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Sufficient conditions for n-starlikeness

Viorel Cosmin Holhoş

Abstract

In this paper we obtain a sufficient condition for n-starlikeness of the form:

$$\alpha \frac{D^{n+2}f\left(z\right)}{D^{n}f\left(z\right)}+\left(\beta-\alpha\right)\frac{D^{n+1}f\left(z\right)}{D^{n}f\left(z\right)}\prec h\left(z\right)$$

where $h\left(z\right)$ is an univalent function in the unit disc U and D^n is the Sălăgean differential operator.

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1 Introduction

Let $\mathcal{A}_n$, $n \in \mathbb{N}^*$ denote the class of functions of the form:

$$f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k$$

which are analytic in the unit disc $U=\{z\ ;\ z\in\mathbb{C},\ |z|<1\}$ and $\mathcal{A}_1=\mathcal{A}$.

We note
$$S^*=\left\{f\in\mathcal{A}:\operatorname{Re}\frac{zf'(z)}{f(z)}>0,\ z\in U\right\}$$
the class of functions $f\in\mathcal{A}$ which are *starlike* in the unit disc.

We denote by K *the* class of functions $f\in\mathcal{A}$ which are *convex* in the unit disc U, that is
$$K=\left\{f\in\mathcal{A}:\operatorname{Re}\left(\frac{zf''(z)}{f'(z)}+1\right)>0,\ z\in\ U\right\}.$$

For $f\in\mathcal{A}_n$ we define the Sălăgean differential operator D^n by
$$\begin{aligned}D^0f(z)&=f(z)\\ D^1f(z)&=Df(z)=zf'(z)\end{aligned}$$
and
$$D^{n+1}f(z)=D\left(D^nf(z)\right);\qquad n\in\mathbb{N}^*.$$

Let $\alpha\in[0,1)$ and $n\in\mathbb{N}$. The class $S_n(\alpha)$ named the class of $n-$*starlike function of order* α is defined by
$$S_n(\alpha)=\left\{f\in\mathcal{A}:\operatorname{Re}\frac{D^{n+1}f(z)}{D^nf(z)}>\alpha,\ z\in U\right\}.$$

Theorem 1 (see [1]). *Let q be a univalent function in U and let the functions θ,ϕ be analytic in a domain D containing $q(U)$, with $\phi(w)\neq 0$, when $w\in q(U)$. Let $Q(z)=zq'(z)\phi(q(z))$, $h(z)=\theta(q(z))+Q(z)$ and suppose that*

(i) *Q is starlike in U*

(ii) $\operatorname{Re}\dfrac{zh'(z)}{Q(z)}=\operatorname{Re}\left[\dfrac{\theta'(q(z))}{\phi(q(z))}+\dfrac{zQ'(z)}{Q(z)}\right]>0,\quad z\in U.$

If p *is analytic in* U, *with* $p(0) = q(0)$, $p(U) \subset D$, *and*

$$\theta\left(p(z)\right) + zp'(z)\phi\left(p(z)\right) \prec \theta\left(q(z)\right) + zq'(z)\phi\left(q(z)\right) = h(z)$$

then $p\left(z\right) \prec q\left(z\right)$ *and* q *is the best dominant.*

2 Main results

Theorem 2. *Let* $n \in \mathbb{N}$, *let* q *be a convex function in* U, *with* $q\left(0\right) = 1$,

(1) $$\operatorname{Re} q\left(z\right) > \frac{\alpha - \beta}{2\alpha}, \ \alpha > 0, \ \alpha + \beta > 0$$

and let $f(z) \in \mathcal{A}$, *with* $\dfrac{f\left(z\right)}{z} \neq 0$, *that satisfy*

$$\alpha \frac{D^{n+2} f\left(z\right)}{D^{n} f\left(z\right)} + \left(\beta - \alpha\right) \frac{D^{n+1} f\left(z\right)}{D^{n} f\left(z\right)} \prec h\left(z\right),$$

where

$$h\left(z\right) = \alpha z q'\left(z\right) + \alpha q^{2}\left(z\right) + \left(\beta - \alpha\right) q\left(z\right)$$

then

(2) $$\frac{D^{n+1} f(z)}{D^{n} f\left(z\right)} \prec q\left(z\right), \ z \in U.$$

Proof. Let q be a convex function in U, with $q\left(0\right) = 1$ and in Theorem 1 we choose

$$\begin{aligned} \theta\left(w\right) &= \alpha w^{2} + (\beta - \alpha) w \\ \phi\left(w\right) &= \alpha \neq 0 \\ Q\left(z\right) &= z q'\left(z\right) \phi\left(q\left(z\right)\right) = \alpha z q'\left(z\right) \end{aligned}$$

and we have

(i) $Q\left(z\right) = \alpha z q'(z)$ is starlike in U, because q is convex

$$(ii)\quad \mathrm{Re}\frac{zh'(z)}{Q(z)}=\mathrm{Re}\left[\frac{\theta'(q(z))}{\phi(q(z))}+\frac{zQ'(z)}{Q(z)}\right]=$$
$$=\mathrm{Re}\left[\frac{2\alpha q(z)+\beta-\alpha}{\alpha}+\frac{zQ'(z)}{Q(z)}\right]>0$$

because (1).

The conditions of Theorem 1 are satisfied and for $p(z)=1+p_1z+...$ which satisfies:

$$\alpha p^2(z)+(\beta-\alpha)p(z)+\alpha zp'(z)\prec h(z)$$

we have $p(z)\prec q(z)$ and q is the best dominant.

If we let

$$p(z)=\frac{D^{n+1}f(z)}{D^nf(z)}$$

then

$$\begin{aligned}\alpha p^2(z)+(\beta-\alpha)p(z)+\alpha zp'(z) &= \alpha\frac{D^{n+2}f(z)}{D^nf(z)}+(\beta-\alpha)\frac{D^{n+1}f(z)}{D^nf(z)}\prec\\ &\prec \alpha q^2(z)+(\beta-\alpha)q(z)+\alpha zq'(z)\end{aligned}$$

which implies that

$$\frac{D^{n+1}f(z)}{D^nf(z)}\prec q(z).$$

Remark 3. *For $n=0$ we obtain the result given in [2].*

For $n=1$ we have the following result:

Corollary 4. *Let q be a convex function in U, with $q(0)=1$,*

(3) $$\mathrm{Re}\,q(z)>\frac{\alpha-\beta}{2\alpha},\ \alpha>0,\ \alpha+\beta>0.$$

If

$$\alpha\frac{D^3f(z)}{Df(z)}+(\beta-\alpha)\frac{D^2f(z)}{Df(z)}\prec\alpha q^2(z)+(\beta-\alpha)q(z)+\alpha zq'(z),$$

then

$$\frac{D^2 f(z)}{Df(z)} \prec q(z), z \in U. \tag{4}$$

If we let $\alpha = 1$, $\beta \geq 1$ and $q(z) = \frac{1+z}{1-z}$ then

$$h(z) = \frac{2z}{(1-z)^2} + \left(\frac{1+z}{1-z}\right)^2 + (\beta - 1)\frac{1+z}{1-z}, \; z \in U \tag{5}$$

and from Theorem 2 we have:

Corollary 5. *If $f(z) \in \mathcal{A}$, with $\frac{f(z)}{z} \neq 0$, satisfies*

$$\frac{D^{n+2} f(z)}{D^n f(z)} + (\beta - 1)\frac{D^{n+1} f(z)}{D^n f(z)} \prec h(z)$$

where h is given by (5), then

$$\frac{D^{n+1} f(z)}{D^n f(z)} \prec \frac{1+z}{1-z}, \; z \in U. \tag{6}$$

The relation (6) is equivalent to

$$\mathrm{Re}\, \frac{D^{n+1} f(z)}{D^n f(z)} > 0$$

that is f is n-starlike function. In this case for $n = 0$ we obtain the following starlikeness condition:

Corollary 6. *If $f(z) \in \mathcal{A}$, with $\frac{f(z)}{z} \neq 0$, satisfies*

$$\frac{D^2 f(z)}{Df(z)} + (\beta - 1)\frac{D^1 f(z)}{Df(z)} \prec h(z),$$

where h is given by (5), then

$$\mathrm{Re}\, \frac{zf'(z)}{f(z)} > 0, \; z \in U. \tag{7}$$

For n=1 we obtain the following convexity condition:

Corollary 7. *If* $f(z) \in \mathcal{A}$, *with* $\frac{f(z)}{z} \neq 0$, *satisfies*

$$\frac{D^3 f(z)}{D' f(z)} + (\beta - 1) \frac{D^2 f(z)}{D' f(z)} \prec h(z),$$

where h is given by (5), then

$$\operatorname{Re}\left(1 + \frac{z f''(z)}{f'(z)}\right) > 0, \ z \in U. \tag{8}$$

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Universitatea Ecologică “Traian”
Str. 1 Decembrie 1918, nr. 14
330018 Deva, Romania
E-mail: *holhosv@yahoo.com*