1}{2^j j! \prod\limits_{r=1}^{j}(2r-n)}$.

Proof. By Theorem 5.1 and Lemma 4.1, combining (3.1) with (3.2), we have

$$\begin{aligned} f(a) &= \sum_{j=0}^{k-1}(-1)^j \int\limits_{\partial B(a,t)} H_{j+1}^*(x-a)\mathrm{d}\theta D^j f(x) \\ &= \frac{1}{t^n\omega_n}\sum_{j=0}^{k-1}(-1)^j A_{j+1} \int\limits_{\partial B(a,t)} (\mathbf{x}-\mathbf{a})^{j+1}\,\mathrm{d}\theta D^j f(x) \\ &= \frac{1}{t^n\omega_n}\sum_{j=1}^{k-1}(-1)^j A_{j+1} \int\limits_{\partial B(a,t)} (\mathbf{x}-\mathbf{a})^{j+1}\,\mathrm{d}\theta D^j f(x) + \\ &\qquad + \frac{1}{t^n\omega_n} \int\limits_{B(a,t)} (nf(x) + (\mathbf{x}-\mathbf{a})Df(x))\,\mathrm{d}x^N. \end{aligned}$$

Denote

$$(5.3)\quad \Delta_j = \frac{(-1)^j A_{j+1}}{t^n \omega_n} \int\limits_{\partial B(a,t)} (\mathbf{x}-\mathbf{a})^{j+1}\, \mathrm{d}\theta D^j f(x), \quad j = 1, \cdots, k-1,$$

then by Lemma 4.1, for $m = 1, \cdots, \left[\dfrac{k-1}{2}\right]$, we have

$$(5.4)\quad \Delta_{2m-1} + \Delta_{2m} = \frac{(-1)^{2m-1} A_{2m}}{t^n \omega_n} \int\limits_{\partial B(a,t)} (\mathbf{x}-\mathbf{a})^{2m}\, \mathrm{d}\theta D^{2m-1} f(x) +$$

$$+ \frac{(-1)^{2m} A_{2m+1}}{t^n \omega_n} \int\limits_{\partial B(a,t)} (\mathbf{x}-\mathbf{a})^{2m+1}\, \mathrm{d}\theta D^{2m} f(x) =$$

$$= \frac{t^{2m}}{t^n \omega_n} \int\limits_{B(a,t)} \big(A_{2m+1}(\mathbf{x}-\mathbf{a}) D^{2m+1} f(x) + (nA_{2m+1} - A_{2m})\, D^{2m} f(x) \big)\, \mathrm{d}x^N =$$

$$= \frac{A_{2m+1} t^{2m}}{t^n \omega_n} \int\limits_{B(a,t)} \big((\mathbf{x}-\mathbf{a}) D^{2m+1} f(x) + (n-2m) D^{2m} f(x) \big)\, \mathrm{d}x^N .$$

Obviously, if $k-1$ is an even number, then

$$\begin{aligned} f(a) &= \sum_{m=1}^{\frac{k-1}{2}} (\Delta_{2m-1} + \Delta_{2m}) + \frac{1}{t^n \omega_n} \int\limits_{B(a,t)} (n f(x) + (\mathbf{x}-\mathbf{a}) D f(x))\, \mathrm{d}x^N = \\ &= \sum_{m=0}^{\frac{k-1}{2}} \frac{A_{2m+1} t^{2m}}{t^n \omega_n} \int\limits_{B(a,t)} \big((\mathbf{x}-\mathbf{a}) D^{2m+1} f(x) + (n-2m) D^{2m} f(x) \big)\, \mathrm{d}x^N . \end{aligned}$$

(5.5)

If $k-1$ is an odd number, since f is a left k-regular function in Ω, then by Lemma 4.1, we have

$$(5.6)\qquad \Delta_{k-1} = \frac{(-1)^{k-1} A_k}{t^n \omega_n} \int\limits_{\partial B(a,t)} (\mathbf{x}-\mathbf{a})^k\, \mathrm{d}\theta D^{k-1} f(x) = 0,$$

and so,

$$
\begin{aligned}
f(a) &= \Delta_{k-1} + \sum_{m=1}^{\left[\frac{k-1}{2}\right]} (\Delta_{2m-1} + \Delta_{2m}) + \\
&+ \frac{1}{t^n \omega_n} \int\limits_{B(a,t)} (n f(x) + (\mathbf{x} - \mathbf{a}) D f(x))\, \mathrm{d}x^N = \qquad (5.7)
\end{aligned}
$$

$$
= \sum_{m=0}^{\left[\frac{k-1}{2}\right]} \frac{A_{2m+1} t^{2m}}{t^n \omega_n} \int\limits_{B(a,t)} \big((\mathbf{x} - \mathbf{a}) D^{2m+1} f(x) + (n - 2m) D^{2m} f(x)\big)\, \mathrm{d}x^N.
$$

From (5.6) and (5.7), the result follows.

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